Online Appendix to:
A New Keynesian Q Theory
and the Link between Inflation and the Stock Market

Pierlauro Lopez
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This section explores the theoretical robustness of the NKQ equation and shows reasons for the presence of a specification error, $e_t$. The NKQ equation is robust to several alternative modeling choices such as to firm-specific rather than rented capital (Woodford, 2005), to state-dependent pricing (Gertler and Leahy, 2008), and to time-varying markups (e.g., Leahy, 2011), as the New Keynesian Phillips curve in its form $\pi_t = \beta E_t \pi_{t+1} + \lambda t$ is robust to all these modeling variations. The presence of measurement error and alternative modeling choices that break this form of the New Keynesian Phillips curve will however affect the NKQ equation; the presence of leverage directly modifies the definition of corporate profits, and hence the NKQ equation (expression (V.11)); and the absence of a fiscal policy that corrects the steady-state distortions that owe to monopolistic competition affects the NKQ equation (expression (VI.12)).

Coincidently, this section derives the New Keynesian Phillips curve when capital is firm-specific under investment adjustment costs (expression (II.4)), generalized demand curves (expression (III.9)) and in its hybrid version (expression (IV.10)), thereby extending to these cases the result in Sveen and Weinke (2005) and Woodford (2005). I also remind here their derivation of the New Keynesian Phillips curve under firm-specific capital and capital adjustment costs (CAC).

I Reminder: Phillips curve under capital adjustment costs (CAC)

I show here explicitly the derivation of the New Keynesian Phillips curve under a CAC specification (Sveen and Weinke, 2005; Woodford, 2005). The problem of ith firm is:

$$\min_{\pi_t} \max_{P_t} E_0 \sum_{h=0}^{\infty} \left[ P_{t+h}(i) \bar{P}_{t+h} - (1-\tau) W_{t+h} N_{t+h}(i) - (1-\tau) I_{t+h}(i) - T_{t+h} \right]$$

$$- MC_{t+h}(i) \left[ P_{t+h}(i) - A_{t+h} K_{t+h}(i) N_{t+h}(i)^{1-\alpha} \right]$$

$$- (1-\tau) Q_{t+h}(i) \left[ K_{t+h+1}(i) - (1-\delta) K_{t+h}(i) - \Phi \left( \frac{I_{t+h}(i)}{K_{t+h}(i)} \right) K_{t+h}(i) \right]$$

subject to

$$P_{t+h}(i) = \begin{cases} P^*_t(i) & \text{prob. } 1-\theta \\ P_{t+h-1}(i) & \text{prob. } \theta \end{cases}$$

Optimal price setting implies:

$$\sum_{h=0}^{\infty} (\beta \theta)^h E_t e^{\bar{P}_{t+h}(i)} \bar{P}_{t+h} + \bar{\pi}_{t+h} \left[ M^{-1} e^{\bar{P}_t(i)} - P_{t+h} - MC e^{\bar{m}_{t+h}(i)} \right] = 0$$

where $M \equiv \frac{\varepsilon}{1-\varepsilon}$ is the real markup, and $E^t$ denotes an expectation operator conditional on firm $i$ not resetting in the future.

The homogeneity of the production function implies $\bar{m}_{t+h}(i) = -m \bar{m}_{t+h}(i) = \frac{\alpha}{1-\alpha} \bar{y}_{t+h}(i) - \frac{\alpha}{1-\alpha} \bar{K}_{t+h}(i)$

and $m \bar{p}_{t+h}(i) = \bar{y}_{t+h}(i) - \bar{K}_{t+h}(i) = \frac{1-\alpha}{\alpha} \bar{m}_{t+h}(i)$, and hence

$$\bar{m}_{t+h}(i) = -\varepsilon \bar{p}_{t+h}(i) - \bar{K}_{t+h}(i)$$

$$\bar{m}_{t+h}(i) = -\frac{\alpha \varepsilon}{1-\alpha} \bar{p}_{t+h}(i) - \frac{\alpha}{1-\alpha} \bar{K}_{t+h}(i)$$
It follows that a loglinearization of the optimal price setting equation around the zero-inflation steady state can be written as

$$p_t^*(i) - p_t = \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h} + (1 - \beta \theta) \Theta \sum_{h=0}^{\infty} (\beta \theta)^h E_t [\bar{m}c_{t+h} - \frac{\alpha}{1 - \alpha} \bar{k}_{t+h}(i)]$$

where I used $E_t[p_t^*(i) - p_{t+h}] = p_t^*(i) - E_t p_{t+h} = p_t^*(i) - p_t - \sum_{h=1}^{\infty} E_t \pi_{t+h}$ and where $\Theta \equiv \frac{1 - \alpha}{1 - \alpha + \alpha}$.

Then, guessing a solution

$$\bar{k}_{t+1}(i) = \psi_2 \bar{k}_t(i) + \psi_3 \bar{p}_t(i)$$

one has $E_t[\bar{k}_{t+h}(i)] = \psi_2 E_t[\bar{k}_{t+h-1}(i)] + \psi_3 [\bar{p}_t(i) + p_t^* - p_t - \sum_{h=1}^{\infty} E_t \pi_{t+h}]$ and $\sum_{h=0}^{\infty} (\beta \theta)^h E_t \bar{k}_{t+h}(i) = \bar{k}_t(i) + \beta \theta \psi_2 [\bar{p}_t(i) + p_t^* - p_t - \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h}]$. Plugging this expression into the optimal price setting decision and using the approximate aggregate price dynamics yields

$$\kappa \left( \bar{p}_t^*(i) + \frac{\theta}{1 - \alpha} \bar{p}_t \right) = -\frac{\alpha(1 - \beta \theta)}{(1 - \alpha)(1 - \beta \theta \psi_2)} \bar{k}_t(i) + \kappa \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h} + (1 - \beta \theta) \sum_{h=0}^{\infty} (\beta \theta)^h E_t \bar{m}c_{t+h}$$

where $\kappa \equiv \frac{1}{\bar{\theta}} + \frac{\alpha \beta \psi_3}{(1 - \alpha)(1 - \beta \psi_2)}$. The aggregation of this expression results in the New Keynesian Phillips curve under firm-specific capital accumulation,

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1 - \theta)(1 - \beta \theta)}{\kappa \theta} \bar{m}c_t$$

(1.1)

and hence, subtracting the expression from the individual equation,

$$\bar{p}_t^*(i) = \psi_1 \bar{k}_t(i)$$

(1.2)

where $\psi_1 \equiv -\frac{\alpha(1 - \beta \theta)}{(1 - \alpha)(1 - \beta \psi_2)}$.

Then, a loglinearization of the optimality condition for capital for firm $i$ results in equation

$$\Xi \bar{k}_{t+1}(i) = \beta E_t \bar{k}_{t+2}(i) + \bar{k}_t(i) - \bar{\varepsilon} [\Xi - 1 - \beta] E_t \bar{p}_{t+1}(i)$$

where $\Xi \equiv \frac{(1 + \beta)(1 - \alpha)\eta + \beta \delta}{\eta(1 - \alpha)}$. Noting that, to a first-order approximation, $E_t p_{t+1}(i) = \theta p_t(i) + (1 - \theta) E_t p_t^*(i)$, it follows that

$$E_t \bar{p}_{t+1}(i) = \theta \bar{p}_t(i) + (1 - \theta) E_t \bar{p}_t^*(i)$$

These two expressions identify the coefficients $\psi_2$ and $\psi_3$ as

$$\Xi \psi_2 = 1 + \beta \psi_2^2 + \{ \beta \psi_3 - \bar{\varepsilon} [\Xi - 1 - \beta] \}(1 - \theta) \psi_1 \psi_2$$

$$\Xi \psi_3 = \beta \psi_2 \psi_3 + \{ \beta \psi_3 - \bar{\varepsilon} [\Xi - 1 - \beta] \}[\theta + (1 - \theta) \psi_1 \psi_3]$$

thereby verifying the guessed solution for individual deviations from aggregate choices.

The problem of firms accumulating firm-specific capital under sticky prices adds therefore the two idiosyncratic dynamic equations to the state-space system driving aggregate variables,

$$\begin{bmatrix} E_t \bar{p}_{t+1}(i) \\ \bar{k}_{t+1}(i) \end{bmatrix} = \begin{bmatrix} \theta + (1 - \theta) \psi_1 \psi_3 & (1 - \theta) \psi_1 \psi_2 \\ \psi_3 & \psi_2 \end{bmatrix} \begin{bmatrix} \bar{p}_t(i) \\ \bar{k}_t(i) \end{bmatrix}$$

(1.3)

which, to have a unique bounded solution, must have both eigenvalues within the unit circle.

**II Robustness: Investment adjustment costs (IAC) (Christiano et al., 2005)**

The New Keynesian Q equation is only partially robust to using capital adjustment costs (CAC) or investment adjustment costs (CAC) $K_{t+1} = (1 - \delta) K_t + [1 - \Psi(\Delta I)] I_t$ with $\Psi(1) = \Psi'(1) = 0$ and $\hat{\eta} \equiv \Psi''(1) > 0$. The only difference is the relationship between marginal Q and the investment-capital ratio. The NKQ equation under IAC is:

$$s_t = \eta_i k_t - \frac{\beta \theta}{\lambda} E_t \pi_{t+1} + \epsilon_t, \quad \epsilon_t = -\eta_i (1 - \delta) i k_{t-1} + \beta E_t i k_{t+1}$$

(II.4)

where I used the approximate relation $\Delta k_{t+1} = \delta i k_t$. 

2
II.1 Flexible prices

I first look at the standard Q theory under IAC. Corporate profits become:

\[ V_{t}^{IAC} = \min_{\{Q_{t+h}\}_{h=0}^{\infty}} \frac{\max_{K_{t+h+1}}}{\min_{I_{t+h}}^{\infty}} E_{t} \sum_{h=1}^{\infty} \frac{\beta^{h} \Lambda_{t+h}}{\Lambda_{t}} (MPK_{t+h}K_{t+h} - I_{t+h}) + E_{t} \sum_{h=1}^{\infty} \frac{\beta^{h} \Lambda_{t+h}}{\Lambda_{t}} Q_{t+h}[(1 - \delta)K_{t+h} + [1 - \Psi(\Delta I_{t+h})]I_{t+h} - K_{t+h+1}] \]

with necessary conditions for \( s \geq t \):

\[ \Lambda_{s}I_{s} = \Lambda_{s}I_{s}Q_{s}[1 - \Psi(\Delta I_{s}) - \Psi'(\Delta I_{s})\Delta I_{s}] + E_{s}\beta\Lambda_{s+1}I_{s+1}Q_{s+1}\Psi'(\Delta I_{s+1})\Delta I_{s+1} \]
\[ = \Lambda_{s}Q_{s}[K_{s+1} - (1 - \delta)K_{s}] - \Lambda_{s}Z_{s} + E_{s}\beta\Lambda_{s+1}Z_{s+1} \]
\[ \Lambda_{s}Q_{s} = E_{s}\beta\Lambda_{s+1}[MPK_{s+1} + (1 - \delta)Q_{s+1}] \]

where \( Z_{t} = I_{t}Q_{t}s'(\Delta I_{t})\Delta I_{t} \). It follows that:

\[ V_{t}^{IAC} = MPK_{t}K_{t} - I_{t} + E_{t} \sum_{h=1}^{\infty} \frac{\beta^{h} \Lambda_{t+h}}{\Lambda_{t}} (MPK_{t+h}K_{t+h} - I_{t+h}) \]
\[ = MPK_{t}K_{t} - I_{t} + \frac{1}{\Lambda_{t}} E_{t} \sum_{h=1}^{\infty} \left[ \beta^{h-1} \Lambda_{t+h-1}Q_{t+h-1}K_{t+h-1} - \beta^{h} \Lambda_{t+h}Q_{t+h}(1 - \delta)K_{t+h} \right] \]
\[ + \beta^{h} \Lambda_{t+h}Z_{t+h} - \beta^{h+1} \Lambda_{t+h+1}Z_{t+h+1} - \beta^{h} \Lambda_{t+h}Q_{t+h}K_{t+h+1} + \beta^{h} \Lambda_{t+h}Q_{t+h}K_{t+h+1} \]
\[ = Q_{t}K_{t+1} + MPK_{t}K_{t} - I_{t} + E_{t}M_{t+1}Z_{t+1} \]

Let the dividend \( D_{t} = MPK_{t}K_{t} - I_{t} + E_{t}M_{t+1}Z_{t+1} \), that includes the discounted benefit of current investment in relaxing tomorrow’s IAC constraint. The ex-dividend value of the firm as a fraction of book value is therefore:

\[ S_{t} = \frac{V_{t}^{IAC} - D_{t}}{K_{t+1}} = Q_{t} \]

II.2 Sticky prices

Joint optimality of investment and capital accumulation decisions imply:

\[ \Lambda_{s}P_{s}I_{s} = \Lambda_{s}P_{s}I_{s}Q_{s}[1 - \Psi(\Delta I_{s}) - \Psi'(\Delta I_{s})\Delta I_{s}] + E_{s}\beta\Lambda_{s+1}P_{s+1}I_{s+1}Q_{s+1}s'(\Delta I_{s+1})\Delta I_{s+1} \]
\[ = \Lambda_{s}P_{s}Q_{s}[K_{s+1} - (1 - \delta)K_{s}] - \Lambda_{s}P_{s}Z_{s+1} + E_{s}\beta\Lambda_{s+1}P_{s+1}Z_{s+1} \]
\[ \Lambda_{s}P_{s}Q_{s} = E_{s}\beta\Lambda_{s+1}P_{s+1}[\hat{MC}_{s+1}MPK_{s+1} + (1 - \delta)Q_{s+1}] \]

where \( I_{t} = I_{t}s'(\Delta I_{t})\Delta I_{t} \).

By the degree-one homogeneity of the production function and the optimality conditions,

\[ \mathcal{P}_{t}(i)Y_{t}(i) - (1 - \tau) \frac{W_{t}}{P_{t}}N_{t}(i) + \tau I_{t}(i) - T_{t} = \mathcal{P}_{t}(i)Y_{t}(i) - \frac{W_{t}}{P_{t}}N_{t}(i) + \mathcal{T}_{t}(i) \]
\[ = \hat{MC}_{t}(i)MPK_{t}(i)K_{t}(i) + [\mathcal{P}_{t}(i) - \hat{MC}_{t}(i)]Y_{t}(i) + \mathcal{T}_{t}(i) \]

where \( \mathcal{P}_{t}(i) = P_{t}(i)/P_{t} \) and \( \mathcal{T}_{t}(i) = \tau W_{t}[N_{t}(i) - N_{t}] + \tau[I_{t}(i) - I_{t}] \) is a term that aggregates to zero.
It follows that the discounted value of profits of the $i$th firm is:

$$V^i_t = \sum_{h=0}^{\infty} \beta^h E_t \frac{\Lambda^t h P_t^{i+h}}{N^t h} \left[ \mathcal{P}_t^{i+h} Y_t^{i+h} - (1 - \tau) \frac{W_t^{i+h} N_t^{i+h} - (1 - \tau) Y_t^{i+h} - T_t^{i+h}}{P_t^{i+h}} \right]$$

$$= \sum_{h=1}^{\infty} \beta^h E_t \Lambda^{t+h} P_t^{i+h} \left[ \hat{MC}^{i+h} M PK_t^{i+h} K_t^{i+h} - I_t^{i+h} + \mathcal{P}_t^{i} \hat{MC}^{i} Y_t^{i+h} + T_t^{i+h} \right]$$

$$= \hat{MC}^i M PK_t^{i} K_t^{i} - I_t^{i} + \frac{1}{\Lambda_t} E_t \sum_{h=1}^{\infty} \left[ \beta^{h-1} \hat{A}_{t+h-1} Q_t^{i+h-1} K_t^{i} - \beta^{h} \hat{A}_{t+h} Q_t^{i+h} (1 - \delta) K_t^{i+h} + \beta^{h+1} \hat{A}_{t+h+1} Z_t^{i+h+1} - \beta^{h} \hat{A}_{t+h} Q_t^{i+h} K_t^{i+h+1} + \beta^{h} \hat{A}_{t+h+1} Z_t^{i+h+1} \right] + \Phi^i_t$$

where $\hat{A}_t \equiv \Lambda_t P_t$ and $\Phi^i_t \equiv \sum_{h=1}^{\infty} E_t M_{t+h} \left\{ \left[ \mathcal{P}_t^{i} \hat{MC}^{i} Y_t^{i+h} + T_t^{i+h} \right] \right\}$.

Let the dividend $D_t^{i} = \hat{MC}^i M PK_t^{i} K_t^{i} - I_t^{i} + E_t M_{t+1} Z_{t+1}$, that includes the discounted benefit of current investment in relaxing tomorrow’s IAC constraint. The ex-dividend value of the firm as a fraction of book value is therefore:

$$S_t^{i} = \frac{V_t^{i} - D_t^{i}}{K_t^{i+1}} = Q_t^{i} + \frac{1}{K_t^{i+1}} \sum_{h=1}^{\infty} E_t M_{t+h} \left\{ \left[ \mathcal{P}_t^{i} \hat{MC}^{i} Y_t^{i+h} + T_t^{i+h} \right] \right\}$$

A linear approximation around the deterministic steady state yields:

$$s_t = q_t = \beta E_t(s_{t+1} - q_{t+1}) - \beta \theta E_t \hat{m}_t c_{t+1}$$

$$= - \theta \hat{E} \sum_{j=0}^{\infty} \beta^j E_t \hat{m}_t c_{t+j+1}$$

$$= - \frac{\beta \theta}{\lambda} E_t \hat{m}_t c_{t+1}$$

where the last step uses the New Keynesian Phillips curve. Note that IAC does not change the price-setting optimality condition and hence the aggregate New Keynesian Phillips curve. However, it does affect the determinacy conditions for the idiosyncratic dynamics of prices and capital in the firm-specific problem.

### II.3 Firm-specific New Keynesian Phillips curve under IAC

Sveen and Weinke (2005) and Woodford (2005) derive the firm-specific New Keynesian Phillips curve under CAC. I extend their result to IAC.

I define the expectations operator $E^i$ conditional on firm $i$ not resetting in the future. Then, a loglinearization of the optimal price setting equation around the zero-inflation steady state can be written as

$$p_t^{i*} - p_t = \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h} + (1 - \beta \theta) \Theta \sum_{h=0}^{\infty} (\beta \theta)^h E_t \left[ \hat{m}_{t+h} - \frac{1 - \omega}{1 - \alpha - \omega} \hat{m}_{t+h} \right]$$

where I used $E_t[p_t^{i*} - p_t] = p_t^{i*} - E_t p_t + p_t^{i*} - p_t - \sum_{j=1}^{h} E_t \pi_{t+j}$ and where $\Theta \equiv \frac{1 - \alpha}{1 - \alpha + \omega}$. I guess a solution for state variables:

$$E_t \begin{bmatrix} \bar{k}_{t+1}^{i} \\ \bar{p}_{t+1}^{i} \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix} \begin{bmatrix} k_{t}^{i} \\ p_{t}^{i} \end{bmatrix}$$

and note that $E_t \bar{p}_{t+1}^{i} = \bar{p}_{t+1}^{i} + p_t^{i*} - p_t - \sum_{j=1}^{h} E_t \pi_{t+j}$:

$$E_t \begin{bmatrix} \bar{k}_{t+1}^{i} \\ \bar{z}_{t+1}^{i} \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{13} \\ \psi_{31} & \psi_{33} \end{bmatrix} E_t \begin{bmatrix} k_{t+1}^{i} \\ z_{t+1}^{i} - 1 \end{bmatrix} + \begin{bmatrix} \psi_{12} \\ \psi_{32} \end{bmatrix} E_t \bar{p}_{t+1}^{i}$$

$$= \Psi_t^{h} z_t^{i} + \sum_{j=1}^{h} \Psi_t^{j-1} \Gamma_t E_t \bar{p}_{t+1}^{i}$$

(II.5)
where \( E_t^i p_{t+h-1} = \tilde{p}_t^i + p_t^* - E_t p_{t+h-1} \), and hence \( E_t^i z_{t+h}^i = \Psi^h z_t^i + [I - \Psi]\Gamma (\tilde{p}_t^i + p_t^*) - \sum_{j=0}^{h-1} \Psi^j \Gamma E_t p_{t+h-j} \). It follows that

\[
\sum_{h=0}^{\infty} (\beta \theta)^h E_t z_{t+h}^i = \sum_{h=0}^{\infty} (\beta \theta \Psi)^h z_t^i + \sum_{h=1}^{\infty} (\beta \theta)^h \sum_{j=1}^{h} \Psi^{j-1} \Gamma E_t p_{t+h-j} \\
= [I - \beta \theta \Psi]^{-1} \left[ z_t^i + \sum_{h=1}^{\infty} (\beta \theta)^h \Gamma (\tilde{p}_t^i + p_t^* - p_t - \sum_{j=1}^{h} E_t \pi_{t+j}) \right] \\
= [I - \beta \theta \Psi]^{-1} \left[ z_t^i + \Gamma \left( \tilde{p}_t^i + p_t^* - p_t - \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h}^i \right) \right]
\]

Plugging this expression into the optimal price setting decision:

\[
\tilde{p}_t^i + p_t^* = p_t + \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h}^i + \frac{1 - \beta \theta}{\kappa} \sum_{h=0}^{\infty} (\beta \theta)^h E_t \pi_{t+h}^i + \frac{\alpha (1 - \beta \theta)}{(1 - \alpha) \kappa \xi} ((\beta \theta \psi_{33} - 1) \tilde{k}_t^i + \beta \theta \psi_{13} \tilde{z}_t^i)
\]

where \( \kappa = \frac{1}{\alpha} + \frac{\alpha}{1 - \alpha} \frac{(\beta \theta \psi_{33} - 1) \psi_{12} + \beta \theta \psi_{13} \psi_{32}}{\xi} \) and \( \xi = (\psi_{11} \psi_{33} - \psi_{13} \psi_{31}) (\beta \theta)^2 - (\psi_{11} + \psi_{33}) \beta \theta + 1 \).

Using the approximate aggregate price dynamics \( p_t = \theta p_{t-1} + (1 - \theta) \hat{p}_t^i \):

\[
\tilde{p}_t^i + \frac{\theta}{1 - \theta} \tilde{\pi}_t = \sum_{h=1}^{\infty} (\beta \theta)^h E_t \pi_{t+h}^i + \frac{1 - \beta \theta}{\kappa} \sum_{h=0}^{\infty} (\beta \theta)^h E_t \pi_{t+h}^i - \frac{\alpha (1 - \beta \theta)}{(1 - \alpha) \kappa \xi} ((\beta \theta \psi_{33} - 1) \tilde{k}_t^i + \beta \theta \psi_{13} \tilde{z}_t^i - 1)
\]

The aggregation of this expression results in the New Keynesian Phillips curve under firm-specific capital accumulation and IAC,

\[
\pi_t = \frac{\beta E_t \pi_{t+1}^i + (1 - \theta)(1 - \beta \theta)}{\kappa \theta} \pi_{t+1}^i \quad (II.6)
\]

and hence, subtracting the expression from the individual equation,

\[
\tilde{p}_t^i = - \frac{\alpha (1 - \beta \theta)}{\kappa \theta} (\beta \theta \psi_{33} - 1) \tilde{k}_t^i - \frac{\alpha (1 - \beta \theta)}{(1 - \alpha) \kappa \xi} \beta \theta \psi_{13} \tilde{z}_t^i - 1 \quad (II.7)
\]

Relative to CAC, IAC injects an additional idiosyncratic state and it scales the slope of the New Keynesian Phillips curve by a factor \( \kappa \) that differs from the CAC specification.

### II.4 Determinacy of idiosyncratic dynamics

Using \( \pi_{t+1}^i + \beta \tilde{\pi} t_{t+1}^i = - \frac{\alpha}{1 - \alpha} \tilde{p}_t^i - \frac{1}{1 - \alpha} \tilde{k}_t^i \), a linearization of the optimality conditions for investment, capital accumulation and the dynamics of capital under IAC results in the steady-state relations \( Q = 1, IK = \delta \) and \( YK = \theta \) and the approximate idiosyncratic dynamics:

\[
\tilde{k}_{t+1}^i = (1 - \delta) \tilde{k}_t^i + \delta_t^i \\
\tilde{q}_t^i = \beta (1 - \delta) E_t \tilde{q}_{t+1}^i - \frac{1 - \beta (1 - \delta)}{1 - \alpha} E_t (\alpha \tilde{p}_{t+1}^i + \tilde{k}_t^i) \\
\tilde{z}_t^i = \frac{1}{1 + \beta} \tilde{z}_{t-1}^i + \frac{1}{1 + \beta} E_t \tilde{z}_{t+1}^i + \frac{1}{\eta (1 + \beta)} \tilde{q}_t^i \\
E_t \tilde{p}_{t+1}^i = \theta \pi_t^i + (1 - \theta) E_t p_{t+1}^i
\]

where the last equality follows from the first-order approximation \( E_t p_{t+1}^i (i) = \theta p_t^i (i) + (1 - \theta) E_t p_{t+1}^i (i) \). This information, combined with (II.7), identifies the solution (which underlined conjecture (II.5) \( \tilde{k}_{t+1}^i = \psi_{11} \tilde{k}_t^i + \psi_{12} \tilde{p}_t^i + \psi_{13} \tilde{z}_{t-1}^i, E_t \tilde{p}_{t+1}^i = \psi_{21} \tilde{k}_t^i + \psi_{22} \tilde{p}_t^i + \psi_{23} \tilde{z}_{t-1}^i, \tilde{t}_t^i = \psi_{31} \tilde{k}_t^i + \psi_{32} \tilde{p}_t^i + \psi_{33} \tilde{z}_{t-1}^i \) and \( \tilde{Q}_t^i = \psi_{41} \tilde{k}_t^i + \psi_{42} \tilde{p}_t^i + \psi_{43} \tilde{z}_{t-1}^i \) provided the associated saddle-path stability conditions are verified. The problem of firms accumulating firm-specific capital under sticky prices and IAC adds therefore these saddle-path stability conditions to the state-space system driving aggregate variables.
### III Robustness: Generalized demand function (Leahy, 2011)

A generalized demand function \( Y_t(i) = D(P_t(i), Y_t) \), with notation \( X_t(i) \equiv X_t(i)/X_t \), can give rise to time-varying markups with a nonzero elasticity of individual markups to aggregate demand. I maintain the steady-state property \( D_2 = 1 \) and define the elasticity \( \varepsilon_t(i) \equiv -[\partial \ln Y_t(i)/\partial \ln P_t(i)] \).

The problem of ith firm becomes:

\[
\begin{align*}
\min \quad & E_0 \sum_{t=0}^{\infty} M_{t,t+h} \left[ \bar{P}_{t+h}(i) D \left( \bar{P}_{t+h}(i), Y_{t+h} \right) - (1 - \tau) \frac{W_{t+h}}{P_{t+h}} N_{t+h}(i) + \left( 1 - \tau \right) I_{t+h}(i) - T_{t+h} \right] \\
\text{subject to} \quad & \left( 1 - \tau \right) Q_{t+h}(i) \left[ K_{t+h+1}(i) - (1 - \delta) K_{t+h}(i) - \Phi \left( \frac{I_{t+h}(i)}{K_{t+h}(i)} \right) K_{t+h}(i) \right] \\
\end{align*}
\]

subject to

\[
\begin{align*}
P_{t+h}(i) = \begin{cases} P^*_t(i) & \text{prob. } 1 - \theta \\ P_{t+h-1}(i) & \text{prob. } \theta \end{cases}
\end{align*}
\]

Optimal price setting implies:

\[
\sum_{h=0}^{\infty} (\beta \theta)^h E^i_e \hat{m}_{t+h} - \hat{p}_{t+h} D_1 \left( e^{\hat{p}_t(i) - p_t} Y_e \hat{y}_{t+h} \right) \left[ \mathcal{M}^{-1} e^{-\hat{p}_{t+h}(i) + p^*_t(i) - p_t} - MC e^{\hat{m}_{t+h}(i)} \right] = 0
\]

where \( M_t(i) \equiv \frac{\varepsilon_t(i)}{\varepsilon_t(i) + 1} \) is the real markup, and \( E^i \) denotes an expectation operator conditional on firm \( i \) not resetting in the future.

After a loglinearization around the deterministic steady state of the demand curve, \( \bar{Y}_{t+h}(i) = \frac{D_2}{D_1} p_{t+h}(i) + (D_2 - 1) \bar{y}_{t+h} = -\bar{p}_{t+h}(i) \), the homogeneity of the production function implies \( \hat{m}_{t+h}(i) = -\hat{m}_t(i) = \frac{\alpha}{1 - \alpha} \bar{y}_{t+h}(i) - \frac{1}{1 - \alpha} \bar{K}_{t+h}(i) \) and \( \hat{m}_t(i) = \bar{y}_{t+h}(i) - \bar{K}_{t+h}(i) = \frac{1 - \alpha}{\alpha} \hat{m}_{t+h}(i) \), and hence

\[
\begin{align*}
\hat{m}_t(i) &= -\hat{p}_{t+h}(i) - \bar{K}_{t+h}(i) \\
\hat{m}_{t+h}(i) &= -\left( 1 - \alpha \right) \hat{p}_{t+h}(i) + \frac{\alpha}{1 - \alpha} \hat{K}_{t+h}(i)
\end{align*}
\]

Thus, since \( M_{t+h}(i) = M(e^{\hat{p}_t(i) - p_t}, Y_e \hat{y}_{t+h}) \) one has, up to a second-order term, \( \hat{m}_{t+h}(i) = \frac{\partial \hat{m}}{\beta \theta} \hat{p}_t(i) - p_t + p^*_t(i) - p_t + \sum_{j=1}^{h} E_t \hat{m}_{t+j} \) and where \( \Theta \equiv \frac{\alpha}{1 - \alpha - \xi_2} \).

Then, guessing a solution \( \hat{K}_{t+h}(i) = \psi_2 \bar{K}_t(i) + \psi_3 \bar{P}_t(i) \) one has \( E_t \bar{K}_{t+h}(i) = \psi_2 E_t \bar{K}_{t+h-1}(i) + \psi_3 E_t \bar{P}_{t+h-1}(i) = \psi_2 E_t \bar{K}_{t+h-1}(i) + \psi_3 \bar{P}_t(i) + p^*_t - p_t - \sum_{j=1}^{h} E_t \hat{m}_{t+j} \) and

\[
\begin{align*}
\kappa \left( \bar{p}_t(i) + \frac{\theta}{1 - \theta} \bar{p}_t \right) &= -\frac{\alpha (1 - \beta \theta)}{(1 - \alpha)(1 - \beta \theta \psi_2)} \bar{K}_t(i) + \kappa \sum_{h=1}^{\infty} (\beta \theta)^h E_t \hat{m}_{t+h} + (1 - \beta \theta) \sum_{h=0}^{\infty} (\beta \theta)^h E_t \hat{m}_{t+h} + \xi_2 \hat{y}_{t+h}
\end{align*}
\]

where \( \kappa \equiv \frac{1}{\beta \theta} \left( \frac{\alpha \hat{p}_t(i)}{1 - \alpha - \xi_2 (1 - \beta \theta \psi_2)} \right) \). The aggregation of this expression results in the New Keynesian Phillips curve under firm-specific capital accumulation,

\[
\pi_t = \beta E_t \pi_{t+1} + \frac{(1 - \theta)(1 - \beta \theta)}{\kappa \theta} \hat{m}_t + \xi_2 \hat{y}_t
\]

and hence, subtracting the expression from the individual equation,

\[
\bar{p}_t(i) = \psi_1 \bar{K}_t(i)
\]
where \( \psi_1 \equiv -\frac{\alpha(1-\beta)}{\alpha(1-\alpha)(1-\beta\psi_2)} \).

Then, a loglinearization of the optimality condition for capital for firm \( i \) results in equation

\[
\Xi E_t \bar{p}_{t+1} = \beta E_t \bar{K}_{t+1} + \bar{K}_t - \varepsilon [\Xi - 1 - \beta E_t \bar{p}_{t+1}]
\]

where \( \Xi \equiv \frac{[1+\beta][1-\alpha]+\delta}{\eta[1-\alpha]} \). Noting that, to a first-order approximation, \( E_t \bar{p}_{t+1} = \theta \bar{p}_t + (1 - \theta) E_t \bar{p}_{t+1} \), it follows that

\[
E_t \bar{p}_{t+1} = \theta \bar{p}_t + (1 - \theta) E_t \bar{p}_{t+1}
\]

These two expressions identify the coefficients \( \psi_2 \) and \( \psi_3 \) as

\[
\Xi \psi_2 = 1 + \beta \psi_2^2 + \{ \beta \psi_3 - \varepsilon [\Xi - 1 - \beta] \} (1 - \theta) \psi_1 \psi_2
\]

\[
\Xi \psi_3 = \beta \psi_2 \psi_3 + \{ \beta \psi_3 - \varepsilon [\Xi - 1 - \beta] \} [\theta + (1 - \theta) \psi_1 \psi_3]
\]

thereby verifying the guessed solution for individual deviations from aggregate choices.

The problem of firms accumulating firm-specific capital under sticky prices adds therefore the two idiosyncratic dynamic equations to the state-space system driving aggregate variables,

\[
\begin{bmatrix}
E_t \bar{p}_{t+1} \\
\Xi E_t \bar{p}_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\theta + (1 - \theta) \psi_1 \psi_3 \\
(1 - \theta) \psi_1 \psi_2
\end{bmatrix}
\begin{bmatrix}
\bar{p}_t \\
\bar{K}_t
\end{bmatrix}
\]

which, to have a unique bounded solution, must have both eigenvalues within the unit circle.

Provided this determinacy condition is met, the derivation of the NKQ equation is then unchanged. Thus, given the New Keynesian Phillips curve associated with the generalized demand function, it follows that

\[
s_t = q_t - \frac{\beta \theta}{\lambda} E_t \pi_{t+1} + e_t, \quad e_t = \vartheta \xi_y \sum_{j=0}^{\infty} \beta^j E_t y_{t+j+1}
\]

### IV Robustness: Presence of a fraction of backward-looking firms (Galí and Gertler, 1999)

The presence of backward-looking firms give rise to a hybrid New Keynesian Phillips curve and implies that the aggregate firm is a weighted average of forward- and backward-looking firms. Suppose a fraction \( \omega \in [0, 1] \) of firms are backward-looking in the sense that they set their prices, \( p_t^b \), according to rule \( p_t^b = p_t^f + \pi_t \) (see Galí and Gertler, 1999, for a motivation), while fraction \( 1 - \omega \) of firms set prices \( p_t^f = p_t^i \). The idiosyncratic determinacy conditions under firm-specific capital characterize the decisions of forward-looking firms, and hence remain the same in this context. The average price level in this economy still evolves according to \( p_t = \theta p_{t-1} + (1 - \theta) p_t^i \), where (1 - \theta) is the fraction of firms (whether backward- or forward-looking) that will reset their prices. But, since the fraction \( \omega \) is backward-looking, the average newly set price in period \( t \) is

\[
p_t^i = \omega p_t^i + (1 - \omega) p_t^f
\]

\[
= \omega (p_{t-1}^i + \pi_{t-1}) + (1 - \omega)(1 - \beta\theta) \sum_{h=0}^{\infty} (\beta\theta)^h E_t \left[ \kappa^{-1}(\hat{m}_c + \hat{y}) + \pi_{t+h} \right]
\]

where \( \kappa^{-1} \neq \Theta \) embeds the aggregate effect of firm-specific capital accumulation. Rearranging:

\[
p_t^i - p_t^b = \frac{1}{1 + \beta \theta \omega} \left[ \frac{\omega}{\theta} \pi_{t-1} - \omega (1 + \beta \theta) \pi_t + \frac{\beta \theta}{1 - \theta} E_t \pi_{t+1} + \frac{(1 - \omega)(1 - \beta\theta)}{\kappa} (\hat{m}_c + \hat{y}) \right]
\]

where the last equality uses the approximate price dynamics. The equation can be rewritten as the hybrid New Keynes Phillips curve under firm-specific capital accumulation,

\[
\pi_t = \beta \gamma f E_t \pi_{t+1} + \frac{\gamma b}{\beta} \pi_{t-1} + \lambda (\hat{m}_c + \hat{y})
\]

where \( \gamma f = \frac{\theta}{\theta + \omega[1 - (1 - \beta)]} \), \( \gamma b = \frac{\beta \omega}{\theta + \omega[1 - (1 - \beta)]} \) and \( \lambda = \frac{\theta(1 - \omega)}{\theta + \omega[1 - (1 - \beta)]} \frac{(1 - \beta\theta)(1 - \theta)}{\kappa \theta} \).

It follows that the market-book ratio of the aggregate firm is

\[
s_t = q_t - \frac{\beta \theta}{\lambda} E_t \pi_{t+1} + e_t, \quad e_t = \frac{\beta \theta}{\lambda} \left[ -\beta \gamma f E_t \pi_{t+1} + (1 - \gamma f - \gamma b) \sum_{j=1}^{\infty} \beta^j E_t \pi_{t+j+1} - \frac{\gamma b}{\beta} \pi_t \right]
\]

where I relied on a Dixit-Stiglitz demand curve, \( \xi_y = 0 \).
V Robustness: Corporate debt (Belo, Collin-Dufresne and Goldstein, 2015)

When firms are financed by debt $B_t$, there is a difference between corporate profits (EBIT) and dividends,

$$D_t = Ebit_t + B_t - R^b_{t-1}B_{t-1}$$

Following Belo et al. (2015), assume the stationarity of the logarithm of leverage $L_t = B_t/S_tK_{t+1}$ with $S_t = E_tM_{t+1}(S_{t+1} + D_{t+1})$. The first-order effect of leverage is to amplify the volatility of corporate profits in dividends:

$$\hat{d}_t = \frac{Ebit}{Ebit - r^bB}e^{\hat{b}t} + \frac{B}{Ebit - r^bB}b_t - \frac{(1 + r^b)B}{Ebit - r^bB}(r^b_t + \hat{b}_t - 1)$$

as $Ebit/(Ebit - r^bB) > 1$. Using the no-arbitrage condition $E_tM_{t+1}R^b_t = 1$, I find the valuation:

$$s_t = q_t - \frac{\beta\vartheta}{\alpha}E_t\pi_{t+1} + e_t, \quad e_t = -\frac{L}{1 + L}I_t - \frac{L}{1 + L} \sum_{j=0}^{\infty} \beta^{j+1}t_{t+j}$$

where I used an unchanged New Keynesian Phillips curve.

VI Robustness: Distorted steady state

The approximate relation between marginal and average Q in a New Keynesian model with nominal rigidities is

$$\hat{\pi}_t = q_t - \frac{\beta\vartheta}{\alpha}E_t\pi_{t+1} + e_t, \quad e_t = -\kappa_1q_t + E_t\sum_{j=1}^{\infty} \beta^{j}[\kappa_2q_{t+j} + \kappa_3m_{t+j} - \vartheta\kappa_4\ell_{t+j}]$$

where $\kappa_1 \equiv \frac{1-M}{\alpha M}(1 - \frac{\alpha\beta}{1-\beta}/\left(1 + \frac{1-M\beta}{M(1-\beta)}\right))$, $\kappa_2 \equiv \frac{1-M}{\alpha M}\frac{\alpha\beta}{1-\beta}/\left(1 + \frac{1-M\beta}{M(1-\beta)}\right)$, $\kappa_3 \equiv \frac{1-M}{\alpha M}\frac{1 + \frac{\alpha\beta}{1-\beta}}{1 + \frac{1-M\beta}{M(1-\beta)}}$, and $\kappa_4 \equiv 1/\left(M + \frac{1-M\beta}{1-\beta}\right)$, with the wedge that distorts the steady state $M \equiv MC/(1 - \tau)$.

It is easy to verify that equation (VI.12) reduces to the NKQ equation under the corrective employment subsidy, $\tau = 1 - MC = 1/\varepsilon$.

VII Robustness: Rental market for capital

I specify a general-equilibrium New Keynesian model under the assumption of an economy-wide rental market for capital.

VII.1 Households

Identical households choose consumption demand $\{C_t\}$, labor supply $\{N_t\}$, investment $\{I_t\}$, demand for one-period nominal risk-free bonds $\{B_{t+1}\}$ that pay nominal interest rate $i^f$, shares of the aggregate firm $\{S_{t+1}\}$ that sell at price $V$ and units of capital stock $\{K_{t+1}\}$ that sell at price $Q$ to maximize the utility function:

$$E_0\sum_{t=0}^{\infty} \beta^t \left( \frac{1}{1 - \gamma}C_t^{1-\gamma} - \frac{1}{1 + \varphi}N_t^{1+\varphi} \right)$$

subject to the budget constraint:

$$\int_0^1 P_t(i)C_t(i)di + \int_0^1 P_t(i)I_t(i)di + B_{t+1} + P_tQ_tV_{t+1} + P_tQ_tK_{t+1} \leq W_tN_t + \exp(i^f_{t-1})B_t + P_t(Q_t + D_t)V_t + P_t\left(R^k_t + \left[1 - \delta - \Phi\left(\frac{I_t}{K_t}\right)\right]Q_t\right)K_t$$

where $D_t$ are aggregate, per-period, corporate profits. Implicit in the budget constraint is the evolution of capital as $K_{t+1} = (1 - \delta)K_t + \Phi(I_t/K_t)K_t$, where $C$ is real consumption, $I$ is real investment, $P$ is the nominal price of a unit of consumption, $N$ are hours worked, $W$ is the nominal wage rate, $D$ are real aggregate, per-period, corporate profits that firms pay to households (who own the firms), and $K$ is the capital stock that households
own and rent to firms at the competitive rental rate $R_t^k$. Parameters $\gamma$ and $\varphi$ are the inverse of the (constant) intertemporal elasticity of substitution and Frisch labor supply elasticity, respectively. Parameter $\delta$ is the average depreciation rate of capital. The capital adjustment cost function $\Phi(\cdot)$ is such that $\Phi(I/K) = 1$ and its curvature $\eta \equiv -[\Phi''(I/K)I/K]/\Phi'(I/K)$ is constant and positive in the steady state.

The price $V_t$ reflects the expected discounted value of future corporate profits that owe to monopolistic competition and nominal rigidities, while $Q_t$ is value of the capital stock.

I assume Calvo pricing. Aggregate real consumption and investment are the Dixit-Stiglitz aggregates of the continuum of goods $i \in [0,1]$, $C_t \equiv \int_0^1 C_t(i) \frac{di}{P_t}$ and $I_t \equiv \int_0^1 I_t(i) \frac{di}{P_t}$, respectively, where $\varepsilon$ is the (constant) elasticity of substitution between consumption of any two goods. Households form consumption and investment units by minimizing costs. The first-order condition for that problem yields the demand curve for good $i$,

$$C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t, \quad I_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} I_t$$

where $P_t \equiv [\int_0^1 P_t(i) 1^{-\varepsilon} di]^{1/(1-\varepsilon)}$ is the price index. The solution implies the optimal expenditure plans

$$P_t C_t = \int_0^1 P_t(i) C_t(i) di, \quad P_t I_t = \int_0^1 P_t(i) I_t(i) di$$

Optimal choices by the household requires that the joint evolution of the processes satisfies:

$$C_t N_t^{\varphi} = \frac{W_t}{P_t}$$

$$E_t M_{t,t+1} 1 + \frac{i_t f}{\Pi_{t,t+1}} = 1$$

$$Q_t = E_t M_{t+1} \left( R_{t+1}^k + \left(1 - \delta \right) + \Phi \left( \frac{I_{t+1}}{K_{t+1}} \right) - \Phi' \left( \frac{I_{t+1}}{K_{t+1}} \right) \frac{I_{t+1}}{K_{t+1}} \right) Q_{t+1} \right)$$

$$Q_t = \left[ \Phi' \left( \frac{I_t}{K_t} \right) \right]^{-1}$$

where $Q$ is marginal Tobin’s Q (the marginal value of the capital stock), $M_{t+1} = \beta(C_{t+1}/C_t)^{-\gamma}$ is the stochastic discount factor for real payoffs, and $\Pi_{t,t+1} \equiv P_{t+1}/P_t$ is the gross inflation rate between periods $t$ and $t + 1$.

### VII.2 Firms

A continuum of monopolistically competitive firms $i \in [0,1]$ choose nominal prices $\{P^*_t(i)\}$, labor demand $\{N_t(i)\}$ and investment $\{I_t(i)\}$ to maximize the expected discounted value of profits:

$$E_0 \sum_{t=0}^{\infty} M_{0,t} \left[ \frac{P_t(i)}{P_t} Y_t(i) - (1 - \tau) \frac{W_t}{P_t} N_t(i) - (1 - \tau) R_t^k K_t(i) - T_t \right]$$

subject to the production function $Y_t(i) = A_t K_t(i)^{\alpha} N_t(i)^{1 - \alpha}$, where $Y$ is output, $A$ is technology and parameter $1 - \alpha$ is the labor share of value added, to the demand curve $Y_t(i) = (P_t(i)/P_t)^{-\varepsilon} Y_t$, and to the price resetting technology. Namely, Calvo-type price stickiness,

$$P_t(i) = \begin{cases} P^*_t(i), & \text{with probability } (1 - \theta) \\
I_{t-1}(i), & \text{with probability } \theta \end{cases}$$

implies that firms can reset prices at any given time only with probability $1 - \theta$.

The government levies lump-sum taxes, $T_t$, to finance an employment subsidy, $\tau$. The subsidy $\tau = 1/\varepsilon$ would offset all deterministic steady-state distortions caused by monopolistic competition.

Firms rent the required capital stock entirely at each period on a competitive rental market at the common rental rate, $R_t^k$. Therefore, all firms that reset prices at period $t$ face an identical problem. In particular, the firms’ first-order conditions for optimal price setting, and for optimal labor and capital purchases are:

$$\sum_{h=0}^{\infty} \theta^h E_t M_{t,t+h} P^x_{t+h} Y_{t+h}(i) \left[ P^*_t(i) - MC_{t+h}(i) P_{t+h} \right] = 0$$

$$MC_t(i) M P N_t(i) = (1 - \tau) \frac{W_t}{P_t}$$

$$MC_t(i) M P K_t(i) = (1 - \tau) R_t^k$$
where $MPN$ is the marginal productivity of labor, $MC$ is the real marginal cost of output, $MPK$ is the marginal productivity of capital, and $\bar{MC} \equiv \bar{M}MC$, with $\bar{M} \equiv \epsilon/(\epsilon - 1)$ the average price markup that firms charge over marginal costs under monopolistic competition in the goods market.

It follows that $MPN_t(i)/MPK_t(i) = W_t/(P_t R^F_t)$, i.e., the optimal choice of labor and capital implies that all firms produce with the same capital-labor ratio. Therefore, the cross-sectional dispersion in aggregate marginal costs and productivities is zero $\bar{MC}C_t(i) = R^F_t MPK_t(i) = \alpha R^F_t A_t[K_t(i)/N_t(i)]^{\alpha - 1} = \alpha R^F_t A_t[K_t/N_t]^{\alpha - 1}$, i.e., the market for goods clears—for each firm $i$ supply equals demand, $Y_t(i) = C_t(i) + I_t(i)$, where $I_t(i)$ denotes the demand for good $i$ made for investment purposes.

The government runs a balanced budget, $T_t = \tau(W_t/P_t)N_t + \tau R^k_t K_t$, and sets the nominal interest rate following a simple rule that reacts to inflation:

$$i_t^f = \ln(\beta) + \phi_x \pi_t, \quad \phi_x > 1$$ (VII.20)

**VII.3 Government**

The government runs a balanced budget, $T_t = \tau(W_t/P_t)N_t + \tau R^k_t K_t$, and sets the nominal interest rate following a simple rule that reacts to inflation:

$$i_t^f = \ln(\beta) + \phi_x \pi_t, \quad \phi_x > 1$$ (VII.20)

The market for goods clears—for each firm $i$, supply equals demand, $Y_t(i) = C_t(i) + I_t(i)$, where $I_t(i)$ denotes the demand for good $i$ made for investment purposes.

The labor market also clears; labor supply equals average labor demand, $N_t = \int_0^1 N_t(i)di$.

**VII.5 Loglinearization around the deterministic steady state**

Lower-case letters denote logarithms and hat terms deviations from the deterministic steady state. Equations (VII.13) and (VII.18) combine as

$$\sigma \hat{c}_t + \varphi \hat{n}_t = \hat{m}c_t + \hat{y}_t - \hat{n}_t$$

The approximate aggregate production function is

$$\hat{y}_t = a_t + \alpha \hat{k}_t + (1 - \alpha)\hat{n}_t$$

The approximate market clearing conditions is

$$\hat{y}_t = \gamma_c \hat{c}_t + \gamma_i \hat{i}_t$$ (VII.21)

where $\gamma_i \equiv \alpha \beta \delta/[1 - \beta(1 - \delta)]$ is the investment share of output and $\gamma_c = 1 - \gamma_i$ is the consumption share of output. Equation (VII.16) is approximately

$$\hat{i}_k_t \equiv i_t - \hat{k}_t = \frac{1}{\eta} \hat{q}_i$$ (VII.22)

which links the real economy to financial markets.

Equation (VII.14) holds approximately as the dynamic IS equation

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(\hat{g}_t^f - E_t \pi_{t+1})$$ (VII.23)

The approximate no-arbitrage condition (VII.15) is the forward-looking equation for $Q$

$$\hat{q}_t = \beta E_t \hat{q}_{t+1} + \vartheta(\hat{m}c_{t+1} + \hat{y}_{t+1} - \hat{k}_{t+1}) - (\hat{g}_t^f - E_t \pi_{t+1})$$ (VII.24)

The New Keynesian Phillips curve relating inflation to marginal costs is

$$\pi_t = \beta E_t \pi_{t+1} + \lambda \hat{m}c_t, \quad \lambda \equiv \frac{(1 - \theta)(1 - \beta \theta)}{\theta}$$ (VII.25)
which is the loglinearization of equation (VII.17) and uses approximate aggregate price dynamics \( p_t = \theta p_{t-1} + (1-\theta)p^*_t \). Finally, real marginal costs relate to the other variables as

\[
\hat{mc}_t = \left( \frac{\sigma}{\gamma_c} + \frac{\alpha + \varphi}{1 - \alpha} \right) \hat{y}_t - \frac{\sigma\gamma_i}{\gamma_c \eta} \hat{q}_t - \left( \frac{\sigma\gamma_i}{\gamma_c} + \frac{\alpha (1 + \varphi)}{1 - \alpha} \right) \hat{k}_t - \frac{1 + \varphi}{1 - \alpha} \delta_t \tag{VII.26}
\]

The approximate equation for capital accumulation is

\[
\hat{k}_{t+1} = (1 - \delta) \hat{k}_t + \delta \hat{r}_t \tag{VII.27}
\]

The interest-rate rule (VII.20) provides determinate dynamics and a zero-inflation deterministic steady state. Finally, technology follows the process \( a_{t+1} = \rho a_t + \epsilon_{t+1} \) with \( \epsilon_t \sim N iid(0, \sigma^2) \).

### VII.6 Competitive equilibrium

The competitive equilibrium consists in the exogenous process \( \{a_t\}_{t=0}^{\infty} \) and the stochastic processes \( \{i_t^f; \pi_t; q_t; y_t; c_t; i_t; mc_t;\} \) that satisfy equations (VII.20) to (VII.27).

### VII.7 Pricing a synthetic firm

In this rental-market context, the value of the firm is:

\[
V_t(i) = \sum_{h=1}^{\infty} E_t M_{t,t+h} \left[ \mathbb{P}_{t+h}(i) Y_{t+h}(i) - (1 - \tau) W_{t+h} N_{t+h}(i) - (1 - \tau) R^k_{t+h} K_{t+h}(i) - T_{t+h} \right]
\]

with \((1 - \tau)W_t = P_t MC_t(i) MPN_t(i)\) and \((1 - \tau)R^k_t = P_t MC_t(i) MPN_t(i)\). Therefore:

\[
V_t = \sum_{h=1}^{\infty} E_t M_{t,t+h} \left[ \left( \mathbb{P}_{t+h}(i) - \frac{MC_{t+h}(i)}{1 - \tau} \right) Y_{t+h}(i) - T_{t+h} \right]
\]

\[
V_t = \int_0^1 V_t(i) di = \sum_{h=1}^{\infty} E_t M_{t,t+h} \left( 1 - \frac{MC_{t+h}}{1 - \tau} \right) Y_{t+h} = E_t M_{t+1} (V_{t+1} + D_{t+1})
\]

where \( D_t \equiv [1 - MC_t/(1 - \tau)] Y_t \). If \( \tau = \epsilon^{-1} \), then \( DK_t \approx -YK \hat{mc}_t = -\delta \hat{mc}_t \) and

\[
\xi_t = \frac{V_t}{K_{t+1}} = E_t M_{t+1} (\xi_{t+1} \Delta K_{t+2} + DK_{t+1}) \approx \beta E_t \xi_{t+1} - \beta \delta E_t \hat{mc}_{t+1} = -\frac{\beta \delta}{\chi} E_t \pi_{t+1}
\]

where the last equality uses the New Keynesian Phillips curve under homogeneous capital (VII.25).

Then, one can construct a synthetic claim to capital and dividends under the assumption of no arbitrage, whose price will be:

\[
S_t = \frac{Q_t K_{t+1} + V_t}{K_{t+1}} = Q_t + \xi_t
\]

with \( S = Q = 1 \). Therefore, I recover the NKQ equation under a rental-market assumption:

\[
s_t = q_t + \xi_t = \eta i_t k_t - \frac{\beta \delta}{\chi} E_t \pi_{t+1} \tag{VII.28}
\]

which is equivalent to the NKQ equation under firm-specific capital, except for the link between \( \lambda \) and the deep parameters.

Note, however, how the legitimacy of the interpretation of this synthetic claim as the value of the firm owes ultimately to the firm-specific problem in partial equilibrium.


