Does Uncertainty Vanish in the Small? The Smooth Ambiguity Case

Fabio Maccheroni^a, Massimo Marinacci^a, Doriana Ruffino^b

^aDepartment of Decision Sciences, Dondena, and IGIER, Università Bocconi

^bDepartment of Finance, University of Minnesota

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Abstract

We study orders of risk and model uncertainty aversion in the smooth ambiguity model proposed by Klibanoff, Marinacci, and Mukerji (2005). We consider a quadratic approximation of their model and we show that both risk and model uncertainty attitudes have at most a second order effect. Specifically, the order depends on the properties of the support of the decision maker's limit prior, which we fully characterize. We find that model uncertainty attitudes have a second order effect unless the support is a singleton, that is, unless model uncertainty fades away in the limit. Specializing our general analysis to the case of a binomial state space allows relating the results to important applications in stochastic calculus and mathematical finance.

1 Introduction

In this paper we study the order of convergence of risk and model uncertainty attitudes in a quadratic approximation of the smooth ambiguity functional form of Klibanoff, Marinacci, and Mukerji [7] as uncertainty becomes smaller and smaller.

We show that, under standard differentiability assumptions, both risk and model uncertainty attitudes have at most a second order effect. The order depends on the properties of the support of the decision maker's limit prior. In particular, model uncertainty attitudes have a second order effect, unless the support is a singleton, that is, unless model uncertainty fades away in the limit.

Consider an act f defined on a finite state space Ω . Its certainty equivalent is given by

$$C(f) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) \right) d\mu(q) \right), \quad (1)$$

where μ is the decision maker's prior on probability models $q \in \Delta$, and u and v represent the decision maker's risk and model uncertainty attitudes, respectively.

By standard reduction, the prior μ induces a model $\bar{q} \in \Delta$, with $\bar{q}(\omega) = \int_{\Delta} q(\omega) d\mu(q)$ for each $\omega \in \Omega$. Let w be a constant, possibly interpreted as current wealth. Given any act h , and using $\bar{q}$, it is possible to prove that the quadratic approximation of the uncertainty premium of $w + h$

$$\pi(w + h) = E_{\bar{q}}(w + h) - C(w + h)$$

is given by

$$\pi(w + h) = \frac{1}{2} \lambda_u(w) (\sigma_{\bar{q}}^2(h) - \sigma_{\mu}^2(E(h))) + \frac{1}{2} \lambda_v(w) \sigma_{\mu}^2(E(h)) + o(\|h\|_{\bar{q}}^2). \quad (2)$$

Here λ_u and λ_v are the Arrow-Pratt coefficients of u and v , respectively, and $\sigma_{\mu}^2(E(h))$ is the variance under μ of the function $E(h) : \Delta \rightarrow \mathbb{R}$ that associates to each model q its expected value $E_q(h)$ with respect to h . This variance represents the model uncertainty perceived by the decision maker.

As detailed in Section 3, to model small uncertainty we consider nets $\{h_t, \mu_t\}_{t \in (0,1]}$ that suitably vanish as t goes to zero.¹ For example, consider a binomial state space $\Omega = \{1, 2\}$ and the act W_t such that

$$W_t(\omega) = \begin{cases} t & \text{if } \omega = 1 \\ -t & \text{if } \omega = 2 \end{cases}$$

with $t \in (0, 1]$. Consider the net $\{W_t, \mu_t\}_{t \in (0,1]}$ and suppose that μ_t converges to some μ (i.e., $\mu_t = \mu$ for all t). In Section 7 we show that approximation (2) becomes

$$\pi_t(w + W_t) = 2\lambda_u(w) (\bar{q}_t - \vartheta_t) t^2 + 2\lambda_v(w) (\vartheta_t - \bar{q}_t^2) t^2 + o(t^2), \quad (3)$$

where $\vartheta_t = \int_{\Delta} q^2 d\mu_t(q)$. Thus, the effects of risk and model uncertainty attitudes depend on the terms $\Lambda_u(t) = (\bar{q}_t - \vartheta_t) t^2$ and $\Lambda_v(t) = (\vartheta_t - \bar{q}_t^2) t^2$, respectively. These terms may have either a second order effect or a negligible effect. Risk attitudes, for instance, have a second order effect when $\Lambda_u(t) \sim t^2$ and a negligible one when $\Lambda_u(t) = o(t^2)$. While in the former case they matter in approximation (3), in the latter they do not.

Propositions 42 and 43 show that the order effect of $\Lambda_u(t)$ and $\Lambda_v(t)$ critically depends on the properties of the support of μ . Specifically, if this support is a singleton, then $\Lambda_v(t) = o(t^2)$ and model uncertainty attitudes are negligible. However, when the support is not a singleton, this is no longer the case and model uncertainty persists as t goes to zero. As a matter of fact, both risk and model uncertainty attitudes have a second order effect when the support is nonsingleton.

In Section 8, the heart of the paper, the results of Section 7 on the net $\{W_t, \mu_t\}_{t \in (0,1]}$ are generalized to any net $\{h_t, \mu_t\}_{t \in (0,1]}$ defined on a binomial state space. Theorem 48, our main result, fully characterizes the conditions under which risk and model uncertainty attitudes have either a second order or a negligible effect. Even in this more general case the properties of the support of μ play a key role. In particular, risk and model uncertainty attitudes have a similar effect of order $\|h_t\|_{\bar{q}_t}^2$ when this support is nonsingleton.²

In Section 9 we consider a special case of our setting, which is related to Skiadas [13]. We show that Skiadas' findings on the negligible effect of model uncertainty attitudes depend on the singleton nature of the support of μ that he implicitly considers. Finally, Section C in the Appendix completes the paper and outlines the multinomial case.

A final remark is in order before continuing. The approach of our paper is analytical, rather than behavioral. We consider the properties of the functional form proposed by [7] ignoring their behavioral foundations. Some of these can be found in [7] (e.g., the properties of the Arrow-Pratt indexes), others are left to future research.

¹The index t has no specific temporal meaning.

²Notice that $\|W_t\|_{\bar{q}_t}^2 = t^2$, so that the second order effects discussed for W_t become more generally effects of order $\|h_t\|_{\bar{q}_t}^2$.

2 Preliminaries

2.1 Mathematical setup

Throughout the paper we consider a finite state space Ω of cardinality n and a base probability measure $P : 2^\Omega \rightarrow \mathbb{R}$ defined for simplicity on the power set 2^Ω (all our results actually hold in any algebra of subsets of Ω).

The collection of all functions $f : \Omega \rightarrow \mathbb{R}$ can be thus identified with $\mathbb{R}^n$. Given an interval $I \subseteq \mathbb{R}$, we set

$$L(I) = \{f \in \mathbb{R}^n : f(\omega) \in I \text{ for almost all } \omega \in \Omega\}.$$

Throughout the paper $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^n$. Recall that in $\mathbb{R}^n$ all norms are equivalent; i.e., given any norm $\|\cdot\|'$ there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 \|f\| \leq \|f\|' \leq c_2 \|f\|, \quad \forall f \in \mathbb{R}^n. \quad (4)$$

We denote by $E_P(f)$ and $\sigma_P^2(f)$ the expectation and variance of a function $f : \Omega \rightarrow \mathbb{R}$, respectively.

Barycenters The set of all probability measures $q : 2^\Omega \rightarrow [0, 1]$ is the simplex of $\mathbb{R}^n$, which we denote by Δ . We endow Δ with the Borel σ -algebra $\mathcal{B}(\Delta)$. Each Borel probability measure $\mu : \mathcal{B}(\Delta) \rightarrow [0, 1]$ induces a measure $\bar{q} \in \Delta$ given by

$$\bar{q}(\omega) = \int_{\Delta} q(\omega) d\mu(q), \quad (5)$$

called the *barycenter* of μ . It follows that,

$$\sum_{\omega \in \Omega} f(\omega) \bar{q}(\omega) = \int_{\Delta} \left(\sum_{\omega \in \Omega} f(\omega) q(\omega) \right) d\mu(q), \quad \forall f \in \mathbb{R}^n. \quad (6)$$

The barycenter $\bar{q}$ has a natural interpretation in terms of reduction of compound lotteries. In fact, if $\text{supp}\mu = \{q_1, \dots, q_m\}$ is finite³ and $\mu(q_i) = \mu_i$ for $i = 1, \dots, m$, then (5) becomes

$$\bar{q}(\omega) = \mu_1 q_1(\omega) + \dots + \mu_m q_m(\omega), \quad \forall \omega \in \Omega.$$

Hence, μ can be seen as a lottery whose outcomes are all possible models q , which in turn can be seen as lotteries that determine the state.

Orders of convergence Orders of convergence as $t \downarrow 0$ (that is $t \rightarrow 0^+$) are cardinal to our analysis. Given two functions $\varphi, \psi : (0, 1] \rightarrow \mathbb{R}$,

- (i) $\varphi(t) = o(\psi(t))$ if for all $M > 0$ there exist δ such that $|\varphi(t)| \leq M |\psi(t)|$ for $t < \delta$;⁴
- (ii) $\varphi(t) = O(\psi(t))$ if there exist $\delta, M > 0$ such that $|\varphi(t)| \leq M |\psi(t)|$ for $t < \delta$;⁵
- (iii) $\varphi(t) \asymp \psi(t)$ if $\varphi(t) = O(\psi(t))$ and $\psi(t) = O(\varphi(t))$;⁶
- (iv) $\varphi(t) \sim \psi(t)$ if $\lim_{t \downarrow 0} \varphi(t) / \psi(t) = 1$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

³A *carrier* of μ is any Borel subset of Δ having full measure. If the intersection of all closed carriers is a carrier, it is called *support* of μ and denoted by $\text{supp}\mu$.

⁴In other words, $\lim_{t \downarrow 0} \varphi(t) / \psi(t) = 0$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

⁵In other words, $\limsup_{t \downarrow 0} |\varphi(t) / \psi(t)| < \infty$, provided $\psi(t) \neq 0$ for t sufficiently close to 0.

⁶Equivalently, there exist $\delta, c_1, c_2 > 0$ such that $c_1 |\psi(t)| \leq |\varphi(t)| \leq c_2 |\psi(t)|$ for $t < \delta$.

It is important to recall that $\varphi(t) = o(\psi(t))$ and $\psi(t) = O(\varphi(t))$ are abuses for $\varphi(t) \in o(\psi(t))$ and $\psi(t) \in O(\varphi(t))$.

The following property plays a key role in the paper.

Lemma 1 *Given any two functions $\varphi, \psi : (0, 1] \rightarrow \mathbb{R}$, it holds $\varphi = O(\psi)$ if and only if*

$$\xi = o(\varphi) \implies \xi = o(\psi) \quad (7)$$

for any function $\xi : (0, 1] \rightarrow \mathbb{R}$.

Weak and quadratic convergence Some of our results will use the standard notion of weak convergence of Borel probability measures on $\mathcal{B}(\Delta)$. Specifically, a net $\{\mu_\alpha\}_\alpha$ of Borel probability measures $\mu_\alpha : \mathcal{B}(\Delta) \rightarrow [0, 1]$ *weakly converges* to some Borel probability measure μ , written $\mu_\alpha \xrightarrow{w} \mu$, if

$$\lim_\alpha \int_\Delta \varphi(q) d\mu_\alpha(q) = \int_\Delta \varphi(q) d\mu(q), \quad \forall \varphi \in C(\Delta), \quad (8)$$

where $C(\Delta)$ is the space of continuous functions $\varphi : \Delta \rightarrow \mathbb{R}$.⁷ Weak convergence implies the following moments' convergence.

Lemma 2 *If $\mu_\alpha \xrightarrow{w} \mu$, then, for each $\omega \in \Omega$, $\lim_\alpha \int_\Delta q^n(\omega) d\mu_\alpha(q) = \int_\Delta q^n(\omega) d\mu(q)$ for each $n \geq 1$.*

In the special case in which Ω is a doubleton, $\Delta = \{(q, 1 - q) : q \in [0, 1]\}$ can be identified with the unit interval $[0, 1]$. In this case, the converse of Lemma 2 is also true.

Lemma 3 *$\mu_\alpha \xrightarrow{w} \mu$ on $[0, 1]$ if and only if $\lim_\alpha \int_{[0,1]} q^n d\mu_\alpha(q) = \int_{[0,1]} q^n d\mu(q)$ for each $n \geq 1$.*

In view of Lemmas 2 and 3, we give the following definition.

Definition 4 *A net $\{\mu_\alpha\}_\alpha$ of probability measures $\mu_\alpha : \Delta \rightarrow [0, 1]$ *quadratically converges* to a probability $\mu : \Delta \rightarrow [0, 1]$, written $\mu_\alpha \xrightarrow{sq} \mu$, if for each $\omega \in \Omega$,*

$$\lim_\alpha \int_\Delta q(\omega) d\mu_\alpha(q) = \int_\Delta q(\omega) d\mu(q) \quad \text{and} \quad \lim_\alpha \int_\Delta q^2(\omega) d\mu_\alpha(q) = \int_\Delta q^2(\omega) d\mu(q).$$

Thus, quadratic convergence only requires convergence of the first two moments. It is straightforward to construct nets of probability measures that quadratically converge, but do not weakly converge.⁸

2.2 Decision theoretic setup

Given any nonsingleton interval $I \subseteq \mathbb{R}$ of monetary outcomes, we consider decision makers who behave according to the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji [7]. These decision makers' preferences $\succsim$ over monetary acts are represented by the preference functional $V : L(I) \rightarrow \mathbb{R}$ defined by

$$V(f) = \int_\Delta \phi \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) d\mu(q), \quad \forall f \in L(I), \quad (9)$$

where μ is a Borel probability measure on Δ , and $u : I \rightarrow \mathbb{R}$ and $\phi : u(I) \rightarrow \mathbb{R}$ are strictly increasing and continuous functions.

⁷For details on weak convergence we refer the reader to Aliprantis and Border (2006).

⁸Finally notice that since Δ is a compact metric space, weak convergence coincides with weak* convergence when probability measures on Δ are regarded as continuous linear functionals on $C(\Delta)$.

We adopt the same setting of [9]. In particular, since we consider monetary acts, it is natural to refer to monetary certainty equivalents. To this end, set $v = \phi \circ u : I \rightarrow \mathbb{R}$ (see [7, p. 1859]). In Maccheroni, Marinacci, and Ruffino [9] we discuss to what extent v describes attitudes toward model uncertainty. Using v we can rewrite (9) as

$$V(f) = \int_{\Delta} (v \circ u^{-1}) \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) d\mu(q), \quad \forall f \in L(I). \quad (10)$$

The certainty equivalent function $C : L(I) \rightarrow \mathbb{R}$ induced by V is defined by $V(C(f)) = V(f)$ for all acts f , that is,

$$C(f) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) \right) d\mu(q) \right), \quad \forall f \in L(I). \quad (11)$$

This is the composition of two monetary certainty equivalents,

$$c(f, q) = u^{-1} \left(\int_{\Omega} u(f(\omega)) dq(\omega) \right) \quad \text{and} \quad v^{-1} \left(\int_{\Delta} v(c(f, q)) d\mu(q) \right).$$

Consider the Arrow-Pratt coefficients $\lambda_u(w) = -u''(w)/u'(w)$ and $\lambda_v(w) = -v''(w)/v'(w)$. By [7, Prop. 1], the decision maker is ambiguity averse (as defined in [7, p. 1863]) if and only if ϕ is concave, that is, if and only if $\lambda_v \geq \lambda_u$. Ambiguity neutrality corresponds to $\phi(x) = x$, that is, $u = v$ (up to a normalization). In this case,

$$C(f) = u^{-1} \left(\int_{\Delta} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) d\mu(q) \right) = u^{-1}(E_{\bar{q}}(u \circ f)) = c(f, \bar{q}), \quad \forall f \in L(I) \quad (12)$$

where $\bar{q}$ is the barycenter of μ given in (5). Under ambiguity neutrality we thus get back to the certainty equivalent of a subjective expected utility (SEU) decision maker with prior $\bar{q}$. The same happens when the support of μ is a singleton $\{q\}$, that is, when the decision maker does not perceive any ambiguity. In this case it trivially holds $\bar{q} = q$ and again $C(f) = u^{-1}(E_{\bar{q}}(u \circ f))$.

In the ambiguity neutral and risk neutral case $u(x) = v(x) = x$, the certainty equivalent (11) reduces to the expected value $E_{\bar{q}}(f)$ of f under $\bar{q}$.

The *uncertainty premium* $\pi(f)$ of act f for the decision maker described by (11) is

$$\pi(f) = E_{\bar{q}}(f) - C(f), \quad \forall f \in L(I).$$

Throughout the paper we maintain the following assumption.

Assumption 1 *The functions $u, v : I \rightarrow \mathbb{R}$ are continuous, strictly increasing, and concave.*

Under this assumption the certainty equivalent (10) is well defined and $\pi(f) \geq 0$ for each $f \in L(I)$.

2.3 Quadratic approximation

Let $w \in \text{int } I$ be a scalar that we interpret as current wealth. To ease notation, we also denote by w the degenerate random variable $w1_{\Omega}$. Given $h \in \mathbb{R}^n$ such that $w + h \in L(I)$, the certainty equivalent $C(w + h)$ of $w + h$ is

$$C(w + h) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{\omega \in \Omega} u(w + h) q(\omega) \right) \right) d\mu(q) \right). \quad (13)$$

For all $h \in \mathbb{R}^n$, the function

$$E(h) : q \mapsto \sum_{\omega \in \Omega} h(\omega) q(\omega)$$

is continuous and bounded on Δ . Its variance with respect to μ is given by

$$\int_{\Delta} \left(\sum_{\omega \in \Omega} h(\omega) q(\omega) \right)^2 d\mu(q) - \left(\int_{\Delta} \left(\sum_{\omega \in \Omega} h(\omega) q(\omega) \right) d\mu(q) \right)^2,$$

and it is denoted by $\sigma_{\mu}^2(E(h))$. This variance reflects the uncertainty on the expectation $E(h)$ as it is perceived by the decision maker. Thus, higher values of $\sigma_{\mu}^2(E(h))$ correspond to a higher incidence of model uncertainty in the valuation of $E(h)$.

We now report the second order approximation of the certainty equivalent (13), a special case of a more general approximation derived by [9]. The approximation is based on the Arrow-Pratt coefficients $\lambda_u(w)$ and $\lambda_v(w)$. The Peano remainder $o(\|h\|_{\bar{q}}^2)$ is in terms of the $L^2(\bar{q})$ norm $\|h\|_{\bar{q}}^2 = E_{\bar{q}}(h^2)$.

Proposition 5 *Let $\mu : \mathcal{B}(\Delta) \rightarrow [0, 1]$ be a Borel probability measure on Δ and $u, v : I \rightarrow \mathbb{R}$ be twice continuously differentiable with $u', v' > 0$. Then,*

$$C(w+h) = w + E_{\bar{q}}(h) - \frac{1}{2} \lambda_u(w) \sigma_{\bar{q}}^2(h) - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \sigma_{\mu}^2(E(h)) + o(\|h\|_{\bar{q}}^2) \quad (14)$$

as $\|h\|_{\bar{q}} \rightarrow 0$.

Thus, the sign and magnitude of the effect of perceived model uncertainty on the certainty equivalent depend on the difference $\lambda_v(w) - \lambda_u(w)$.

For the study of risk and model uncertainty attitudes, it is useful to rearrange (14) as

$$C(w+h) = w + E_{\bar{q}}(h) - \frac{1}{2} \lambda_u(w) (\sigma_{\bar{q}}^2(h) - \sigma_{\mu}^2(E(h))) - \frac{1}{2} \lambda_v(w) \sigma_{\mu}^2(E(h)) + o(\|h\|_{\bar{q}}^2),$$

where we group terms according to the Arrow-Pratt coefficients. Risk and model uncertainty attitudes are thus weighted by $\sigma_{\bar{q}}^2(h) - \sigma_{\mu}^2(E(h))$ and $\sigma_{\mu}^2(E(h))$, respectively.

Similarly, the quadratic approximation of the uncertainty premium $\pi(w+h)$ of $w+h$ is

$$\pi(w+h) = \frac{1}{2} \lambda_u(w) (\sigma_{\bar{q}}^2(h) - \sigma_{\mu}^2(E(h))) + \frac{1}{2} \lambda_v(w) \sigma_{\mu}^2(E(h)) + o(\|h\|_{\bar{q}}^2). \quad (15)$$

Since

$$C(w+h) = w + E_{\bar{q}}(h) - \pi(w+h),$$

the effects of risk and model uncertainty attitudes on the second order approximation (14) can be studied directly via the premium (15). For this reason in what follows we focus on (15), although our main interest is ultimately in the certainty equivalent approximation (14).

2.4 First remarks on attitudes

In view of the quadratic approximation (15) of the uncertainty premium $\pi(w+h)$ we can make a few preliminary remarks on risk and model uncertainty attitudes. Specifically, according to the properties of the support of μ we consider four basic cases:

Case 1: The support of μ is neither a singleton nor a collection of Dirac measures. In this case both risk and model uncertainty attitudes typically matter.

Case 2: The support of μ is a singleton; i.e., $\text{supp } \mu = \{p\}$ for some $p \in \Delta$. In this case, $\bar{q} = p$ and only risk attitudes may matter. Specifically,

$$\pi(w+h) = \frac{1}{2} \lambda_u(w) \sigma_p^2(h) + o(\|h\|_p^2).$$

Case 3: The support of μ consists of Dirac measures; i.e., $\mu = \sum_{i=1}^n \mu_i \delta_{\omega_i}$ with $(\mu_1, \dots, \mu_n) \in \Delta$. Simple computation delivers $(\bar{q}_1, \dots, \bar{q}_n) = (\mu_1, \dots, \mu_n)$ and hence $\sigma_\mu^2(E(h)) = \sigma_{\bar{q}}^2(h)$. In this case only model uncertainty attitudes may matter. In fact, we can write

$$\pi(w + h) = \frac{1}{2} \lambda_v(w) \sigma_\mu^2(E(h)) + o(\|h\|_{\bar{q}}^2) = \frac{1}{2} \lambda_v(w) \sigma_{\bar{q}}^2(h) + o(\|h\|_{\bar{q}}^2).$$

Case 4: The support of μ is a singleton consisting of a Dirac measure; i.e., $\text{supp } \mu = \{\delta_\omega\}$. Neither risk nor model uncertainty attitudes matter in this case (which is the intersection of the two previous ones) and actually $\pi(w + h) = 0$.

Case 1 is the “normal” case, while Cases 2-4 are special. They are all based on the properties of the support of μ , an information trait. In the asymptotic analysis that we will carry out in the paper the limit behavior of h will also matter, along with that of μ . This will give rise to the Limit Cases 1-4 of Sections 7 and 8.

3 Small risks

3.1 Notion and characterization

The notion of *small risk* is cardinal to this paper. Since a definition of absolute risk smallness is elusive, we follow Pratt [11] in studying it via limit behavior. We first consider the case of a given probability measure $q \in \Delta$. Our analysis will be based on nets $\{h_t, q\}_{t \in (0,1]}$ where each $h_t : \Omega \rightarrow \mathbb{R}$ is a monetary act. We call them *risky monetary nets* when $\limsup_{t \downarrow 0} E_q(|h_t|) < \infty$. This limit condition put some discipline on how outcomes $h_t(\omega)$ can diverge as t goes to zero.

In a coin toss example with $\Omega = \{H, T\}$, the net $\{h_t, q\}_{t \in (0,1]}$ consists of acts whose outcomes $h_t(H)$ and $h_t(T)$ depend on whether heads or tails come up, with probabilities $q(H)$ and $q(T)$. The index t has no temporal meaning. The net $\{h_t, q\}_{t \in (0,1]}$ should not be regarded as a process, but as a sequence of static settings – “snapshots” – that at each t feature an act h_t whose “riskiness” depends on q .⁹

In this risk setting the certainty equivalent (11) reduces to

$$C(f) = u^{-1} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right). \quad (16)$$

In terms of the smooth model, this can be due to either $\phi(x) = x$ (ambiguity neutrality) or a singleton support for μ (no perceived ambiguity). Either way, here we are in a standard SEU setting.

We can now present the key notion of small risk.

Definition 6 A risky monetary net $\{h_t, q\}_{t \in (0,1]}$ is a small risk if $\lim_{t \downarrow 0} E_q(h_t^2) = 0$.

We illustrate this notion with few important examples.

Example 7 A net $\{h_t\}_{t \in (0,1]}$ such that

$$\lim_{t \downarrow 0} h_t(\omega) = 0, \quad \forall \omega \in \Omega \quad (17)$$

or, equivalently, such that $\lim_{t \downarrow 0} \|h_t\| = 0$, forms a small risk when paired with any $q \in \Delta$. In view of (4), it does not matter what norm of $\mathbb{R}^n$ is used to measure outcomes’ sizes. $\blacktriangle$

⁹Few remarks on notation: (i) as previously mentioned, the index t has no temporal meaning; (ii) we choose the letter h because small risks will be relevant for the approximations (14) and (15); (iii) the reason why we mention explicitly q in the net $\{h_t, q\}_{t \in (0,1]}$ will become clear later in the paper.

Example 8 Adapted to our setting, Pratt [11]’s analysis considers a net $\{h_t\}_{t \in (0,1]}$ of monetary acts that are actuarially neutral with respect to q and whose variances vanish; i.e., $E_q(h_t) = 0$ and $\lim_{t \downarrow 0} \sigma_q^2(h_t) = 0$. In other words, the net $\{h_t\}_{t \in (0,1]}$ goes to zero in $L^2(q)$ norm. The net $\{h_t, q\}_{t \in (0,1]}$ is a small risk. $\blacktriangle$

Example 9 Adapted to our setting, Segal and Spivak [12]’s analysis is based on a fixed monetary act h that determines a net $\{h_t\}_{t \in (0,1]}$ so that each h_t has the same distribution under q as th , for all $t \in (0, 1]$. That is,

$$q(h_t \in E) = q(th \in E) \quad (18)$$

for all $t \in (0, 1]$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Since $E_q(h_t^2) = t^2 E_q(h^2)$, then the net $\{h_t, q\}_{t \in (0,1]}$ is a small risk. $\blacktriangle$

Example 10 A special case of the previous example is $h_t = th$, q almost surely, for some monetary act h . We call *directional small risks* the nets $\{h_t, q\}_{t \in (0,1]}$ with $h_t = th$. $\blacktriangle$

The next result – a version for nets of a known result for sequences – characterizes small risks by considering three types of limit behavior and showing their equivalence to the notion of small risk. In particular, in (ii) limit behavior is in the sense of convergence in probability and acts’ values eventually vanish with probability one according to q . In (iii) limit behavior is in the sense of $L^n(p)$ convergence and acts’ moments eventually vanish.¹⁰ Finally, in (iv) limit behavior is in the sense of almost sure convergence and acts’ outcomes eventually vanish in each state that belongs to the support of q .

Proposition 11 *For a risky monetary net $\{h_t, q\}_{t \in (0,1]}$, the following conditions are equivalent:*

- (i) $\{h_t, q\}_{t \in (0,1]}$ is a small risk;
- (ii) $\lim_{t \downarrow 0} q(\omega : |h_t(\omega)| < \varepsilon) = 1$ for each $\varepsilon > 0$;
- (iii) $\lim_{t \downarrow 0} E_q(|h_t|^n) = 0$ for all $n \geq 1$;
- (iv) $\lim_{t \downarrow 0} h_t(\omega) = 0$ for all $\omega \in \text{supp } q$.

Condition (iv) easily implies (iii), which implies (i). In turn, (i) easily implies (ii). By the Chebyshev inequality, for all $\varepsilon > 0$

$$0 \leq q(\omega : |h_t(\omega)| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E_q(h_t^2) \rightarrow 0 \text{ as } t \downarrow 0.$$

Showing that (ii) implies (iv) is less immediate and is proved in the Appendix.

3.2 Order effects

Small risks can be used to model small risky deviations h_t from a sure prospect w and to study their effects on the risk premium

$$\pi(w + h_t) = w + E_q(h_t) - C(w + h_t). \quad (19)$$

Consider the following classification: risk attitudes have a

- (i) *first order effect* at $\{h_t, q\}_{t \in (0,1]}$ if $\pi(w + h_t) \asymp \|h_t\|_q$;
- (ii) *second order effect* at $\{h_t, q\}_{t \in (0,1]}$ if $\pi(w + h_t) \asymp \|h_t\|_q^2$;

¹⁰The condition $\limsup_{t \rightarrow 0} E_q(|f_t|) = \lim_{\varepsilon \downarrow 0} \sup_{t \in (0, \varepsilon)} E_q(|f_t|) < \infty$ only requires that the expectation $E_q(|f_t|)$ be eventually bounded.

(iii) (quadratically) negligible effect at $\{h_t, q\}_{t \in (0,1]}$ if $\pi(w + h_t) = o(\|h_t\|_q^2)$.

This classification revisits in our setup the orders of risk aversion that Segal and Spivak [12] studied in a lottery setting (see Montesano [10] for related ideas and Machina [8] for a comprehensive analysis). It is straightforward to see that (i)-(iii) are mutually exclusive alternatives, though not exhaustive.

The next result is related with [12, Proposition 1] and provides a behavioral characterization of these effects. To this end, set

$$k_* = \liminf_{t \downarrow 0} \frac{E_q^2(h_t)}{E_q(h_t^2)} \quad \text{and} \quad k^* = \limsup_{t \downarrow 0} \frac{E_q^2(h_t)}{E_q(h_t^2)}$$

Since it always holds $E_q^2(h_t) \leq E_q(h_t^2)$, we have $k_* \leq k^* \leq 1$. For instance, for the small risk of Example 9 it holds $k_* = k^* = E_q^2(h)/E_q(h^2)$.

Proposition 12 Consider a small risk $\{h_t, q\}_{t \in (0,1]}$ with $E_q(h_t) > 0$ for all t .

(i) If $\pi(w + h_t) \sim k \|h_t\|_q$ and $k > \sqrt{k^*}$, then for all t small enough

$$C(w + h_t) < w.$$

(ii) If $\pi(w + h_t) \sim k \|h_t\|_q$ and $0 < k < \sqrt{k_*}$, then for all t small enough

$$C(w + h_t) > w.$$

(iii) If $\pi(w + h_t) = o(\|h_t\|_q)$, then for all t small enough

$$C(w + h_t) > w.$$

Points (i) and (ii) consider first order effects. Point (i) shows that when k is high enough (e.g., $k \geq 1$), then $\pi(w + h_t) \sim k \|h_t\|_q$ implies that for all t small enough the decision maker strictly prefers the sure amount w over the risky one $w + h_t$. Since $E_q(h_t) > 0$, a risk neutral decision maker would prefer the opposite. Hence, in this case even for small risks the decision maker is not risk neutral.

Points (ii) and (iii) show that this is no longer the case when either k is small enough or $\pi(w + h_t) = o(\|h_t\|_q)$, that is, the effect is of second or higher order. In these cases, for t small enough the decision maker compares $w + h_t$ and w as if he were risk neutral. That is, risk attitudes do not matter.

In sum, Proposition 12 shows that risk attitudes that have a first order effects may or may not matter when comparing $w + h_t$ and w ; with second or higher order they do not matter.

3.3 Differential case

Throughout this subsection we make the following assumption.

Assumption 2 The utility function u is twice continuously differentiable with $u' > 0$.

By Proposition 5, under this assumption a small risk $\{h_t, q\}_{t \in (0,1]}$ has the quadratic approximation of its premium $\pi(w + h_t)$ given by

$$\pi(w + h_t) = \frac{1}{2} \lambda_u(w) \sigma_q^2(h_t) + o(\|h_t\|_q^2). \quad (20)$$

Remarkably, this approximation makes it possible to study risk attitudes' order effects through the variance $\sigma_q^2(h_t)$, an insight that goes back to Pratt (1964) and Arrow (1970). In particular, these attitudes have a first order effect at $\{h_t, q\}_{t \in (0,1]}$ if and only if $\sigma_q^2(h_t) \asymp \|h_t\|_q$, a second order effect if and only if $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$, and a negligible effect if and only if $\sigma_q^2(h_t) = o(\|h_t\|_q^2)$.

Since $\sigma_q^2(h_t) \leq \|h_t\|_q^2$, we have $\sigma_q^2(h_t) = o(\|h_t\|_q)$, and so risk attitudes do not have a first order effect under Assumption 2. They can have a second or higher effects, as we show next.

Lemma 13 *It holds $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$ if and only if $k^* < 1$ and $\sigma_q^2(h_t) = o(\|h_t\|_q^2)$ if and only if $k_* = 1$.*

Hence, risk attitudes have a second order effect at $\{h_t, q\}_{t \in (0,1]}$ when $k^* < 1$ and a negligible one when $k_* = 1$. Finally, there are no clear cut effects when $k_* < k^* = 1$.

Since $k_* \leq k^* \leq 1$, condition $k_* = 1$ is equivalent to $k_* = k^* = 1$, that is,

$$\lim_{t \downarrow 0} \frac{E_q^2(h_t)}{E_q(h_t^2)} = 1. \quad (21)$$

This condition holds when $\{h_t, q\}_{t \in (0,1]}$ is *asymptotically constant*, that is,

$$h_t(\omega) \sim h_t(\omega') \quad \forall \omega, \omega' \in \text{supp } q. \quad (22)$$

Intuitively, small risks that satisfy this condition are asymptotically riskless (notice that condition (22) trivially holds when $\text{supp } q$ is a singleton).

Under a mild regularity condition, asymptotic condition (22) is also necessary for (21).

Proposition 14 *Let $\{h_t, q\}_{t \in (0,1]}$ be a small risk, with $\lim_{t \downarrow 0} h_t(\omega) / h_t(\omega') \in [-\infty, \infty]$ for all $\omega, \omega' \in \text{supp } q$. Under Assumption 2, risk attitudes are negligible at $\{h_t, q\}_{t \in (0,1]}$ if and only if it is asymptotically constant.*

In other words, risk attitudes are negligible when risk asymptotically vanishes, that is, for small risks that are asymptotically constant, and thus asymptotically riskless. In this case,

$$\pi(w + h_t) = o(\|h_t\|_q^2).$$

Instead, when a small risk is not asymptotically constant (and we are not in the indeterminate case $k_* < k^* = 1$) we have

$$\pi(w + h_t) \sim \frac{1}{2} \lambda_u(w) \sigma_q^2(h_t),$$

that is, when risk attitudes have a second order effect, then the risk premium is asymptotically equivalent to $2^{-1} \lambda_u(w) \sigma_q^2(h_t)$. In this term, risk attitudes captured by $\lambda_u(w)$ are multiplicatively separated from the riskiness of the net $\{h_t, q\}_{t \in (0,1]}$ captured by $\sigma_q^2(h_t)$.

Example 15 (i) Consider a small risk $\{h_t, q\}_{t \in (0,1]}$ defined by (18); for example, a directional small risk. It holds $k^* = E_q^2(h) / E_q(h^2) < 1$ if h is not constant q -a.e. In this case $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$ and so risk attitudes have a second order effect. If, instead, h is constant q -a.e., then $k_* = E_q^2(h) / E_q(h^2) = 1$ and so risk attitudes have a negligible effect (it actually holds $\pi(w + h_t) = 0$ for each $0 < t \leq 1$).

(ii) Consider a space $\Omega = \{1, 2\}$ and a risky monetary net $\{h_t, q\}_{t \in (0,1]}$ with $0 < q(1) < 1$ and

$$h_t(\omega) = \begin{cases} t + t^2 & \text{if } \omega = 1 \\ t & \text{if } \omega = 2 \end{cases}.$$

This net is an asymptotically constant small risk (i.e., $k_* = 1$). Hence, $\pi(w + h_t) = o(\|h_t\|_q^2)$ and risk attitudes have a negligible effect at this small risk. $\blacktriangle$

3.4 Global effects

Till now we studied order effects at some small risk $\{h_t, q\}_{t \in (0,1]}$. It is a “local” standpoint. A global one would require that the order be consistent across all small risks. For this reason, we say that risk attitudes have a

- (i) *first order effect* if there is an open ball B such that $\pi(w + h) \asymp \|h\|_q$ for all $h \in B$;¹¹
- (ii) *second order effect* if there is an open ball B such that $\pi(w + h) \asymp \|h\|_q^2$ for all $h \in B$;
- (iii) *(quadratically) negligible effect* if there is an open ball B such that $\pi(w + h) = o(\|h\|_q^2)$ for all $h \in B$.

Clearly, if risk attitudes have a first, second, or negligible order effect, the same applies to all small risks. Indeed, any small risk $\{h_t, q\}_{t \in (0,1]}$ is such that $h_t \in B$ eventually.

The next result shows that, unless q is trivial, there is always some small risk at which risk attitudes are not negligible.

Proposition 16 *Under Assumption 2, risk attitudes have a negligible effect if and only if q is trivial, i.e., $q = \delta_\omega$ for some $\omega \in \Omega$.*

3.5 Scales of risks

Some classic scales can be used to benchmark the rate at which $E_q(h_t^2)$ converges to zero. The simplest is the power scale

$$..., t^{\frac{1}{n}}, ..., t^{\frac{1}{2}}, t, t^2, ..., t^n, ...$$

Let us consider the case of quadratic speed t^2 (while similar considerations would apply to other powers).

Definition 17 *A risky monetary net $\{h_t, q\}_{t \in (0,1]}$ is a quadratic small risk if $E_q(h_t^2) = O(t^2)$.*

In other words, quadratic small risks feature second moments $E_q(h_t^2)$ that go to zero with at least quadratic speed.

Example 18 The net $\{h_t, q\}_{t \in (0,1]}$ defined by (18) is a quadratic small risk since $E_q(h_t^2) = t^2 E_q(h^2)$. In particular, this is the case for directional small risks. $\blacktriangle$

By Lemma 1, for quadratic small risks the quadratic approximation (20) holds with a Peano remainder $o(t^2)$ in place of $o(\|h_t\|_q^2)$, that is,

$$\pi(w + h_t) = \frac{1}{2} \lambda_u(w) \sigma_q^2(h_t) + o(t^2).$$

Next we characterize quadratic small risks.

Proposition 19 *Given a monetary net $\{h_t\}_{t \in (0,1]}$, the following conditions are equivalent:*

- (i) $\{h_t\}_{t \in (0,1]}$ is a quadratic small risk;
- (ii) $E_q(|h_t|^n) = O(t^n)$ for some $n \geq 1$;
- (iii) $E_q(|h_t|^n) = O(t^n)$ for all $n \geq 1$;

¹¹Open ball B here means any open ball centered at the origin, i.e., $B = \{h \in \mathbb{R}^n : \|h\|_q < \varepsilon\}$ for some $\varepsilon > 0$.

(iv) $h_t(\omega) = O(t)$ for all $\omega \in \text{supp } q$.

Example 20 For the net $\{h_t, q\}_{t \in (0,1]}$ defined by (18) we have $E_q(|h_t|) = tE_q(|h|)$. ▲

Points (iv) of Propositions 11 and 19 show that the difference between small risks and quadratic ones can be seen by looking at the limit behavior of $h_t(\omega)$ for each $\omega \in \text{supp } q$. Small risks correspond to their convergence to zero, i.e., $h_t(\omega) \rightarrow 0$, while quadratic small risks require that this convergence occurs at least at linear speed, i.e., $h_t(\omega) = O(t)$.

4 Small uncertainties

The earlier risk analysis can be extended to account for ambiguity. We call *uncertain monetary nets*, indicated $\{h_t, \mu\}_{t \in (0,1]}$, nets of monetary acts $h_t : \Omega \rightarrow \mathbb{R}$ and probability distributions μ on Δ , with $\lim_{t \downarrow 0} E_{\bar{q}}(|h_t|) < \infty$. In a coin toss example, μ is the decision maker's prior on the probability models q that determine how likely heads and tails are.

Now the relevant certainty equivalent is (11). As in (5), we set

$$\bar{q}(\omega) = \int_{\Delta} q(\omega) d\mu(q).$$

Hence, an uncertain monetary net $\{h_t, \mu\}_{t \in (0,1]}$ induces a risky monetary net $\{h_t, \bar{q}\}_{t \in (0,1]}$. Using these induced nets we can extend to uncertain monetary nets some of the notions that we previously established for risky ones.

Definition 21 An uncertain monetary net $\{h_t, \mu\}_{t \in (0,1]}$ is a *small uncertainty* if $\lim_{t \downarrow 0} E_{\bar{q}}(h_t^2) = 0$.

It is straightforward to check that an uncertain monetary net $\{h_t, \mu\}_{t \in (0,1]}$ is a small uncertainty if and only if the induced risky monetary net $\{h_t, \bar{q}\}_{t \in (0,1]}$ is a small risk. This fact is important to extend to small uncertainties the analysis carried out for small risks. A first consequence is that Proposition 11 holds verbatim with $\bar{q}$ in place of q . This means that, for example, $\{h_t, \mu\}_{t \in (0,1]}$ is a small uncertainty if and only if $\lim_{t \downarrow 0} h_t(\omega) = 0$ for all $\omega \in \text{supp } \bar{q}$.

Regarding order effects, we now have uncertainty attitudes in place of just risk ones. In particular, uncertainty attitudes have a

- (i) *first order effect* at $\{h_t, \mu\}_{t \in (0,1]}$ if $\pi(w + h_t) \asymp \|h_t\|_{\bar{q}}$;
- (ii) *second order effect* at $\{h_t, \mu\}_{t \in (0,1]}$ if $\pi(w + h_t) \asymp \|h_t\|_{\bar{q}}^2$;
- (iii) *(quadratically) negligible effect* at $\{h_t, \mu\}_{t \in (0,1]}$ if $\pi(w + h_t) = o(\|h_t\|_{\bar{q}}^2)$.

Proposition 12 also holds verbatim with $\bar{q}$ in place of q . Hence, the order effects of uncertainty attitudes depend on the limit behavior of the ratio $E_{\bar{q}}(h_t) / \|h_t\|_{\bar{q}}$.

More can be said assuming differentiability. Throughout the rest of the section we give the following version of Assumption 1 (in which strict inequalities simplify our analysis).

Assumption 3 The functions $u, v : I \rightarrow \mathbb{R}$ are twice differentiable with $u', v' > 0$.

Under Assumption 3 we can consider the quadratic approximation

$$\pi(w + h_t) = \frac{1}{2} \lambda_u(w) (\sigma_{\bar{q}}^2(h_t) - \sigma_{\mu}^2(E(h_t))) + \frac{1}{2} \lambda_v(w) \sigma_{\mu}^2(E(h_t)) + o(\|h_t\|_{\bar{q}}^2) \quad (23)$$

of the premium $\pi(w + h_t)$ of a small uncertainty $\{h_t, \mu\}_{t \in (0,1]}$. This makes it possible to separate the effects of risk and model uncertainty attitudes. To this end, define the functions Λ_u and Λ_v as

$$\Lambda_u(h, \mu) = \sigma_{\bar{q}}^2(h) - \sigma_{\mu}^2(E(h)) \quad \text{and} \quad \Lambda_v(h, \mu) = \sigma_{\mu}^2(E(h)). \quad (24)$$

These functions describe the effect of risk and model uncertainty attitudes on approximation (23), which can be rewritten as

$$\pi(w + h_t) = \frac{1}{2}\lambda_u(w)\Lambda_u(h_t, \mu) + \frac{1}{2}\lambda_v(w)\Lambda_v(h_t, \mu) + o\left(\|h_t\|_{\bar{q}}^2\right). \quad (25)$$

In the present differential setting first order effects never arise with small uncertainties.

Lemma 22 *It holds $\Lambda_u(h_t, \mu) = o\left(\|h_t\|_{\bar{q}}\right)$ and $\Lambda_v(h_t, \mu) = o\left(\|h_t\|_{\bar{q}}\right)$ for each small uncertainty $\{h_t, \mu\}_{t \in (0,1]}$.*

In view of this lemma, in the differential case we only need to consider second and higher order effects. In particular, risk attitudes have a

- (i) *second order effect* at $\{h_t, \mu\}_{t \in (0,1]}$ if $\Lambda_u(h_t, \mu) \asymp \|h_t\|_{\bar{q}}^2$;
- (ii) *(quadratically) negligible effect* at $\{h_t, \mu\}_{t \in (0,1]}$ if $\Lambda_u(h_t, \mu) = o\left(\|h_t\|_{\bar{q}}^2\right)$.

A similar classification holds for model uncertainty attitudes, with Λ_v in place of Λ_u .

If risk and model uncertainty attitudes both have a second order effect, then overall uncertainty attitudes have a second order effect. If, instead, risk and model uncertainty attitudes have effects of different orders, the overall uncertainty attitudes have effects of the lower order. In particular, uncertainty attitudes have a second order effect either if risk attitudes have a second order effect and model uncertainty attitudes have a higher order effect or if the opposite is true. In this case we have

$$\pi(w + h_t) = \frac{1}{2}\lambda_u(w)\Lambda_u(h_t, \mu) + o\left(\|h_t\|_{\bar{q}}^2\right) \quad \text{or} \quad \pi(w + h_t) = \frac{1}{2}\lambda_v(w)\Lambda_v(h_t, \mu) + o\left(\|h_t\|_{\bar{q}}^2\right),$$

respectively. Finally, $\pi(w + h_t) = o\left(\|h_t\|_{\bar{q}}^2\right)$ if both risk and model uncertainty attitudes have negligible effects.

Example 23 Consider the net $\{h_t, \mu\}_{t \in (0,1]}$ such that the induced net $\{h_t, \bar{q}\}_{t \in (0,1]}$ is defined by (18). For example, a directional small uncertainty with $h_t = th$, $\bar{q}$ -a.e., for a given a monetary act h . We have $\sigma_{\bar{q}}^2(h_t) = t^2\sigma_{\bar{q}}^2(h)$ and $\sigma_{\mu}^2(E(h_t)) = t^2\sigma_{\mu}^2(E(h))$. Hence,

$$\Lambda_u(h_t, \mu) = t^2\left(\sigma_{\bar{q}}^2(h) - \sigma_{\mu}^2(E(h))\right) \quad \text{and} \quad \Lambda_v(h_t, \mu) = t^2\sigma_{\mu}^2(h).$$

Suppose that h is such that $\sigma_{\bar{q}}^2(h) > 0$ and $\sigma_{\mu}^2(h) > 0$. Since $\|h_t\|_{\bar{q}}^2 = t^2\|h\|_{\bar{q}}^2$, both risk and model uncertainty attitudes have second order effects if $\sigma_{\bar{q}}^2(h) \neq \sigma_{\mu}^2(E(h))$ while only uncertainty attitudes have a second order effect if $\sigma_{\bar{q}}^2(h) = \sigma_{\mu}^2(E(h))$. $\blacktriangle$

We defer to Sections 7 and 8 a detailed analysis (for the binomial case) of the asymptotic versions of the four cases discussed in Section 2.4. Here we consider global effects. In particular, in the differential case, risk attitudes have

- (i) *second order effect* if there is an open ball B such that $\Lambda_u(h, \mu) \asymp \|h\|_{\bar{q}}^2$ for all $h \in B$;
- (ii) *(quadratically) negligible effect* if there is an open ball B such that $\Lambda_u(h, \mu) = o\left(\|h\|_{\bar{q}}^2\right)$ for all $h \in B$.

A similar classification holds for model uncertainty attitudes, with Λ_v in place of Λ_u . The following result extends Proposition 16 to uncertainty.

Proposition 24 (i) *Model uncertainty attitudes have a negligible effect if and only if $\text{supp } \mu = \{p\}$ for some $p \in \Delta$.* (ii) *Risk attitudes have a negligible effect if and only if $\text{supp } \mu$ consists of Dirac measures.* (iii) *Both risk and model uncertainty attitudes have a negligible effect if and only if $\text{supp } \mu = \{\delta_\omega\}$ for some $\omega \in \Omega$.*

In other words, risk attitudes are “globally” negligible if and only if the support of μ consists of Dirac measures on Ω , while model uncertainty attitudes are globally negligible if and only if μ is itself a Dirac measure on Δ (i.e., it gives full weight to a single model $p \in \Delta$). Finally, both attitudes are negligible when μ is a Dirac measure on a Dirac measure, that is, $\text{supp } \mu = \{\delta_\omega\}$. These cases are asymptotic global versions of Cases 2-4 discussed in Section 2.4.

Finally, we can define quadratic small uncertainties when $E_{\bar{q}}(h_t^2) = O(t^2)$. The analysis is similar, up to obvious modifications, to the one carried out for the risk case. For brevity we thus omit it.

5 Risk varying setting

5.1 Small risks

We extend our analysis to a risk varying setting in which q may depend on t . Thus, we now focus on nets $\{h_t, q_t\}_{t \in (0,1]}$ of monetary acts $h_t : \Omega \rightarrow \mathbb{R}$ and probability distributions $q_t \in \Delta$ that we call *risky monetary nets* provided $\limsup_{t \downarrow 0} E_{q_t}(|h_t|) < +\infty$ and $q_t \rightarrow q$ for some $q \in \Delta$; that is, $\lim_{t \downarrow 0} q_t(\omega) = q(\omega)$ for all $\omega \in \Omega$. These two conditions put some discipline on how outcomes and probabilities can vary as t goes to zero.

This notion of risky monetary nets $\{h_t, q_t\}_{t \in (0,1]}$ reduces to the earlier one when $q_t = q$ for all $t \in (0,1]$. Here, each act h_t delivers outcomes under possibly different risk conditions determined by q_t . We therefore relax the assumption that each static situation t features the same risk conditions. In a coin toss example, the probability $q_t(H)$ and $q_t(T)$ with which heads and tails come up may depend on the index t .¹²

The notion of small risk, introduced in Definition 6, is readily extended to the present more general setting.

Definition 25 *A risky monetary net $\{h_t, q_t\}_{t \in (0,1]}$ is a small risk if $\lim_{t \downarrow 0} E_{q_t}(h_t^2) = 0$.*

There is a key novelty in this notion of small risk for the risk varying case: now there are two (complementary) sources of smallness, outcomes and probabilities. That is, we may have $E_{q_t}(h_t^2) \rightarrow 0$ because either outcomes $h_t(\omega)$ or their probabilities $q_t(\omega)$ (or both) are becoming smaller and smaller as t goes to zero. For example, consider $\Omega = \{1, 2\}$ and

$$h_t(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = 1 \\ \psi(t) & \text{if } \omega = 2 \end{cases}$$

Set $q_t = q_t(1) \rightarrow q = q(1)$. If $q \in (0, 1)$, then $E_{q_t}(h_t^2) \rightarrow 0$ provided $\psi(t) \rightarrow 0$ and $\varphi(t) \rightarrow 0$. In contrast, if $q = 0$ then $E_{q_t}(h_t^2) \rightarrow 0$ provided only $\psi(t) \rightarrow 0$, without any requirement on $\varphi(t)$.

¹²The risk varying setting is more general than the previous one, even in terms of induced distributions on outcomes. For example, given $\Omega = \{1, 2\}$, let $\{h_t, q_t\}_{t \in (0,1]}$ be such that $q_t(1) \neq q_{t'}(1)$ for each $t \neq t'$ and $h_t(1) \neq h_t(2)$ for each t . Given any $q \in \Delta$, it holds $q(h_t = x) \in \{q(1), 1 - q(1)\}$ for all t . Hence, there is no $q \in \Delta$ such that $q(h_t = x) = q_t(h_t = x)$ for all t .

To characterize small risks we extend Proposition 11 to the risk varying setting. To this end, set¹³

$$\limsup_{t \downarrow 0} (\text{supp } q_t) = \lim_{t \downarrow 0} \bigcup_{\tau \leq t} \text{supp } q_\tau.$$

Then, it is easy to see that $\text{supp } q \subseteq \limsup_{t \downarrow 0} (\text{supp } q_t)$. In addition, set

$$\Gamma_q = \limsup_{t \downarrow 0} (\text{supp } q_t) - \text{supp } q,$$

that is, Γ_q is the difference between the sets $\limsup_{t \downarrow 0} (\text{supp } q_t)$ and $\text{supp } q$. Then, $\Gamma_q = \emptyset$ when $q_t = q$ for all $t \in (0, 1]$. In other words, the nonemptiness of Γ_q is peculiar to the risk varying setting.

Lemma 26 *It holds $\omega \notin \limsup_{t \downarrow 0} (\text{supp } q_t)$ if and only if $q_t(\omega) = 0$ eventually.*

This lemma has two important implications. First, $q_t(\limsup_{t \downarrow 0} (\text{supp } q_t)) = 1$ for each t small enough. Second, $\Gamma_q = \{\omega \notin \text{supp } q : q_t(\omega) > 0 \text{ i.o.}\}$, in which we used the “infinitely often” terminology of Probability Theory (see [2, p. 40]).

We now characterize small risks by generalizing Proposition 11, which is the special case $q_t = q$ for all $t \in (0, 1]$.

Proposition 27 *For a risky monetary net $\{h_t, q_t\}_{t \in (0, 1]}$, the following conditions are equivalent:*

- (i) $\{h_t, q_t\}_{t \in (0, 1]}$ is a small risk;
- (ii) $\lim_{t \downarrow 0} q_t(\omega : |h_t(\omega)| > \varepsilon) = 0$ for each $\varepsilon > 0$;
- (iii) $\lim_{t \downarrow 0} E_{q_t}(|h_t|^n) = 0$ for all $n \geq 1$;
- (iv) $\lim_{t \downarrow 0} h_t(\omega) = 0$ for each $\omega \in \text{supp } q$ and $\lim_{t \downarrow 0} h_t(\omega) q_t(\omega) = 0$ for each $\omega \in \Gamma_q$;
- (v) $\lim_{t \downarrow 0} h_t(\omega) = 0$ for each $\omega \in \text{supp } q$.

Immediately, (iii) implies (i), and (iv) implies (v). Moreover, (iii) easily implies (ii): by the Chebyshev inequality

$$\lim_{t \downarrow 0} q_t(\omega : |h_t(\omega)| > \varepsilon) \leq \frac{1}{\varepsilon^n} \lim_{t \downarrow 0} E_{q_t}(|h_t|^n) = 0, \quad \forall \varepsilon > 0,$$

for each $n \geq 1$. The remaining implications are less straightforward and are proved in the Appendix.

Let us adapt Examples 7-9 to the present setting.

Example 28 (i) A net $\{h_t\}_{t \in (0, 1]}$ that statewise converges as in (17) forms a small risk $\{h_t, q_t\}_{t \in (0, 1]}$ along with any net $\{q_t\}_{t \in (0, 1]}$ that quadratically converges to some $q \in \Delta$.

(ii) In a risk varying setting, Pratt [11]’s analysis amounts to consider a risky monetary net $\{h_t, q_t\}_{t \in (0, 1]}$ with $E_{q_t}(h_t) = 0$ and $\lim_{t \downarrow 0} E_{q_t}(h_t^2) = 0$.

(iii) As to Segal and Spivak [12], let $\{q_t\}_{t \in (0, 1]}$ be any net such that $q_t \rightarrow q$. Fix a monetary act $h \in \mathbb{R}^n$ and define a risky monetary net $\{h_t, q_t\}_{t \in (0, 1]}$ such that, at each $t \in (0, 1]$, h_t has the same distribution under q_t as th_1 under q_1 . That is,

$$q_t(h_t \in E) = q_1(th_1 \in E), \quad \forall t \in (0, 1], \quad (26)$$

¹³The limit is meaningful since $\bigcup_{\tau \leq t} \text{supp } q_\tau \subseteq \bigcup_{\tau \leq t'} \text{supp } q_\tau$ if $t < t'$ (see, e.g., [6, p. 23] and [2, p. 39]).

for all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Since $E_{q_t}(h_t^2) = t^2 E_{q_1}(h^2)$, this net is a small risk. For example, given $\Omega = \{1, 2\}$ let (h_1, q_1) be such that $h_1(1) = -h_1(2) = 1$ and $q_1(1) = 1/3$. Then,

$$q_1(tf_1 = x) = \begin{cases} \frac{1}{3} & \text{if } x = t \\ \frac{2}{3} & \text{if } x = -t \end{cases}$$

If h_t is such that $h_t(1) = -h_t(2) = t$, then $\{h_t, q_t\}_{t \in (0,1]}$ is a risky monetary net that satisfies (26).

(iv) A special case of the previous example are the directional small risks $\{h_t, q_t\}_{t \in (0,1]}$, where $h_t = h$, q_t -a.e., for a given monetary act $h \in \mathbb{R}^n$. $\blacktriangle$

5.2 Order effects

To study the order effects of small risks in the present risk varying setting we need to consider the indexed certainty equivalent

$$C_t(f) = u^{-1} \left(\sum_{\omega \in \Omega} u(f(\omega)) q_t(\omega) \right), \quad (27)$$

Relative to (16), the model q_t may now change with t . In particular, the risk premium (19) takes now the form

$$\pi_t(w + h) = w + E_{q_t}(h) - C_t(w + h).$$

We can define order effects in the usual way: risk attitudes have a

- (i) *first order effect* at $\{h_t, q_t\}_{t \in (0,1]}$ if $\pi_t(w + h_t) \asymp \|h_t\|_{q_t}$;
- (ii) *second order effect* at $\{h_t, q_t\}_{t \in (0,1]}$ if $\pi_t(w + h_t) \asymp \|h_t\|_{q_t}^2$;
- (iii) *(quadratically) negligible effect* at $\{h_t, q_t\}_{t \in (0,1]}$ if $\pi_t(w + h_t) = o(\|h_t\|_{q_t}^2)$.

If we define

$$k_* = \liminf_{t \downarrow 0} \frac{E_{q_t}^2(h_t)}{E_{q_t}(h_t^2)} \quad \text{and} \quad k^* = \limsup_{t \downarrow 0} \frac{E_{q_t}^2(h_t)}{E_{q_t}(h_t^2)} \quad (28)$$

we can extend Proposition (12) to the present setting, up to obvious changes (i.e., add index t whenever needed). Therefore, for all t small enough it holds

$$C_t(w + h_t) < w$$

if $\pi_t(w + h_t) \sim k \|h_t\|_{q_t}$ and $k > \sqrt{k^*}$, while

$$C_t(w + h_t) > w.$$

if either $\pi_t(w + h_t) \sim k \|h_t\|_{q_t}$ and $k < \sqrt{k_*}$ or $\pi_t(w + h_t) = o(\|h_t\|_{q_t})$. As we observed in the constant risk case, here as well risk attitudes with first order effects may or may not matter when comparing $w + h_t$ and w ; with second or higher order they do not matter.

Substantially more delicate is the extension of the differential case to this risk varying setting. Indeed, because of the dependence on t , we need to improve substantially the quadratic approximation (20) to control variations in t . To see why this is the case, consider a risky monetary net $\{h_t, q\}_{t \in (0,1]}$ in which q_t does not depend on t . In this case $\lim_{t \downarrow 0} E_q(h_t^2) = 0$ and so the quadratic approximation of the uncertainty premium is given by (20), that is,

$$\pi(w + h_t) = \frac{1}{2} \lambda_u(w) \sigma_q^2(h_t) + o(\|h_t\|_q^2) \quad (29)$$

When q_t depends on t , we need a similar approximation based on the time varying certainty equivalent $C_t(w + h_t)$ that at each t may feature a different model q_t . This a nontrivial complication of the risk varying case that has to be carefully addressed.

Definition 29 A small risk $\{h_t, q_t\}_{t \in (0,1]}$ is controllable if $E_{q_t}(h_t^2) \asymp \sum_{\omega \in \text{supp } q_t} h_t^2(\omega)$.

Since $E_{q_t}(h_t^2) \leq \sum_{\omega \in \text{supp } q_t} h_t^2(\omega)$, controllability amounts to require

$$\liminf_{t \downarrow 0} \frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)} > 0. \quad (30)$$

Two remarks are in order before illustrating this notion with a couple of examples. First, observe that by Lemma 26 it holds

$$\liminf_{t \downarrow 0} \frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)} = \liminf_{t \downarrow 0} \frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q \cup \Gamma_q} h_t^2(\omega)} \quad (31)$$

To check condition (30) we can then consider either $\sum_{\omega \in \text{supp } q \cup \Gamma_q} h_t^2(\omega)$ or $\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)$ in the denominator, as more convenient. Second, recall that, by Proposition 27, a small risk is characterized by the condition $\lim_{t \downarrow 0} h_t(\omega) = 0$ for all $\omega \in \text{supp } q$. Since $E_{q_t}(h_t^2) \rightarrow 0$, in view of (31) a necessary condition for a small risk to be controllable is that it also holds $\lim_{t \downarrow 0} h_t(\omega) = 0$ for all $\omega \in \Gamma_q$.

Example 30 (i) Let $\Omega = \{1, 2\}$ and

$$h_t(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = 1 \\ \psi(t) & \text{if } \omega = 2 \end{cases}$$

with $\varphi(t) \sim \psi(t)$. If $0 < q_t < 1$, then

$$\frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)} = \frac{q_t \varphi^2(t) + (1 - q_t) \psi^2(t)}{\varphi^2(t) + \psi^2(t)} \sim \frac{\varphi^2(t)}{2\varphi^2(t)} = \frac{1}{2}$$

and so this risky monetary net $\{h_t, q_t\}_{t \in (0,1]}$ is controllable. If, instead, $\psi(t) = o(\varphi(t))$, then

$$\frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)} \sim q_t \rightarrow q$$

and so in this case $\{h_t, q_t\}_{t \in (0,1]}$ is controllable if $q > 0$.

(ii) Directional small risks are controllable. For, fix h and consider a risk pattern $\{q_t\}_{t \in (0,1]}$ with $q_t \rightarrow q$. Let $\{h_t, q_t\}_{t \in (0,1]}$ be such that $h_t = th$, q -a.e., for each $t \in (0, 1]$. Then,

$$\frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q \cup \Gamma_q} h_t^2(\omega)} = \frac{E_{q_t}(h^2)}{\sum_{\omega \in \text{supp } q \cup \Gamma_q} h^2(\omega)} \rightarrow \frac{E_q(h^2)}{\sum_{\omega \in \text{supp } q \cup \Gamma_q} h^2(\omega)}$$

and so $\{h_t, q_t\}_{t \in (0,1]}$ is controllable when $E_q(h^2) > 0$, i.e., $h \neq 0$ q -a.e. ▲

The next result, a special case of Theorem 38 below, shows that in the controllable case we can suitably extend the quadratic approximation (29) to risky monetary nets.

Theorem 31 Let $\{h_t, q_t\}_{t \in (0,1]}$ be a controllable small risk. If Assumption 2 holds with u three time differentiable, then

$$C_t(w + h_t) = w + E_{q_t}(h_t) - \frac{1}{2} \lambda_u(w) \sigma_{q_t}^2(h_t) + o(\|h_t\|_{q_t}^2) \quad (32)$$

for all $w \in \mathbb{R}$.

In other words, the risk premium takes the form

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \sigma_{q_t}^2(h_t) + o\left(\|h_t\|_{q_t}^2\right). \quad (33)$$

that is,

$$\frac{\pi_t(w + h_t) - \frac{1}{2} \lambda_u(w) \sigma_{q_t}^2(h_t)}{\|h_t\|_{q_t}^2} \rightarrow 0$$

where $\|h_t\|_{q_t}^2 \rightarrow 0$ because $\{h_t, q_t\}_{t \in (0,1]}$ is a small risk. Approximation (29) is the special case of (33) where $q_t = q$ for all t .

Thanks to the quadratic approximation (32) we can study order effects through the variances $\sigma_{q_t}^2(h_t)$. Since Lemma 13 is easily extended to the risk varying case, with k_* and k^* given by (28), we have:

- (i) risk attitudes do not have a first order effect at $\{h_t, q_t\}_{t \in (0,1]}$ since $\sigma_{q_t}^2(h_t) \leq \|h_t\|_{q_t}^2$ implies $\sigma_{q_t}^2(h_t) = o\left(\|h_t\|_{q_t}\right)$;
- (ii) risk attitudes have a second order effect at $\{h_t, q_t\}_{t \in (0,1]}$ if and only if $k^* < 1$;
- (iii) risk attitudes have a negligible effect at $\{h_t, q_t\}_{t \in (0,1]}$ if and only if $k_* = 1$, that is,

$$\lim_{t \downarrow 0} \frac{E_{q_t}^2(h_t)}{E_{q_t}(h_t^2)} = 1. \quad (34)$$

We defer a detailed analysis of these cases to Sections 7 and 8, where we study the binomial case in the more general setting of varying uncertainty that we introduce in the next section. Now we extend Proposition 14 to show that risk attitudes are negligible when risk asymptotically vanishes.

Proposition 32 *Under Assumption 2, risk attitudes are negligible at controllable small risks $\{h_t, q_t\}_{t \in (0,1]}$ that are asymptotically constant, that is,*

$$h_t(\omega) \sim h_t(\omega') \quad \forall \omega, \omega' \in \text{supp } q \quad (35)$$

Condition (35) may arise either when $\text{supp } q$ is singleton – that is, q is a Dirac and so risk eventually disappears – or when $\text{supp } q$ is nonsingleton but h_t is asymptotically constant. Either way, risk eventually vanishes and risk attitudes do not matter.

Finally, to study global effects in a risk varying setting we need to fix a risk pattern $\{q_t\}_{t \in (0,1]}$ with $q_t \rightarrow q \in \Delta$. By considering open balls B with respect to the limit distribution q – that is, $B = \{h \in \mathbb{R}^n : \|h\|_q < \varepsilon\}$ for some $\varepsilon > 0$ – we can consider verbatim the global notions introduced in Section 3.3. In particular, Proposition 16 is easily generalized to show that risk attitudes have a negligible effect if and only if the limit q is trivial (provided Assumption 2 holds with u three time differentiable).

5.3 Quadratic small risks

The notion of quadratic small risks, introduced in Definition 17, readily extends to the present setting.

Definition 33 *A risky monetary net $\{h_t, q_t\}_{t \in (0,1]}$ is a quadratic small risk if $E_{q_t}(h_t^2) = O(t^2)$.*

Next we characterize quadratic small risks: relative to Proposition 19, here the case of $n = 2$ plays a special role.

Proposition 34 *Given a risky monetary net $\{h_t, q_t\}_{t \in (0,1]}$ the following conditions are equivalent:*

- (i) $\{h_t, q_t\}_{t \in (0,1]}$ is a quadratic small risk;
- (ii) $h_t(\omega) = O(t)$ for all $\omega \in \text{supp } q$ and $h_t^2(\omega) q_t(\omega) = O(t^2)$ for all $\omega \in \Gamma_q$.

Given $\omega \in \Gamma_q$, sufficient conditions for $h_t^2(\omega) q_t(\omega) = O(t^2)$ are $q_t(\omega) = O(t^2)$ and $\limsup_{t \downarrow 0} |h_t(\omega)| < \infty$. That is, (ii) holds – and so $\{h_t, q_t\}_{t \in (0,1]}$ is a quadratic small risk – if $h_t(\omega) = O(t)$ for all $\omega \in \text{supp } q$ and if $q_t(\omega)$ goes to zero fast enough, with $|h_t(\omega)|$ bounded for all $\omega \in \Gamma_q$.

The next example shows that, unlike Proposition 19, condition (ii) in Proposition 34 does not hold for any n .

Example 35 (i) Directional small risks are quadratic. For

$$\frac{E_{q_t}(h_t^2)}{t^2} = E_{q_t}(h^2) \rightarrow E_q(h^2)$$

(ii) Given $\Omega = \{1, 2\}$, let $\{h_t, q_t\}_{t \in (0,1]}$ be given by $h_t(1) = 1$, $h_t(2) = t$, and $q_t(1) = t^3$. Then, $E_{q_t}(h_t^2) = O(t^2)$, but $E_{q_t}(h_t^4) \neq O(t^4)$. Hence, $\{h_t, q_t\}_{t \in (0,1]}$ is a quadratic small risk, although it is not true that $E_{q_t}(|h_t|^n) = O(t^n)$ for all $n \geq 1$. $\blacktriangle$

Finally, when $\{h_t, \mu_t\}_{t \in (0,1]}$ is a quadratic small uncertainty, we can replace $o(\|h_t\|_{\bar{q}_t}^2)$ with $o(t^2)$ in the risk premium approximation (33), that is,

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \sigma_{q_t}^2(h_t) + o(t^2).$$

6 Uncertainty varying setting

In this section we extend our earlier risk analysis to account for ambiguity, as modeled by the smooth ambiguity model (9). We consider a net of probability measures $\{\mu_t\}_{t \in (0,1]}$ on Δ and we allow μ to depend on the index t . As in (5), we set

$$\bar{q}_t(\omega) = \int_{\Delta} q(\omega) d\mu_t(q)$$

We call *uncertain monetary nets*, indicated $\{h_t, \mu_t\}_{t \in (0,1]}$, nets of monetary acts $h_t : \Omega \rightarrow \mathbb{R}$ and probability distributions μ_t on Δ such that $\limsup_{t \downarrow 0} E_{\bar{q}_t}(|h_t|) < +\infty$ and $\mu_t \xrightarrow{sq} \mu$ for some μ on Δ . In a coin toss example, μ_t is the decision maker's prior on the probability models q that determine how likely heads and tails are. This prior now can vary with t .

Set $\bar{q}(\omega) = \int_{\Delta} q(\omega) d\mu(q)$. By Lemma 2, $\lim_{t \downarrow 0} \bar{q}_t(\omega) = \bar{q}(\omega)$ for each $\omega \in \Omega$ if $\mu_t \xrightarrow{sq} \mu$. Hence, an uncertain monetary net $\{h_t, \mu_t\}_{t \in (0,1]}$ induces a risky monetary net $\{h_t, \bar{q}_t\}_{t \in (0,1]}$, and $\mu_t \xrightarrow{sq} \mu$ implies $\bar{q}_t \rightarrow \bar{q}$. Using these induced nets we can readily extend to uncertain monetary nets what we have previously established for risky ones.

Let us begin with the notions of small uncertainties.

Definition 36 *An uncertain monetary net $\{h_t, \mu_t\}_{t \in (0,1]}$ is a small uncertainty if $\lim_{t \downarrow 0} E_{\bar{q}_t}(h_t^2) = 0$.*

This definition extends the scope of Definition 25 to uncertainty. In particular, an uncertain monetary net $\{h_t, \mu_t\}_{t \in (0,1]}$ is a small uncertainty if and only if the induced monetary net $\{h_t, \bar{q}_t\}_{t \in (0,1]}$ is a small risk. For this reason, Proposition 27 holds with $\bar{q}_t$ and $\bar{q}$ in place of q_t and q , respectively.

To study the behavior of small uncertainties we consider the indexed certainty equivalent

$$C_t(f) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{\omega \in \Omega} u(f(\omega)) q(\omega) \right) \right) d\mu_t(q) \right),$$

Relative to (1), the prior μ_t may now change with t .

Through the induced small risk $\{h_t, \bar{q}_t\}_{t \in (0,1]}$ we can easily extend to uncertainty all results on order effects established in Section 5.2 by setting

$$k_* = \liminf_{t \downarrow 0} \frac{E_{\bar{q}_t}(h_t)}{\|h_t\|_{\bar{q}_t}} \quad \text{and} \quad k^* = \limsup_{t \downarrow 0} \frac{E_{\bar{q}_t}(h_t)}{\|h_t\|_{\bar{q}_t}}.$$

For brevity, we omit the details.

As in the risk varying case, the dependence on t requires a substantial improvement of the quadratic approximation (14) to control variations in t . In order to do so we first extend the notion of controllability to uncertainty.

Definition 37 A small uncertainty $\{h_t, \mu_t\}_{t \in (0,1]}$ is controllable if $E_{\bar{q}_t}(h_t^2) \asymp \sum_{\omega \in \text{supp } \bar{q}_t} h_t^2(\omega)$.

For example, directional small uncertainties are controllable provided $h \neq 0$, $\bar{q}$ -a.e.

We can now generalize the quadratic approximation (14). The dependence on t of the priors μ_t substantially complicates the derivation of the following quadratic approximation, which we prove in Appendix B.

Theorem 38 Let $\{h_t, \mu_t\}_{t \in (0,1]}$ be a small uncertainty. If Assumption 3 holds with u and v three-time differentiable, then

$$C_t(w + h_t) = w + E_{\bar{q}_t}(h_t) - \frac{1}{2} \lambda_u(w) (\sigma_{\bar{q}_t}^2(h_t) - \sigma_{\mu_t}^2(E(h_t))) - \frac{1}{2} \lambda_v(w) \sigma_{\mu_t}^2(E(h_t)) + o(\|h_t\|_{\bar{q}_t}^2) \quad (36)$$

for all $w \in \mathbb{R}$.

Thus, the uncertainty premium takes the form

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) (\sigma_{\bar{q}_t}^2(h_t) - \sigma_{\mu_t}^2(E(h_t))) + \frac{1}{2} \lambda_v(w) \sigma_{\mu_t}^2(E(h_t)) + o(\|h_t\|_{\bar{q}_t}^2). \quad (37)$$

Approximation (23) is the special case of (37) where $\mu_t = \mu$ for all t .

The approximation (37) allows to study separately the order effects of risk and model uncertainty attitudes through the functions $\Lambda_u(h, \mu)$ and $\Lambda_v(h, \mu)$ introduced in (24). Through these functions we can write (37) as

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \Lambda_u(h_t, \mu_t) + \frac{1}{2} \lambda_v(w) \Lambda_v(h_t, \mu_t) + o(\|h_t\|_{\bar{q}_t}^2). \quad (38)$$

Notice that relative to their earlier case (25), here also the priors are indexed by t .

Lemma 22 is readily extended to the present setting, so that $\Lambda_u(h_t, \mu) = o(\|h_t\|_{\bar{q}_t})$ and $\Lambda_v(h_t, \mu_t) = o(\|h_t\|_{\bar{q}_t})$ for each small uncertainty $\{h_t, \mu_t\}_{t \in (0,1]}$. As a result, in the differential case we can have only second and higher order effects. In particular, risk attitudes have a

- (i) *second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$* if $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$;
- (ii) *(quadratically) negligible effect at $\{h_t, \mu_t\}_{t \in (0,1]}$* if $\Lambda_u(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$.

A similar classification holds for model uncertainty attitudes, with Λ_v in place of Λ_u . The next two sections will illustrate these notions. In particular, Theorem 48, our paper's main result, will show that the order effect of $\Lambda_u(h_t, \mu_t)$ and $\Lambda_v(h_t, \mu_t)$ critically depends on the limit properties of both outcomes, h_t , and beliefs, μ . The latter dependence is especially interesting: in a nutshell, when the support of μ is trivial, model uncertainty attitudes are negligible; when it is not trivial, model uncertainty persists as t goes to zero. As a matter of fact, both risk and model uncertainty attitudes have a second order effect when the support is not trivial.

Finally, we can define global effects and quadratic small uncertainties by proceeding as in Sections 3.3 and 5.3, *mutatis mutandis*. For example, quadratic small uncertainties are defined by $E_{\bar{q}_t}(h_t^2) = O(t^2)$. In this case in (37) we can replace $o\left(\|h_t\|_{\bar{q}_t}^2\right)$ with $o(t^2)$.

7 A binomial illustration

Consider a binomial model $\Omega = \{1, 2\}$. Let $\{\mu_t\}_{t \in (0,1]}$ be any net of priors that quadratically converges to some μ , and let $W_t : \Omega \rightarrow \mathbb{R}$ be given by $W_t = tW$ with

$$W(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ -1 & \text{if } \omega = 2 \end{cases}$$

In this section we consider directional small uncertainties $\{W_t, \mu_t\}_{t \in (0,1]}$.

Lemma 39 *$\{W_t, \mu_t\}_{t \in (0,1]}$ is a controllable and quadratic small uncertainty.*

Given $q \in \Delta$, to ease notation we set $q = q(1)$. In particular, each μ_t can be viewed as a Borel probability measure on $[0, 1]$. Set $\vartheta_t = \int q^2 d\mu_t(q)$ for each $t > 0$. Then,

$$0 \leq \bar{q}_t^2 \leq \vartheta_t \leq \bar{q}_t \leq 1, \quad (39)$$

and

$$\sigma_{\bar{q}_t}^2(W_t) = E_{\bar{q}_t}(W_t^2) - E_{\bar{q}_t}(W_t)^2 = 4t^2 \bar{q}_t(1 - \bar{q}_t),$$

as well as

$$\sigma_{\mu_t}^2(E(W_t)) = \int (tq + (-t)(1 - q))^2 d\mu_t(q) - \left(\int (tq + (-t)(1 - q)) d\mu_t(q) \right)^2 = 4t^2 (\vartheta_t - \bar{q}_t^2).$$

Hence, for all $w \in \mathbb{R}$ the quadratic approximation (37) of the uncertainty premium $\pi_t(w + W_t)$ takes the form

$$\pi_t(w + W_t) = 2\lambda_u(w)(\bar{q}_t - \vartheta_t)t^2 + 2\lambda_v(w)(\vartheta_t - \bar{q}_t^2)t^2 + o(t^2). \quad (40)$$

Our goal is to study the limit behavior of $\pi_t(w + W_t)$ as t goes to zero, thereby determining the impact of risk and model uncertainty attitudes on the certainty equivalent $C_t(w + W_t)$. To this end, throughout this section we suppose that Assumption 3 holds with u and v three time differentiable.

Here

$$\Lambda_u(W_t, \mu_t) = (\bar{q}_t - \vartheta_t)t^2 \quad \text{and} \quad \Lambda_v(W_t, \mu_t) = (\vartheta_t - \bar{q}_t^2)t^2$$

and so (37) can be rewritten as

$$\pi_t(w + W_t) = 2\lambda_u(w)\Lambda_u(W_t, \mu_t) + 2\lambda_v(w)\Lambda_v(W_t, \mu_t) + o(t^2). \quad (41)$$

Hence risk attitudes have a first order, second order, negligible effect at $\{W_t, \mu_t\}_{t \in (0,1]}$ if, respectively, $\Lambda_u(W_t, \mu_t) \asymp t$, $\Lambda_u(W_t, \mu_t) \asymp t^2$, and $\Lambda_u(W_t, \mu_t) = o(t^2)$. As usual, a similar classification holds for model uncertainty attitudes, with Λ_v in place of Λ_u .

By (39) it holds $0 \leq \bar{q}_t - \vartheta_t, \vartheta_t - \bar{q}_t^2 \leq 1$, and so $0 \leq \Lambda_u(W_t, \mu_t)/t^2, \Lambda_v(W_t, \mu_t)/t^2 \leq 1$. This, in turn, implies that risk and model uncertainty attitudes never have a first order effect.¹⁴ Moreover, $\lim_{t \downarrow 0} (\bar{q}_t - \vartheta_t)$ and $\lim_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2)$ exist finite since μ_t quadratically converges to some μ . As a result, the ratios $\Lambda_u(W_t, \mu_t)/t^2$ and $\Lambda_v(W_t, \mu_t)/t^2$ have a well-defined limit behavior. Summing up:

Lemma 40 *Under Assumption 3, with u and v three-time differentiable, either $\Lambda_u(W_t, \mu_t) \sim t^2$ or $\Lambda_u(W_t, \mu_t) = o(t^2)$. The same result holds for the function $\Lambda_v(W_t, \mu_t)$.*

Thus, risk and model uncertainty attitudes have either a second order or a negligible effect at $\{W_t, \mu_t\}_{t \in (0,1]}$. Depending on which effect prevails, four possible cases ought to be considered:

Limit Case 1: $\Lambda_u(W_t, \mu_t) \sim t^2$ and $\Lambda_v(W_t, \mu_t) \sim t^2$. Both risk and model uncertainty attitudes have a second order effect at $\{W_t, \mu_t\}_{t \in (0,1]}$ and the same relevance in the quadratic approximation (41).

Limit Case 2: $\Lambda_u(W_t, \mu_t) \sim t^2$ and $\Lambda_v(W_t, \mu_t) = o(t^2)$. Risk attitudes have a second order effect at $\{W_t, \mu_t\}_{t \in (0,1]}$, while model uncertainty attitudes have a negligible effect. In this case a KMM [7] decision maker is eventually indistinguishable from a subjective expected utility one.

Limit Case 3: $\Lambda_u(W_t, \mu_t) = o(t^2)$ and $\Lambda_v(W_t, \mu_t) \sim t^2$. Model uncertainty attitudes have a second order effect at $\{W_t, \mu_t\}_{t \in (0,1]}$, while risk attitudes have a negligible effect. In this case a KMM [7] decision maker is eventually risk neutral.

Limit Case 4: $\Lambda_u(W_t, \mu_t) = o(t^2)$ and $\Lambda_v(W_t, \mu_t) = o(t^2)$. Both risk and model uncertainty attitudes have a negligible effect at $\{W_t, \mu_t\}_{t \in (0,1]}$. Neither u nor v play any role in the approximation.

These four fundamental cases are asymptotic versions of the four cases discussed in Section 2.4 and can be characterized through the properties of the limit prior μ . To this end, notice that the barycenter $\bar{q}$ of μ has full support when μ is not a Dirac measure concentrated on either 0 or 1.

Lemma 41 *It holds $\text{supp } \bar{q} = \Omega$ if and only if $\mu \neq \delta_p$, with $p \in \{0, 1\}$.*

Because of this lemma, we consider the cases $\mu \neq \delta_p$ and $\mu = \delta_p$, with $p \in \{0, 1\}$, separately; that is, we consider separately the cases when μ is a Dirac measure concentrated on 0 or 1.

Proposition 42 *Suppose $\mu_t \xrightarrow{sq} \mu \neq \delta_p$, with $p \in \{0, 1\}$. Then*

$$\pi_t(w + W_t) = 2\lambda_u(w)(\bar{q} - \vartheta)t^2 + 2\lambda_v(w)(\vartheta - \bar{q}^2)t^2 + o(t^2). \quad (42)$$

Moreover,

- (i) $\bar{q}^2 < \vartheta < \bar{q}$ if and only if $\text{supp } \mu$ is a nonsingleton subset of $(0, 1)$;
- (ii) $\vartheta = \bar{q}^2$ if and only if $\mu = \delta_p$ for some $p \in (0, 1)$;
- (iii) $\bar{q} = \vartheta$ if and only if $\text{supp } \mu = \{0, 1\}$.

¹⁴Something not surprising in view of the differentiability properties of Assumption 3.

As it will be show momentarily, points (i)-(iii) characterize Limit Cases 1-3. The Limit Case 4 is accounted for by the next result, which considers the Dirac case $\mu = \delta_p$, with $p \in \{0, 1\}$.

Proposition 43 Suppose $\mu_t \xrightarrow{sq} \delta_0$. Then,

$$\pi_t(w + W_t) = o(t^2), \quad (43)$$

provided $\vartheta_t = O(\bar{q}_t^2)$.

A similar result holds when $\mu_t \xrightarrow{sq} \delta_1$.

Let us now study Limit Cases 1-4 in light of Propositions 42 and 43. Thanks to these results, the limit cases can be characterized via sharp properties of μ , thus clarifying what underlies their differences.

Limit Case 1: By Proposition 42-(i), $\Lambda_u(W_t, \mu_t) \sim t^2$ and $\Lambda_v(W_t, \mu_t) \sim t^2$ if and only if $\text{supp } \mu$ is a nonsingleton subset of $(0, 1)$. Both risk and model uncertainty attitudes have a second order effect at $\{W_t, \mu_t\}$ and they both matter in approximation (42).

Limit Case 2: By Proposition 42-(ii), $\Lambda_u(W_t, \mu_t) \sim t^2$ and $\Lambda_v(W_t, \mu_t) = o(t^2)$ if and only if $\mu = \delta_p$ for some $p \in (0, 1)$. That is, Limit Case 2 is characterized by a Dirac μ concentrated on any point p in the interval $[0, 1]$ except the endpoints 0 and 1.

Risk attitudes have a second order effect, while model uncertainty attitudes have a negligible effect. In fact, since μ is a Dirac measure, model uncertainty eventually vanishes “quadratically.” It is therefore natural to expect that attitudes toward a “vanishing phenomenon” would vanish as well. Approximation (42) becomes

$$\pi_t(w + W_t) = 2\lambda_u(w)p(1-p)t^2 + o(t^2). \quad (44)$$

Limit Case 3: By Proposition 42-(iii), $\Lambda_u(W_t, \mu_t) = o(t^2)$ and $\Lambda_v(W_t, \mu_t) \sim t^2$ if and only if $\text{supp } \mu = \{0, 1\}$. Hence, μ attaches positive probability only to both endpoints 0 and 1. Model uncertainty attitudes have a second order effect, while risk attitudes have a negligible effect. Approximation (42) becomes

$$\pi_t(w + W_t) = 2\lambda_v(w)(\vartheta - \bar{q}^2)t^2 + o(t^2). \quad (45)$$

Limit Case 4: By Proposition 43, $\Lambda_u(W_t, \mu_t) = o(t^2)$ and $\Lambda_v(W_t, \mu_t) = o(t^2)$ if and only if $\mu = \delta_p$ for some $p \in \{0, 1\}$, provided $\vartheta_t = O(\bar{q}_t^2)$. Limit Case 4 is characterized by a Dirac μ concentrated on one of the endpoints 0 and 1. Approximation (42) becomes (43) and both risk and model uncertainty attitudes have a negligible effect.

7.1 An example

We now illustrate the four limit cases with an example. Let $0 \leq a_t < b_t \leq 1$, for each $t > 0$. Then, for each Borel subset $\mathcal{B}$ of $[0, 1]$, set

$$\mu_t(\mathcal{B}) = \lambda_{[a_t, b_t]}(\mathcal{B}) = \frac{\lambda(\mathcal{B} \cap [a_t, b_t])}{b_t - a_t}, \quad (46)$$

where λ is the Lebesgue measure on $[0, 1]$. The support of μ_t is $\text{supp } \mu_t = [a_t, b_t]$. Suppose that $a_t \leq a_{t'}$ and $b_{t'} \leq b_t$ for all $t' \leq t$,¹⁵ so that $\text{supp } \mu_{t'} \subseteq \text{supp } \mu_t$ for all $t' \leq t$. Additionally, set $a = \sup_{t>0} a_t$ and $b = \inf_{t>0} b_t$. The next result illustrates how the uniform prior μ_t becomes “sharper” as t goes to zero.

¹⁵That is, $t \mapsto a_t$ is decreasing and $t \mapsto b_t$ is increasing.

Lemma 44 Define μ_t by (46), with $t \mapsto a_t$ decreasing and $t \mapsto b_t$ increasing. Then,

$$\mu_t \xrightarrow{w} \mu = \begin{cases} \lambda_{[a,b]} & \text{if } a < b, \\ \delta_a & \text{if } a = b. \end{cases}$$

In particular,

$$\begin{aligned} \bar{q}_t &= \frac{b_t + a_t}{2} \rightarrow \frac{b + a}{2}, \\ \vartheta_t &= \frac{b_t^3 - a_t^3}{3(b_t - a_t)} = \frac{b_t^2 + a_t b_t + a_t^2}{3} \rightarrow \frac{b^2 + ab + a^2}{3}, \\ \bar{q}_t - \vartheta_t &\rightarrow \frac{3b + 3a - 2b^2 - 2ab - 2a^2}{6}, \\ \vartheta_t - \bar{q}_t^2 &\rightarrow \frac{(b - a)^2}{12}. \end{aligned}$$

By Lemma 44, the cases $a < b$ and $a = b$ ought to be treated separately.

- (i) If $a < b$, then the priors μ_t do not eventually concentrate on a single p and¹⁶

$$\lim_{t \downarrow 0} \frac{\Lambda_u(W_t, \mu_t)}{t^2} = \lim_{t \downarrow 0} (\bar{q}_t - \vartheta_t) > 0 \quad ; \quad \lim_{t \downarrow 0} \frac{\Lambda_v(W_t, \mu_t)}{t^2} = \lim_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) > 0.$$

This is Limit Case 1, where both risk and model uncertainty attitudes have a second order effect.

- (ii) If $a = b$, then the priors μ_t concentrate on a single p , δ_a . Two possibilities arise:

- (1) $a = b \in \{0, 1\}$. In this instance, uncertainty on the model and on the state eventually disappear because the decision maker knows the true state (which is $\omega = 1$ if $a = 1$ and $\omega = 0$ if $a = 0$). Then, $\lim_{t \downarrow 0} (\bar{q}_t - \vartheta_t) = \lim_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) = 0$, and so

$$\lim_{t \downarrow 0} \frac{\Lambda_u(W_t, \mu_t)}{t^2} = \lim_{t \downarrow 0} \frac{\Lambda_v(W_t, \mu_t)}{t^2} = 0$$

and Limit Case 4 occurs. Both risk and uncertainty attitudes have a negligible effect.

- (2) $a = b \in (0, 1)$. Here, while state uncertainty persists (with state 1 assigned probability a), model uncertainty eventually disappears. Then, $\lim_{t \downarrow 0} (\bar{q}_t - \vartheta_t) > \lim_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) = 0$, and so

$$\lim_{t \downarrow 0} \frac{\Lambda_u(W_t, \mu_t)}{t^2} > \lim_{t \downarrow 0} \frac{\Lambda_v(W_t, \mu_t)}{t^2} = 0.$$

This is Limit Case 2, where only risk attitudes have a second order effect.

To conclude, if $a < b$, both attitudes have a second order effect; if $a = b \in \{0, 1\}$, both attitudes have a negligible effect; if $a = b \in (0, 1)$, only risk attitude has a second order effect. Thus, except for the last special case, attitudes toward risk and model uncertainty have effects of the same order. Notice that in this example Limit Case 3 never arises.

¹⁶ Observe that, for $0 \leq a < b \leq 1$, the equation $3b + 3a - 2b^2 - 2ab - 2a^2 = 0$ has no solutions.

8 General binomial analysis

In this section we extend our analysis to a general $h_t : \Omega = \{1, 2\} \rightarrow \mathbb{R}$ such that, for each $t > 0$,

$$h_t(\omega) = \begin{cases} \varphi(t) & \text{if } \omega = 1 \\ \psi(t) & \text{if } \omega = 2 \end{cases},$$

where φ and ψ are nonzero real-valued functions defined on $(0, 1]$. If $\varphi(t) = -\psi(t) = t$, then $h_t = W_t$.

Throughout this section we consider an uncertain monetary net $\{h_t, \mu_t\}_{t \in (0, 1]}$ such that $\bar{q}_t \in (0, 1)$ for all $t \in (0, 1]$. Next we show that such net is a controllable small uncertainty when $\mu \neq \delta_p$ with $p \in \{0, 1\}$.

Lemma 45 *$\{h_t, \mu_t\}_{t \in (0, 1]}$ is a controllable small uncertainty provided $\mu \neq \delta_p$ with $p \in \{0, 1\}$.*

When either $\mu = \delta_0$ or $\mu = \delta_1$, under a mild regularity condition controllable small uncertainties feature $\varphi^2(t)$ and $\psi^2(t)$ that go to zero with the same order.

Lemma 46 *Suppose $\mu = \delta_0$. An uncertain monetary net $\{h_t, \mu_t\}_{t \in (0, 1]}$ is a controllable small uncertainty if $\varphi^2(t) \asymp \psi^2(t)$ and $\lim_{t \downarrow 0} \psi^2(t) = 0$. The converse holds provided $\lim_{t \downarrow 0} \varphi(t)/\psi(t) \in [-\infty, +\infty]$.*

A similar result holds when $\mu = \delta_1$.¹⁷

It is convenient to define $\gamma : (0, 1] \rightarrow \mathbb{R}$ by

$$\gamma(t) = \varphi(t) - \psi(t). \quad (47)$$

For example, $\gamma(t) = 2t$ when $h_t = W_t$. Some algebra shows that

$$\sigma_{\bar{q}_t}^2(h_t) = E_{\bar{q}_t}(h_t^2) - E_{\bar{q}_t}(h_t)^2 = \gamma^2(t) \bar{q}_t(1 - \bar{q}_t),$$

and

$$\sigma_{\mu_t}^2(E(h_t)) = \int (tq + (-t)(1 - q))^2 d\mu_t(q) - \left(\int (tq + (-t)(1 - q)) d\mu_t(q) \right)^2 = \gamma^2(t) (\vartheta_t - \bar{q}_t^2).$$

If $\{h_t, \mu_t\}_{t \in (0, 1]}$ is a controllable small uncertainty, (37) implies:

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) (\bar{q}_t - \vartheta_t) \gamma^2(t) + \frac{1}{2} \lambda_v(w) (\vartheta_t - \bar{q}_t^2) \gamma^2(t) + o\left(\|h_t\|_{\bar{q}_t}^2\right) \quad (48)$$

for all $w \in \mathbb{R}$. When $\{h_t, \mu_t\}_{t \in (0, 1]}$ is a quadratic small uncertainty, approximation (48) becomes

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) (\bar{q}_t - \vartheta_t) \gamma^2(t) + \frac{1}{2} \lambda_v(w) (\vartheta_t - \bar{q}_t^2) \gamma^2(t) + o(t^2),$$

which reduces to (40) when $h_t = W_t$.

Here,

$$\Lambda_u(h_t, \mu_t) = (\bar{q}_t - \vartheta_t) \gamma^2(t) \quad \text{and} \quad \Lambda_v(h_t, \mu_t) = (\vartheta_t - \bar{q}_t^2) \gamma^2(t).$$

¹⁷When $\lim_{t \downarrow 0} \varphi(t)/\psi(t) \in [-\infty, +\infty]$, the condition $\varphi^2(t) \asymp \psi^2(t)$ amounts to require $\varphi^2(t) \sim k\psi^2(t)$ for some $0 \neq k \in \mathbb{R}$.

The interested reader can find in Lemmas 58, 59 and 60 of the Appendix separate characterizations for small, quadratic and controllable uncertainties.

The limit behavior of Λ_u and Λ_v , as t goes to zero, determines the limit relevance of risk and model uncertainty attitudes. In this regard notice that Λ_u and Λ_v depend on h_t via $\gamma(t)$ and on μ_t via $\vartheta_t - \bar{q}_t^2$ and $\bar{q}_t - \vartheta_t$, respectively.

We now study the order effects of risk and model uncertainty attitudes by extending Limit Cases 1-4 to the present general binomial setting. To this end, we generalize Propositions 42 and 43. We begin with Proposition 42, that is, we consider the general case $\mu \neq \delta_p$ with $p \in \{0, 1\}$.

Proposition 47 *Let $\{h_t, \mu_t\}_{t \in (0,1]}$ be a controllable small uncertainty, where $\mu_t \xrightarrow{sq} \mu \neq \delta_p$ with $p \in \{0, 1\}$. Then*

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) (\bar{q} - \vartheta) \gamma^2(t) + \frac{1}{2} \lambda_v(w) (\vartheta - \bar{q}^2) \gamma^2(t) + o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (49)$$

with $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$. Moreover,

- (i) $\bar{q}^2 < \vartheta < \bar{q}$ if and only if $\text{supp } \mu$ is a nonsingleton subset of $(0, 1)$;
- (ii) $\vartheta = \bar{q}^2$ if and only if $\mu = \delta_p$ for some $p \in (0, 1)$;
- (iii) $\bar{q} = \vartheta$ if and only if $\text{supp } \mu = \{0, 1\}$.

First order effects never arise at a small uncertainty $\{h_t, \mu_t\}_{t \in (0,1]}$ when $\mu \neq \delta_p$ with $p \in \{0, 1\}$. Indeed, along with $0 \leq \bar{q}_t - \vartheta_t \leq 1$, condition $\gamma(t) = O\left(\|h_t\|_{\bar{q}_t}\right)$ easily implies $\Lambda_u(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}\right)$. A similar argument shows that $\Lambda_v(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}\right)$.

Thus, only second order and negligible effects are relevant. We study them through the analysis of Limit Cases 1-4. In the special case $h_t = W_t$ these cases readily followed from Proposition 42. However, in this more general setting, their derivation from Proposition 47 is less immediate and requires a formal result. The next theorem, which uses Proposition 47 to fully characterize Limit Cases 1-4 in the present setup, is the paper's main result.

Theorem 48 *Let $\{h_t, \mu_t\}_{t \in (0,1]}$ be a controllable small uncertainty, where $\mu_t \xrightarrow{sq} \mu \neq \delta_p$ with $p \in \{0, 1\}$. Then:*

- (i) *Limit Case 1:* $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ and $\Lambda_v(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ if and only if $\text{supp } \mu$ is a nonsingleton subset of $(0, 1)$ and $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$;
- (ii) *Limit Case 2:* $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ and $\Lambda_v(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$ if and only if $\mu = \delta_p$ for some $p \in (0, 1)$ and $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.
- (iii) *Limit Case 3:* $\Lambda_u(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$ and $\Lambda_v(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ if and only if $\text{supp } \mu = \{0, 1\}$ and $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.
- (iv) *Limit Case 4:* $\Lambda_u(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$ and $\Lambda_v(h_t, \mu_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$ if and only if $\gamma(t) = o\left(\|h_t\|_{\bar{q}_t}\right)$.

In particular, Limit Cases 1-3 also hold with $\sim$ in place of $\asymp$.

In Limit Case 1 both risk and model uncertainty attitudes have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$. This case is characterized by a μ with nonsingleton support. Specifically, when $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$ there exist constants $c, k > 0$ such that approximation (49) becomes

$$\pi_t(w + h_t) = (c\lambda_u(w) + k\lambda_v(w)) \|h_t\|_{\bar{q}_t}^2 + o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (50)$$

with both risk and model uncertainty attitudes proportional to $\|h_t\|_{\bar{q}_t}^2$.

In Limit Case 2 approximation (49) becomes

$$\pi_t(w + W_t) = \frac{1}{2}\lambda_u(w) \bar{q}(1 - \bar{q}) \gamma^2(t) + o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (51)$$

where risk attitudes have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$, while model uncertainty attitudes have a negligible effect. Here, when $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$, there exists a constant $c > 0$ such that (51) becomes

$$\pi_t(w + h_t) = c\lambda_u(w) \|h_t\|_{\bar{q}_t}^2 + o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (52)$$

with risk attitudes proportional to $\|h_t\|_{\bar{q}_t}^2$.

In Limit Case 3 approximation (49) becomes

$$\pi_t(w + W_t) = \frac{1}{2}\lambda_v(w) \bar{q}(1 - \bar{q}) \gamma^2(t) + o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (53)$$

where model uncertainty attitudes have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$, while risk attitudes have a negligible effect. When $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$ there exists a constant $k > 0$ such that (53) becomes

$$\pi_t(w + h_t) = k\lambda_v(w) \|h_t\|_{\bar{q}_t}^2 + o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (54)$$

with model uncertainty attitudes proportional to $\|h_t\|_{\bar{q}_t}^2$.

Finally, in Limit Case 4 approximation (49) becomes

$$\pi_t(w + W_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right). \quad (55)$$

Here, neither u nor v play any role in the approximation since both model uncertainty and risk attitudes have a negligible effect at $\{h_t, \mu_t\}_{t \in (0,1]}$.

Example 49 (i) If $\varphi \sim k\psi$ with $k \neq 1$, then $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$. In this case the properties of μ determine which one, among Limit Cases 1-3, arises. (ii) If $\varphi \sim \psi$, then $\gamma(t) = o\left(\|h_t\|_{\bar{q}_t}\right)$ and so Limit Case 4 arises. $\blacktriangle$

Example 50 We illustrate Limit Cases 1-3 by considering the net $\{W_t, \mu_t\}_{t \in (0,1]}$. Suppose that $\mu_t = \lambda_{[(1/3)+t, (2/3)-t]}$ for each $t \in (0, 1/3]$. Then, $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$ and $\mu_t \xrightarrow{w} \mu = \lambda_{[1/3, 2/3]}$, with $\text{supp } \mu = [1/3, 2/3]$. Hence, (50) takes the form

$$\pi_t(w + h_t) = \left(\frac{13}{108}\lambda_u(w) + \frac{1}{216}\lambda_v(w)\right) t^2 + o(t^2).$$

If, instead, $\mu_t = \lambda_{[2^{-1}-t, 2^{-1}+t]}$ for each $t \in (0, 1/2]$, then $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$ and $\mu_t \xrightarrow{w} \delta_{1/2}$ – the Dirac measure concentrated on $1/2$. In particular, (52) becomes

$$\pi_t(w + W_t) = \frac{1}{2}\lambda_u(w) t^2 + o(t^2).$$

Finally, if $\mu_t = (1/3)\lambda_{[0,t]} + (2/3)\lambda_{[1-t,1]}$, then $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$ and $\mu_t \xrightarrow{w} \mu = (1/3)\delta_0 + (2/3)\delta_1$. Here, (54) becomes

$$\pi_t(w + W_t) = \frac{4}{9}\lambda_v(w)t^2 + o(t^2).$$

▲

Example 51 To illustrate Limit Case 4, consider $\{h_t, \mu_t\}_{t \in (0,1]}$ such that $\mu_t \xrightarrow{sq} \mu \neq \delta_p$, with $p \in \{0, 1\}$, and

$$h_t(\omega) = \begin{cases} t + t^2 & \text{if } \omega = 1 \\ t & \text{if } \omega = 2 \end{cases}.$$

This net, already seen in Example 15, is a small uncertainty with $\gamma(t) = o(\|h_t\|_{\bar{q}_t})$. Here (55) becomes $\pi_t(w + h_t) = o(t^2)$. ▲

We close by considering the Dirac case $\mu = \delta_p$, with $p \in \{0, 1\}$, that is, we generalize Proposition 43. We show that in this case both risk and model uncertainty have a negligible effect, that is, only Limit Case 4 is relevant. This result is not surprising given that a Dirac μ on $\{0, 1\}$ means that both risk and model uncertainty eventually vanish.

Proposition 52 *Let $\{h_t, \mu_t\}_{t \in (0,1]}$ be a controllable small uncertainty, with either $\mu_t \xrightarrow{sq} \delta_0$ or $\mu_t \xrightarrow{sq} \delta_1$. Then*

$$\pi_t(w + h_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (56)$$

provided $\vartheta_t = O(\bar{q}_t^2)$.

Example 53 Suppose $h_t = W_t$ and $\mu_t = \lambda_{[0,t]}$ for each $t > 0$. We have $\|h_t\|_{\bar{q}_t} = t$, $\gamma(t) = 2t$, $\bar{q}_t = t/2$, and $\vartheta_t = t^2/3$. By Proposition 52, $\pi_t(w + h_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$. ▲

9 A special case

Our analysis so far shows that both risk and model uncertainty attitudes are relevant to describe the limit behavior of $C_t(w + W_t)$ and $\pi_t(w + W_t)$ as $t \downarrow 0$, unless μ_t quadratically converges to a Dirac probability measure δ_p . That is, unless decision makers' priors concentrate on a single model p , so that model uncertainty eventually vanishes in terms of quadratic convergence. This is a natural feature of the KMM [7] model that Theorem 48 and Lemma 61 establish in full generality. Intuitively, if the support of the probability μ_t becomes more concentrated, then model uncertainty eventually vanishes. The next result shows that this is indeed the case.

Proposition 54 *Suppose that the collection of convex hulls $\{\text{co}(\text{supp } \mu_t)\}_{t>0}$ has the finite intersection property.¹⁸ If $\text{diam}(\text{supp } \mu_t) \rightarrow 0$ as $t \downarrow 0$, then $\bigcap_t \text{co}(\text{supp } \mu_t)$ contains one point $p \in [0, 1]$ and $\mu_t \xrightarrow{w} \delta_p$.*

By Proposition 47, if $p \in (0, 1)$ and h_t is such that $\lim_{t \downarrow 0} \varphi(t) = \lim_{t \downarrow 0} \psi(t) = 0$, then

$$\pi_t(w + h_t) = \frac{1}{2}\lambda_u(w)p(1-p)\gamma^2(t) + o\left(\|h_t\|_{\bar{q}_t}^2\right). \quad (57)$$

By Lemma 61, if $p = 0$ and h_t is such that $\varphi^2(t) \asymp \psi^2(t)$ and $\lim_{t \downarrow 0} \psi(t) = 0$, then

$$\pi_t(w + h_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right), \quad (58)$$

¹⁸Given any finite index set T , it holds $\bigcap_{t \in T} \text{co}(\text{supp } \mu_t) \neq \emptyset$. For example, $\text{co}(\text{supp } \mu_t) \subseteq \text{co}(\text{supp } \mu_{t'})$ when $t < t'$.

provided $\vartheta_t = O(\bar{q}_t^2)$. A similar result holds when $p = 1$.

In approximation (57) only attitudes toward risk may have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$. That is, only Limit Cases 2 and 4 apply. As we emphasized, this is not surprising given that model uncertainty weakly vanishes because $\mu_t \xrightarrow{w} \delta_p$. When $p = 0$ (or $p = 1$) risk vanishes too; as (58) shows, in this case both risk and model uncertainty attitudes are negligible.

We now provide some examples to show how the properties of the KMM [7] model established in Theorems 4 and 5 of Skiadas [13] represent a special case of (57) and (58). As a consequence, Skiadas' findings that attitudes toward model uncertainty become irrelevant as t goes to 0 must be attributed to the special case that he considers.

Example 55 Given a scalar $\rho \in \mathbb{R}$, let

$$q_t = \frac{1}{2}(1 + \rho t). \quad (59)$$

Then, $1 - q_t = 2^{-1}(1 - \rho t)$ and

$$q_t \in [0, 1] \Leftrightarrow \rho \in \left[-\frac{1}{t}, \frac{1}{t}\right]. \quad (60)$$

Consider a finite collection of scalars $\rho_1 < \dots < \rho_S$. By (59), for each small enough t we have a finite collection of probabilities $\{q_t^s\}_{s=1}^S$. Let μ_t be a discrete probability measure on $[0, 1]$ with $\text{supp } \mu_t = \{q_t^s\}_{s=1}^S$.¹⁹ Since

$$\text{diam}(\text{supp } \mu_t) = \frac{1}{2}(\rho_S - \rho_1)t,$$

we have $\text{diam}(\text{supp } \mu_t) \rightarrow 0$ as $t \downarrow 0$. Moreover, $\text{co}(\text{supp } \mu_{t'}) \subseteq \text{co}(\text{supp } \mu)$ for each $t' < t$. By Proposition 54, $\mu_t \xrightarrow{w} \delta_{1/2}$ so that model uncertainty vanishes and approximation (57) applies. For instance, consider the uncertain monetary net $\{h_t, \mu_t\}_{t \in (0,1]}$, where $h_t = W_t + O(t^2)$ as in [13, Theorem 4]. It is straightforward to verify that this is a quadratic small uncertainty. Hence, (57) takes the form

$$\pi_t(w + h_t) = 2\lambda_u(w)p(1-p)t^2 + o(t^2)$$

because $\gamma^2(t) = 4t^2 + O(t^3)$ and $\|h_t\|_{\bar{q}_t}^2 = t^2 + O(t^3)$. Risk attitudes have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$. $\blacktriangle$

Example 56 Given a scalar $\rho \in \mathbb{R}$ and a function $o(t^2)$, let

$$q_t = (1 + \rho)t^2 + o(t^2). \quad (61)$$

Then, $1 - q_t = 1 - (1 + \rho)t^2 + o(t^2)$ and

$$q_t \in [0, 1] \Leftrightarrow -t^2 \leq \rho t^2 + o(t^2) \leq 1 - t^2. \quad (62)$$

Additionally, this implies $\rho \geq -1$. Consider a finite collection of scalars $-1 < \rho_1 < \dots < \rho_S$ and let $o(t^2)$ be such that (62) holds for all $s = 1, \dots, S$ and all $t > 0$ small enough. Then, through (61), we define a finite collection of probabilities $\{q_t^s\}_{s=1}^S$, for each such t . Let μ_t be a discrete probability measure on $[0, 1]$ with $\text{supp } \mu_t = \{q_t^s\}_{s=1}^S$. Then, $\text{diam}(\text{supp } \mu_t) = (\rho_S - \rho_1)t^2 + o(t^2)$, so that $\text{diam}(\text{supp } \mu_t) \rightarrow 0$ as $t \downarrow 0$. Moreover, $\text{co}(\text{supp } \mu_{t'}) \subseteq \text{co}(\text{supp } \mu)$ for each $t' < t$. By Proposition 54, $\mu_t \xrightarrow{w} \delta_0$ and model uncertainty vanishes.

¹⁹Skiadas uses Q , $\sqrt{h}$, and π instead of q , t , and μ , respectively. This same observation also applies to Example 56.

For ease, set $\mu_t^s = \mu_t(q_s^t)$. Then,

$$\bar{q}_t = \sum_{s=1}^S ((1 + \rho_s) t^2 + o(t^2)) \mu_t^s = t^2 \left(1 + \sum_{s=1}^S \rho_s \mu_t^s \right) + o(t^2) = a(t) t^2 + o(t^2) \quad (63)$$

and

$$\vartheta_t = \sum_{s=1}^S ((1 + \rho_s) t^2 + o(t^2))^2 \mu_t^s = t^4 \sum_{s=1}^S (1 + \rho_s)^2 \mu_t^s + o(t^4) = b(t) t^4 + o(t^4),$$

where $a(t) \in [1 + \rho_1, 1 + \rho_S]$ and $b(t) \in [(1 + \rho_1)^2, (1 + \rho_S)^2]$ for all $t > 0$. Hence, $\vartheta_t = O(\bar{q}_t^2)$ and approximation (58) applies for controllable small uncertainties.

That said, as in [13, Theorem 5] consider the uncertain monetary net $\{h_t, \mu_t\}_{t \in (0,1]}$, where $h_t = h + O(t^2)$ and

$$h(\omega) = \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{if } \omega = 2 \end{cases},$$

This net is a quadratic small uncertainty because, by (63), $\varphi^2(t) \bar{q}_t = \bar{q}_t + \bar{q}_t O(t^2) = O(t^2)$. It is not, however, controllable since $h_t(1)$ does not converge to zero. Hence, we do not know whether it admits the quadratic approximation (37). If we assume – for the sake of the argument – that this is the case, then by (81) in the Appendix this approximation would take the form

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \bar{q}_t + o(t^2)$$

because $\bar{q}_t \gamma^2(t) = \bar{q}_t (1 + O(t^2)) = \bar{q}_t + \bar{q}_t O(t^2) = \bar{q}_t + o(t^2)$ and $\|h_t\|_{\bar{q}_t}^2 = \bar{q}_t + o(t^2)$. Risk attitudes have a second order effect at $\{h_t, \mu_t\}_{t \in (0,1]}$. $\blacktriangle$

Lastly, it may be that $\mu_t \xrightarrow{w} \delta_p$ for some $p \in [0, 1]$ even if $\text{diam}(\text{supp } \mu_t) = 1$ for all $t > 0$. For example, for $t \in (0, 1/3]$, define $\xi_t : [0, 1] \rightarrow \mathbb{R}$ by

$$\xi_t(q) = \begin{cases} t & \text{if } q \in [0, \frac{1}{2} - t) \cup (\frac{1}{2} + t, 1] \\ \frac{1-t+2t^2}{2t} & \text{if } q \in [\frac{1}{2} - t, \frac{1}{2} + t] \end{cases}$$

and set $\mu_t(B) = \int_B \xi_t d\lambda$ for each Borel subset B of $[0, 1]$. Then, $\text{supp } \mu_t = [0, 1]$ and $\mu_t \xrightarrow{w} \delta_{1/2}$. Hence, the conditions on the supports used in Proposition 54 are stronger than the convergence of μ_t to a Dirac probability measure.

A Proofs and Related Analysis

Proof of Lemma 1. Suppose $\varphi \in O(\psi)$, then there are $\eta, M > 0$ such that

$$|\varphi(t)| \leq M |\psi(t)| \quad \forall t < \eta.$$

Let $\xi \in o(\varphi)$. For all $\varepsilon > 0$ there exists $0 < \delta < \eta$ such that

$$|\xi(t)| \leq \frac{\varepsilon}{M} |\varphi(t)| \quad \forall t < \delta.$$

Hence

$$|\xi(t)| \leq \frac{\varepsilon}{M} |\varphi(t)| \leq \varepsilon |\psi(t)| \quad \forall t < \delta$$

and $\xi \in o(\psi)$.

Conversely, suppose $\varphi \notin O(\psi)$. Then for all $\eta, M > 0$ there is $t_{\eta, M} < \eta$ such that

$$|\varphi(t_{\eta, M})| > M |\psi(t_{\eta, M})|.$$

Hence, for all $n \in \mathbb{N}$ (choosing $\eta = 1/n$ and $M = n$) there exists $t_n < 1/n$ such that

$$|\varphi(t_n)| > n |\psi(t_n)|.$$

Without loss of generality, assume that $\{t_n\}$ is strictly decreasing. Set $\xi(t_n) = \psi(t_n)$ for all $n \in \mathbb{N}$ such that $\psi(t_n) \neq 0$, set $\xi(t_n) = \frac{1}{n}\varphi(t_n)$ for all $n \in \mathbb{N}$ such that $\psi(t_n) = 0$, and set $\xi(t) = 0$ otherwise. Let $\varepsilon > 0$ and choose N so that $1/N < \varepsilon$. Set $\delta = t_N$. Now for $t < \delta$ consider the two following cases:

- $t = t_n$ for some $n \in \mathbb{N}$, then $t < t_N$ implies $n > N$, and hence $|\xi(t)| \leq \frac{1}{n} |\varphi(t_n)| \leq \frac{1}{N} |\varphi(t_n)| \leq \varepsilon |\varphi(t)|$;
- $t \neq t_n$ for all $n \in \mathbb{N}$, then $|\xi(t)| = 0 \leq \varepsilon |\varphi(t)|$.

In conclusion, $\xi \in o(\varphi)$. But, letting $M = 1/2$,

$$|\xi(t_n)| > M |\psi(t_n)|$$

for all $n \in \mathbb{N}$. Since $t_n \downarrow 0$ as $n \rightarrow \infty$, there cannot exist $\delta > 0$ such that

$$|\xi(t)| \leq M |\psi(t)| \quad \forall t < \delta.$$

Thus $\xi \in o(\varphi)$ and $\xi \notin o(\psi)$. ■

Proof of Lemma 3 We prove the “if” part, the converse being trivial. Given $\varphi \in C([0, 1])$, let $\varepsilon > 0$. By the Weierstrass Approximation Theorem, there is some polynomial φ_ε on $[0, 1]$ with $\|\varphi - \varphi_\varepsilon\|_\infty \leq \varepsilon$. By hypothesis, there is α_ε such that $\left| \int_{[0, 1]} \varphi_\varepsilon(q) d\mu_\alpha(q) - \int_{[0, 1]} \varphi_\varepsilon(q) d\mu(q) \right| < \varepsilon$ for all $\alpha \geq \alpha_\varepsilon$. Hence, for all $\alpha \geq \alpha_\varepsilon$,

$$\left| \int_{[0, 1]} \varphi(q) d\mu_\alpha(q) - \int_{[0, 1]} \varphi(q) d\mu(q) \right| \leq \|\varphi - \varphi_\varepsilon\|_\infty + \left| \int_{[0, 1]} \varphi_\varepsilon(q) d\mu_\alpha(q) - \int_{[0, 1]} \varphi_\varepsilon(q) d\mu(q) \right| + \|\varphi_\varepsilon - \varphi\|_\infty \leq 3\varepsilon$$

This implies $\lim_{\alpha} \int_{[0, 1]} \varphi(q) d\mu_\alpha(q) = \int_{[0, 1]} \varphi(q) d\mu(q)$. ■

Proof of Proposition 11. We refer to the proof of the more general Proposition 57 for a proof that (ii) implies (iv). ■

Proof of Proposition 12 Consider a small risk $\{h_t, q\}_{t \in (0, 1]}$. Let $\pi(w + h_t) \sim k \|h_t\|_q$ for some $k > 0$. Then

$$w + E_q(h_t) - C(w + h_t) \sim k \|h_t\|_q$$

and so for every $\varepsilon > 0$ there is t_ε such that

$$(1 - \varepsilon) k \|h_t\|_q \leq w + E_q(h_t) - C(w + h_t) \leq (1 + \varepsilon) k \|h_t\|_q \quad \forall t \in (0, t_\varepsilon]$$

that is

$$E_q(h_t) - (1 + \varepsilon) k \|h_t\|_q \leq C(w + h_t) - w \leq E_q(h_t) - (1 - \varepsilon) k \|h_t\|_q \quad \forall t \in (0, t_\varepsilon]$$

(i) Suppose $k > \sqrt{k^*} = \sqrt{\limsup_{t \downarrow 0} E_q^2(h_t) / \|h_t\|_q^2}$. Take ε small enough so that $k(1 - \varepsilon) > \sqrt{k^*}$.

Since $k^* = \lim_{s \downarrow 0} \sup_{t \in (0, s)} \frac{E_q^2(h_t)}{\|h_t\|_q^2}$, there is $t'_\varepsilon \in (0, t_\varepsilon]$ such that

$$\frac{E_q(h_t)}{\|h_t\|_q} < k(1 - \varepsilon) \quad \forall t \in (0, t'_\varepsilon]$$

and hence

$$C(w + h_t) - w \leq E_q(h_t) - (1 - \varepsilon)k \|h_t\|_q < 0 \quad \forall t \in (0, t'_\varepsilon].$$

(ii) Suppose $k < \sqrt{k_*} = \sqrt{\liminf_{t \downarrow 0} E_q^2(h_t) / \|h_t\|_q^2}$. Take ε small enough so that $k(1 + \varepsilon) < \sqrt{k_*}$.

Since $k^* = \lim_{s \downarrow 0} \inf_{t \in (0, s)} \frac{E_q^2(h_t)}{\|h_t\|_q^2}$, there is $t''_\varepsilon \in (0, t_\varepsilon]$ such that

$$\frac{E_q(h_t)}{\|h_t\|_q} > k(1 + \varepsilon) \quad \forall t \in (0, t''_\varepsilon]$$

and hence

$$C(w + h_t) - w \geq E_q(h_t) - (1 + \varepsilon)k \|h_t\|_q > 0 \quad \forall t \in (0, t''_\varepsilon].$$

(iii) Finally, $\pi(w + h_t) = o(\|h_t\|_q)$ and $E_q(h_t) > 0$ imply $\pi(w + h_t) = o(E_q(h_t))$. Therefore

$$C(w + h_t) - w = E_q(h_t) - \pi(w + h_t) = E_q(h_t)(1 + o(1)).$$

The proof is concluded dividing both sides by $E_q(h_t) > 0$. ■

Proof of Lemma 13 Since $\sigma_q^2(h_t) \leq \|h_t\|_q^2$ for all t , it holds $\limsup_{t \downarrow 0} \sigma_q^2(h_t) / \|h_t\|_q^2 \leq 1$. Moreover,

$$\begin{aligned} \liminf_{t \downarrow 0} \frac{\sigma_q^2(h_t)}{\|h_t\|_q^2} = 0 &\Leftrightarrow \liminf_{t \downarrow 0} \left(1 - \frac{(E_q(h_t))^2}{\|h_t\|_q^2}\right) = 0 \Leftrightarrow 1 + \liminf_{t \downarrow 0} \left(-\frac{(E_q(h_t))^2}{\|h_t\|_q^2}\right) \\ &\Leftrightarrow 1 - \limsup_{t \downarrow 0} \frac{(E_q(h_t))^2}{\|h_t\|_q^2} = 0 \Leftrightarrow \limsup_{t \downarrow 0} \frac{(E_q(h_t))^2}{\|h_t\|_q^2} = 1 \end{aligned}$$

Hence, $\sigma_q^2(h_t) \asymp \|h_t\|_q^2$ if and only if $k^* < 1$. On the other hand,

$$\limsup_{t \downarrow 0} \frac{\sigma_q^2(h_t)}{\|h_t\|_q^2} = 0 \Leftrightarrow \liminf_{t \downarrow 0} \frac{(E_q(h_t))^2}{\|h_t\|_q^2} = 1$$

and so $\sigma_q^2(h_t) = o(\|h_t\|_q^2)$ if and only if $k_* = 1$. ■

Proof of Proposition 14 We prove the “only if” part, the converse being easy. To ease notation, wlog set $\text{supp } q = \Omega$. Suppose that risk attitudes are negligible at $\{h_t, q\}_{t \in (0, 1]}$, that is, (21) holds. Suppose $\lim_{t \downarrow 0} h_t(\omega) / h_t(\omega') \in [-\infty, \infty]$ for all $\omega, \omega' \in \text{supp } q$. Assume that, for some $\omega, \omega' \in \text{supp } q$, it holds $h_t(\omega) = o(h_t(\omega'))$. Set $\tilde{q} = 1 - q(\omega')$. Then, $\tilde{q} \in (0, 1)$ and so

$$\frac{E_q^2(h_t)}{\|h_t\|_q^2} = \tilde{q} \frac{\left(\sum_{\omega \neq \omega'} h_t(\omega) \frac{q(\omega)}{\tilde{q}}\right)^2}{\sum_{\omega \neq \omega'} h_t^2(\omega) \frac{q(\omega)}{\tilde{q}}}$$

By (21),

$$\frac{\left(\sum_{\omega \neq \omega'} h_t(\omega) \frac{q(\omega)}{\tilde{q}}\right)^2}{\sum_{\omega \neq \omega'} h_t^2(\omega) \frac{q(\omega)}{\tilde{q}}} \rightarrow \frac{1}{\tilde{q}} > 1$$

which contradicts $\left(\sum_{\omega \neq \omega'} h_t(\omega) (q(\omega) / \tilde{q})\right)^2 \leq \sum_{\omega \neq \omega'} h_t^2(\omega) (q(\omega) / \tilde{q})$. We conclude that $\lim_{t \downarrow 0} |h_t(\omega) / h_t(\omega')| \in (0, \infty)$ for all $\omega, \omega' \in \text{supp } q$. Hence, there are n nonzero scalars $k_1, \dots, k_n$ such that $k_1 = 1$ and $h_t(\omega_i) \sim k_i h_t(\omega_1)$ for $i = 2, \dots, n$. Then,

$$\frac{E_q^2(h_t)}{\|h_t\|_q^2} \sim \frac{h_t^2(\omega_1) (k_1 q(\omega_1) + k_2 q(\omega_2) + \dots + k_n q(\omega_n))^2}{h_t^2(\omega_1) (k_1^2 q(\omega_1) + k_2^2 q(\omega_2) + \dots + k_n^2 q(\omega_n))} = \frac{(k_1 q(\omega_1) + k_2 q(\omega_2) + \dots + k_n q(\omega_n))^2}{k_1^2 q(\omega_1) + k_2^2 q(\omega_2) + \dots + k_n^2 q(\omega_n)}$$

and so

$$\frac{(k_1 q(\omega_1) + k_2 q(\omega_2) + \cdots + k_n q(\omega_n))^2}{k_1^2 q(\omega_1) + k_2^2 q(\omega_2) + \cdots + k_n^2 q(\omega_n)} = 1$$

In turn this implies $k_1 = k_2 = \cdots = k_n = 1$, i.e., (22). $\blacksquare$

Proof of Proposition 16. Let us prove the “only if”, the converse being trivial. Let $h \in \mathbb{R}^n$. There is $t_h > 0$ such that $th \in B$ for all $t \in (0, t_h]$. Hence,

$$\sigma_q^2(h) = \frac{\sigma_q^2(th)}{t^2} = \frac{o(\|th\|_q^2)}{t^2} = \frac{o(t^2)}{t^2} \rightarrow 0$$

and so $\sigma_q^2(h) = 0$. Since this holds for any $h \in \mathbb{R}^n$, there is some $\omega \in \Omega$ such that $q = \delta_\omega$. $\blacksquare$

Proof of Proposition 19 (i) implies (iv) By Lemma 1, $E_q(h_t^2) = O(t^2)$. Hence, there is $a > 0$ such that, for each $t > 0$ small enough and each $\omega \in \text{supp } q$,

$$E_q(h_t^2) = O(t^2) \Leftrightarrow \frac{|h_t(\omega)|^2}{t^2} \leq \frac{a}{\min_{\omega \in \text{supp } q} q(\omega)} \Leftrightarrow \frac{|h_t(\omega)|}{t} \leq \left(\frac{a}{\min_{\omega \in \text{supp } q} q(\omega)} \right)^{\frac{1}{n}}$$

Hence, $h_t(\omega) = O(t)$ for each $\omega \in \text{supp } q$.

(iv) implies (iii) For t small enough there is $a > 0$ such that $|h_t(\omega)|/t \leq a$ for all $\omega \in \text{supp } q$. Hence, $|h_t(\omega)|^n/t^n \leq a^n$ for all $\omega \in \text{supp } q$, and so $E_q(|h_t|^n)/t^n \leq a^n$.

Since (iii) trivially implies (ii), it remains to prove that (ii) implies (i) There is $a > 0$ such that, for each $t > 0$ small enough and each $\omega \in \text{supp } q$,

$$0 \leq \frac{|h_t(\omega)|^n}{t^n} q(\omega) \leq \frac{E_q(|h_t|^n)}{t^n} = \sum_{\omega \in \text{supp } q} \frac{|h_t|^n(\omega)}{t^n} q(\omega) \leq a.$$

Hence, $E_q(h_t^2)/t^2 \leq a^{\frac{2}{n}} / \min_{\omega \in \text{supp } q} q(\omega)$, and so

$$0 \leq \frac{|o(E_q(h_t^2))|}{t^2} = \frac{|o(E_q(h_t^2))|}{E_q(h_t^2)} \frac{E_q(h_t^2)}{t^2} \leq \frac{a^{\frac{2}{n}}}{\min_{\omega \in \text{supp } q} q(\omega)} \frac{o(E_q(h_t^2))}{E_q(h_t^2)},$$

which implies $o(E_q(h_t^2)) = o(t^2)$. $\blacksquare$

Proof of Lemma 22 Since $\sigma_q^2(h_t) \leq \|h_t\|_q^2$, it holds $\sigma_q^2(h_t) = o(\|h_t\|_q)$. Let $\vartheta : \Omega \times \Omega \rightarrow [0, 1]$ be given by $\vartheta(\omega, \omega') = \int_{\Delta} q(\omega) q(\omega') d\mu(q)$. It holds $\sum_{\omega, \omega'} \vartheta(\omega, \omega') = 1$ and $\sum_{\omega'} \vartheta(\omega, \omega') = \bar{q}(\omega)$. In other words, $\bar{q}$ is the marginal of ϑ . It is easy to check that

$$\begin{aligned} \Lambda_v(t) &= \sum_{\omega, \omega'} h_t(\omega) h_t(\omega') \vartheta(\omega, \omega') \\ \Lambda_u(t) &= \sum_{\omega} h_t^2(\omega) [\bar{q}(\omega) - \vartheta(\omega, \omega)] - \sum_{\omega \neq \omega'} h_t(\omega) h_t(\omega') \vartheta(\omega, \omega') \end{aligned}$$

Let $\|h\|_\infty = \max_{\omega \in \text{supp } \bar{q}} |h(\omega)|$. By (4) there is $c > 0$ such that $\|h_t\|_\infty \leq c \|h_t\|_{\bar{q}_t}$ for all $t \in (0, 1]$. Moreover, it holds

$$\begin{aligned} \frac{\sigma_\mu^2(E(h_t))}{\|h_t\|_{\bar{q}}} &= \frac{\sum_{\omega, \omega'} h_t(\omega) h_t(\omega') \vartheta(\omega, \omega')}{\|h_t\|_{\bar{q}}} \leq \frac{\sum_{\omega, \omega'} |h_t(\omega)| |h_t(\omega')| \vartheta(\omega, \omega')}{\|h_t\|_{\bar{q}}} \\ &\leq \frac{\|h\|_\infty^2}{\|h_t\|_{\bar{q}}} \leq c \frac{\|h_t\|_{\bar{q}_t}^2}{\|h_t\|_{\bar{q}_t}} = c \|h_t\|_{\bar{q}_t} \rightarrow 0 \end{aligned}$$

and so $\sigma_\mu^2(E(h_t)) = o(\|h_t\|_{\bar{q}})$. Along with $\sigma_{\bar{q}}^2(h_t) = o(\|h_t\|_{\bar{q}})$, it implies the result. $\blacksquare$

Proof of Proposition 24 We prove the “only if”, the converses being trivial.

(ii) Let $h \in \mathbb{R}^n$. There is $t_h > 0$ such that $th \in B$ for all $t \in (0, t_h]$. Hence,

$$\sigma_{\bar{q}}^2(h) - \sigma_\mu^2(E(h)) = \frac{\sigma_{\bar{q}}^2(th)}{t^2} - \frac{\sigma_\mu^2(E(th))}{t^2} = \frac{\Lambda_u(th, \mu)}{\|th\|_q^2} \|h\|_q^2 = \frac{o(\|th\|_q^2)}{t^2} \|h\|_q^2 = \frac{o(t^2)}{t^2} \rightarrow 0$$

that is, $\sigma_{\bar{q}}^2(h) = \sigma_\mu^2(E(h))$ for all $h \in \mathbb{R}^n$. In turn this is equivalent to $\int E_q(h^2) d\mu = \int E_q^2(h) d\mu(q)$. Since $E_q(h^2) \leq E_q^2(h)$ for all $h \geq 0$, we conclude that $E_q(h^2) = E_q^2(h)$ μ -a.e. In particular, $E_q(h^2) = E_q^2(h)$ for all $q \in \text{supp } \mu$. By the Jensen inequality, $E_q(h^2) = E_q^2(h)$ holds if and only if h is constant q -a.e. Hence, it can hold for all $h \geq 0$ if and only if q is a Dirac measure. We conclude that $\text{supp } \mu$ consists of only Dirac measures.

(i) Similarly,

$$\sigma_\mu^2(E(h)) = \frac{\sigma_\mu^2(E(th))}{t^2} = \frac{\Lambda_u(th, \mu)}{\|th\|_q^2} \|h\|_q^2 = \frac{o(\|th\|_q^2)}{t^2} = \frac{o(t^2)}{t^2} \rightarrow 0$$

and so $\sigma_\mu^2(E(h)) = 0$ for all $h \in \mathbb{R}^n$. Hence, there is some $p \in \Delta$ such that $\mu = \delta_p$.

(iii) Finally, $\sigma_{\bar{q}}^2(h) - \sigma_\mu^2(E(h)) = \sigma_\mu^2(E(h)) = 0$ if and only if $\sigma_{\bar{q}}^2(h) = 0$. Since this holds for any $h \in \mathbb{R}^n$, there is some $\omega_0 \in \Omega$ such that $\bar{q}(\omega_0) = 1$, i.e., $\bar{q} = \delta_{\omega_0}$. Hence, $q = \delta_{\omega_0}$ μ -a.e. and so $\text{supp } \mu = \{\delta_{\omega_0}\}$ since the singleton $\{\delta_{\omega_0}\}$ is a closed set with full measure under μ . $\blacksquare$

Proof of Lemma 26 By [6, p. 25], $\omega \notin \limsup_{t \downarrow 0} \text{supp } q_t$ if and only if

$$\omega \in \liminf_{t \downarrow 0} (\text{supp } q_t)^c = \bigcup_{t \in (0,1]} \bigcap_{\tau \leq t} (\text{supp } q_\tau)^c. \quad (64)$$

Since $\bigcap_{\tau \leq t} (\text{supp } q_\tau)^c \subseteq \bigcap_{\tau \leq t'} (\text{supp } q_\tau)^c$ if $t' \leq t$, (64) holds if and only if there is $t_\omega \in (0, 1]$ such that

$\omega \in \bigcap_{\tau \leq t} (\text{supp } q_\tau)^c$ for all $t \leq t_\omega$. That is, if and only if $q_\tau(\omega) = 0$ for all $t \leq t_\omega$. $\blacksquare$

The next result is a slightly more general version of Proposition 27.

VERIFICARE CHE NE DISCENDA PURE (ii) $\Rightarrow$ (iv) in Proposition 11.

Proposition 57 *Given a risky monetary net $\{h_t, q_t\}_{t \in (0,1]}$, with $q_t \rightarrow q$, consider the following:*

(i) $\limsup_{t \downarrow 0} E_{q_t}(|h_t|) < \infty$ and $\lim_{t \downarrow 0} q_t(\omega : |h_t(\omega)| > \varepsilon) = 0$ for each $\varepsilon > 0$;

(ii) $\{h_t, q_t\}_{t \in (0,1]}$ is a small risk

(iii) $\lim_{t \downarrow 0} E_{q_t}(|h_t|^n) = 0$ for all $n \geq 1$;

(iv) $\lim_{t \downarrow 0} h_t(\omega) = 0$ for each $\omega \in \text{supp } q$ and $\lim_{t \downarrow 0} |h_t(\omega)| q_t(\omega) = 0$ for each $\omega \in \Gamma_q$;

(v) $\lim_{t \downarrow 0} h_t(\omega) = 0$ for each $\omega \in \text{supp } q$.

It holds

$$(i) \Leftarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v).$$

All properties are equivalent if $\limsup_{t \downarrow 0} |h_t(\omega)| < \infty$ for each $\omega \in \Gamma_q$.

Proof Consider as (ii) the more general property: $\lim_{t \downarrow 0} E_{q_t}(|h_t|^n) = 0$ for some $n \geq 1$.

(ii) implies (iv) For each $\omega \in \Omega$, $|h_t(\omega)|^n q_t(\omega) \leq E_{q_t}(|h_t|^n)$ implies $\lim_{t \downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$. But, $0 \leq |h_t(\omega)|^n q_t(\omega) \leq |h_t(\omega)|^n q_t(\omega)$, and so $\lim_{t \downarrow 0} |h_t(\omega)| q_t(\omega) = 0$ for each $\omega \in \Omega$.

Let $\omega \in \text{supp } q$. Suppose per contra that there is a sequence $\{t_n\}_n$, with $t_n \downarrow 0$, such that $h_{t_n}(\omega) \geq \varepsilon$ for some $\varepsilon > 0$. Wlog, suppose $q(\omega) > \varepsilon$. Since $\lim_{t \downarrow 0} q_t(\omega) > 0$, for n large enough it holds $q_{t_n}(\omega) \geq q - \varepsilon$. Hence, for all such n it holds $h_{t_n}(\omega) q_{t_n}(\omega) \geq \varepsilon(q - \varepsilon) > 0$, which contradicts $\lim_{t \downarrow 0} h_t(\omega) q_t(\omega) = 0$.

(iv) implies (iii) Let $n \geq 1$. Let $\omega \in \text{supp } q$. There is $t_\omega \in (0, 1]$ such that $q_t(\omega) > 0$ for all $t \leq t_\omega$. Hence, $0 \leq \lim_{t \downarrow 0} |h_t(\omega)|^n q_t(\omega) \leq \lim_{t \downarrow 0} |h_t(\omega)|^n = 0$, and so $\lim_{t \downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$. If $\omega \notin \limsup_{t \downarrow 0} \text{supp } q_t$, by Lemma 26 there is $t_\omega \in (0, 1]$ such that $q_t(\omega) = 0$ for all $t \leq t_\omega$. Hence, $\lim_{t \downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$. Since $\lim_{t \downarrow 0} |h_t(\omega)| q_t(\omega) = 0$ for each $\omega \in \Gamma_q$, this implies

$$\lim_{t \downarrow 0} E_{q_t}(|h_t|^n) = \lim_{t \downarrow 0} \left(\sum_{\omega \in \Omega} |h_t(\omega)|^n q_t(\omega) \right) = \sum_{\omega \in \Omega} \lim_{t \downarrow 0} |h_t(\omega)|^n q_t(\omega) = 0$$

as desired. This completes the proof of the equivalence among (ii)-(iv).

In the rest of the proof we assume

$$\limsup_{t \downarrow 0} |h_t(\omega)| < \infty, \quad \forall \omega \in \Gamma_q. \quad (65)$$

(v) implies (iv) Let $\omega \in \Gamma_q$. By (65) there is $M > 0$ and $t_M \in (0, 1)$ such that $|h_t(\omega)| \leq M$ for all $t \in (0, t_M)$. Since $\lim_{t \downarrow 0} q_t(\omega) = 0$, this implies $\lim_{t \downarrow 0} |h_t(\omega)| q_t(\omega) = 0$.

(i) implies (iii) Let $\omega \in \text{supp } q$. For t large enough, $q_t(\omega) > 0$. Hence, $|h_t(\omega)| \leq E_{q_t}(|h_t|)/q_t(\omega)$, so that $\limsup_{t \downarrow 0} |h_t(\omega)| \leq \limsup_{t \downarrow 0} E_{q_t}(|h_t|)/q_t(\omega)$. Along with (65) this implies $\limsup_{t \downarrow 0} |h_t(\omega)| < \infty$ for each $\omega \in \limsup_{t \downarrow 0} \text{supp } q_t$. Hence, for some $t^* \in (0, 1]$ there is $M > 0$ such that $|h_t(\omega)| \leq M$ for each $t \in (0, t^*]$ and $\omega \in \limsup_{t \downarrow 0} \text{supp } q_t$. By Lemma 26, there is $t^{**} \in (0, t^*]$ such that $q_t(\omega) = 0$ for each $\omega \notin \limsup_{t \downarrow 0} \text{supp } q_t$. We conclude that $q_t(|h_t(\omega)| \leq M) = 1$ for all $t \in (0, t^{**}]$.

As a result, the probability distributions p_t of each h_t under q_t are defined over $[-M, M]$ for each $t \in (0, t^{**}]$; i.e., $p_t(E) = q_t(h_t \in E)$ for all Borel subsets $\mathcal{B}$ of $[-M, M]$. We want to show that p_t weakly converges to δ_0 , the Dirac measure on 0. That is,

$$\lim_{t \downarrow 0} \int_{[-M, M]} \varphi(x) dp_t(x) = \int_{[-M, M]} \varphi(x) d\delta_0(x) = \varphi(0), \quad (66)$$

given any continuous function $\varphi : [-M, M] \rightarrow \mathbb{R}$. Set $K = \sup_{x \in [-M, M]} |\varphi(x) - \varphi(0)|$ and fix $\varepsilon > 0$. By continuity there is $\delta_\varepsilon > 0$ such that $|\varphi(x) - \varphi(0)| < \varepsilon$ for all $x \in B_{\delta_\varepsilon}(0)$. By (i), $\lim_{t \downarrow 0} q_t(|h_t| > \varepsilon/2) = \lim_{t \downarrow 0} p_t(|h_t| > \varepsilon/2) = 0$. Hence, there is $\delta'_\varepsilon > 0$ such that $q_t(|h_t| > \varepsilon/2) < \varepsilon/K$ for all $t \in (0, \delta'_\varepsilon)$. For each $t \in (0, \delta'_\varepsilon)$ it holds

$$\begin{aligned} \left| \int_{[-M, M]} (\varphi(x) - \varphi(0)) dp_t(x) \right| &\leq \int_{[-M, M]} |\varphi(x) - \varphi(0)| dp_t(x) \\ &= \int_{B_{\delta_\varepsilon}(0)} |\varphi(x) - \varphi(0)| dp_t(x) \\ &\quad + \int_{B_{\delta_\varepsilon}(0)^c} |\varphi(x) - \varphi(0)| dp_t(x) \\ &\leq \varepsilon + K p_t(B_{\delta_\varepsilon}(0)^c) = \varepsilon + K q_t(|h_t| \geq \varepsilon) \\ &\leq \varepsilon + K q_t(|h_t| > \varepsilon/2) \leq 2\varepsilon. \end{aligned}$$

In turn this easily implies (66). Set $\varphi(x) = |x|^n$. Then, (66) implies

$$\lim_{t \downarrow 0} E_{q_t}(|h_t|^n) = \lim_{t \downarrow 0} \int_{[-M, M]} |x|^n dp_t(x) = 0,$$

as desired. ■

Proof of Theorem 31 See the more general proof of Theorem 38. ■

Proof of Proposition 32 Let $\{h_t, q_t\}_{t \in (0,1]}$ be a controllable small risk that is asymptotically constant. Since it is controllable, eventually

$$\frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)} \geq k$$

for some $k \in (0, 1)$. Since

$$\frac{E_{q_t}(h_t^2)}{\sum_{\omega \in \text{supp } q_t} h_t^2(\omega)} \sim \frac{\sum_{\omega \in \text{supp } q} h_t^2(\omega) q_t(\omega) + \sum_{\omega \in \Gamma_q} h_t^2(\omega) q_t(\omega)}{\sum_{\omega \in \text{supp } q} h_t^2(\omega) + \sum_{\omega \in \Gamma_q} h_t^2(\omega)}$$

it holds

$$\frac{\sum_{\omega \in \text{supp } q} h_t^2(\omega) q_t(\omega) + \sum_{\omega \in \Gamma_q} h_t^2(\omega) q_t(\omega)}{\sum_{\omega \in \text{supp } q} h_t^2(\omega) + \sum_{\omega \in \Gamma_q} h_t^2(\omega)} \geq k$$

i.e.,

$$\sum_{\omega \in \text{supp } q} h_t^2(\omega) q_t(\omega) + \sum_{\omega \in \Gamma_q} h_t^2(\omega) q_t(\omega) \geq k \left(\sum_{\omega \in \text{supp } q} h_t^2(\omega) + \sum_{\omega \in \Gamma_q} h_t^2(\omega) \right)$$

i.e.,

$$\sum_{\omega \in \text{supp } q} h_t^2(\omega) (q_t(\omega) - k) \geq \sum_{\omega \in \Gamma_q} h_t^2(\omega) (k - q_t(\omega))$$

In turn this implies

$$\sum_{\omega \in \text{supp } q} h_t^2(\omega) (1 - k) \geq k \sum_{\omega \in \Gamma_q} h_t^2(\omega)$$

and so

$$\sum_{\omega \in \Gamma_q} h_t^2(\omega) = O \left(\sum_{\omega \in \text{supp } q} h_t^2(\omega) \right). \quad (67)$$

Let $\tilde{\omega} \in \text{supp } q$. Since $\{h_t, q_t\}_{t \in (0,1]}$ is asymptotically constant, (67) implies

$$h_t^2(\omega) = O(h_t^2(\tilde{\omega})) \quad \forall \omega \in \Gamma_q \quad (68)$$

Hence, there is $k > 0$ such that, eventually, $h_t^2(\omega) = O(h_t^2(\tilde{\omega}))$ for each $\omega \in \Gamma_q$. Hence,

$$\begin{aligned} \frac{\sum_{\omega \in \Gamma_q} h_t^2(\omega) q_t(\omega)}{\sum_{\omega \in \text{supp } q} h_t^2(\omega) q_t(\omega)} &\sim \frac{\sum_{\omega \in \Gamma_q} h_t^2(\omega) q_t(\omega)}{h_t^2(\tilde{\omega}) q_t(\text{supp } q)} = \frac{1}{q_t(\text{supp } q)} \sum_{\omega \in \Gamma_q} \frac{h_t^2(\omega)}{h_t^2(\tilde{\omega})} q_t(\omega) \\ &\leq \frac{k}{q_t(\text{supp } q)} q_t(\Gamma_q) \rightarrow 0 \end{aligned}$$

We conclude that $\sum_{\omega \in \Gamma_q} h_t^2(\omega) q_t(\omega) = o \left(\sum_{\omega \in \text{supp } q} h_t^2(\omega) q_t(\omega) \right)$. Moreover,

$$\begin{aligned} \left| \frac{\sum_{\omega \in \Gamma_q} h_t(\omega) q_t(\omega)}{\sum_{\omega \in \text{supp } q} h_t(\omega) q_t(\omega)} \right| &\sim \left| \frac{\sum_{\omega \in \Gamma_q} h_t(\omega) q_t(\omega)}{h_t(\tilde{\omega}) q_t(\text{supp } q)} \right| = \frac{1}{q_t(\text{supp } q)} \left| \sum_{\omega \in \Gamma_q} \frac{h_t(\omega)}{h_t(\tilde{\omega})} q_t(\omega) \right| \\ &\leq \frac{1}{q_t(\text{supp } q)} \sum_{\omega \in \Gamma_q} \left| \frac{h_t(\omega)}{h_t(\tilde{\omega})} \right| q_t(\omega) \leq \frac{\sqrt{k}}{q_t(\text{supp } q)} q_t(\Gamma_q) \rightarrow 0 \end{aligned}$$

so that $\sum_{\omega \in \Gamma_q} h_t(\omega) q_t(\omega) = o\left(\sum_{\omega \in \text{supp } q} h_t(\omega) q_t(\omega)\right)$. Hence,

$$\frac{E_{q_t}^2(h_t)}{E_{q_t}(h_t^2)} \sim \frac{\left(\sum_{\omega \in \text{supp } q} h_t(\omega) q_t(\omega)\right)^2}{\sum_{\omega \in \text{supp } q} h_t^2(\omega) q_t(\omega)} \sim \frac{h_t^2(\tilde{\omega}) q_t^2(\text{supp } q)}{h_t^2(\tilde{\omega}) q_t(\text{supp } q)} = q_t(\text{supp } q) \rightarrow 1$$

and so $E_{q_t}^2(h_t) \sim E_{q_t}(h_t^2)$. ■

Proof of Proposition 34 To see that (i) implies (iii) it is enough to observe that there is $a > 0$ such that, for each t small enough,

$$\frac{h_t^2(\omega) q_t(\omega)}{t^2} \leq \frac{E_{q_t}(h_t^2)}{t^2} \leq a, \quad \forall \omega \in \Omega.$$

(ii) implies (i). For, there are $a, b > 0$ such that, for t small enough,

$$0 \leq \frac{E_{q_t}(h_t^2)}{t^2} = \sum_{\omega \in \text{supp } q} \frac{h_t^2(\omega)}{t^2} q_t(\omega) + \sum_{\omega \in \Gamma_q} \frac{h_t^2(\omega)}{t^2} q_t(\omega) \leq a + b.$$

Hence, $E_{q_t}(h_t^2) = O(t^2)$. ■

Proof of Theorem 38 By (86) of Appendix B, given an open interval $(a, b) \subseteq \mathbb{R}$ there is $L > 0$ such that

$$|R_2(h, \bar{q})| \leq \frac{L}{|\Omega|^{3/2}} \left(\sum_{\omega \in \text{supp } \bar{q}} |h(\omega)| \right)^3 \quad \forall \mu. \quad (69)$$

for all h such that $h(\omega) \in (a, b)$ for each $\omega \in \Omega$. In particular,

$$|R_2(h_t, \bar{q}_t)| \leq \frac{L}{|\Omega|^{3/2}} \left(\sum_{\omega \in \text{supp } \bar{q}_t} |h_t(\omega)| \right)^3 \quad \forall t \in (0, 1]. \quad (70)$$

By controllability, there is $\varepsilon > 0$ and $t_\varepsilon \in (0, 1]$ such that

$$\sqrt{\sum_{\omega \in \text{supp } \bar{q}_t} h_t^2(\omega)} \leq \varepsilon \sqrt{E_{\bar{q}_t}(h_t^2)} \quad \forall t \in (0, t_\varepsilon].$$

Hence,

$$\sum_{\omega \in \text{supp } \bar{q}_t} |h_t(\omega)| \leq |\text{supp } \bar{q}_t|^{1/2} \sqrt{\sum_{\omega \in \text{supp } \bar{q}_t} h_t^2(\omega)} \leq \varepsilon |\Omega|^{1/2} \sqrt{E_{\bar{q}_t}(h_t^2)} = \varepsilon |\Omega|^{1/2} \|h_t\|_{\bar{q}_t}$$

and so

$$|R_2(h_t, \bar{q}_t)| \leq \frac{L}{|\Omega|^{3/2}} \varepsilon^3 |\Omega|^{3/2} \|h_t\|_{\bar{q}_t}^3 = \varepsilon^3 L \|h_t\|_{\bar{q}_t}^3$$

In turn this implies $R_2(h_t, \bar{q}_t) = o\left(\|h_t\|_{\bar{q}_t}^2\right)$ as $t \downarrow 0$. ■

Proof of Lemma 39 It holds $W_t(\omega) = O(t)$ for each $\omega \in \text{supp } \bar{q}$ and $W_t^2(\omega) \bar{q}_t(\omega) = t^2 \bar{q}_t(\omega) = o(t^2)$ for each $\omega \in \Gamma_{\bar{q}}$. The result then follows from Proposition ??.

As to controllability, it is enough to observe that

$$\frac{E_{\bar{q}_t}(W_t^2)}{\sum_{\omega \in \text{supp } \bar{q}_t} W_t^2(\omega)} \geq \frac{t^2}{2t^2} = \frac{1}{2}$$

■

Proof of Lemma 44. For all $f \in C^+([0, 1])$ and all $t > 0$, by the Mean Value Theorem there exists $c_t \in [a_t, b_t]$ such that

$$\int f d\mu_t = \frac{1}{b_t - a_t} \int_{a_t}^{b_t} f(q) dq = f(c_t).$$

If $a = b$, then $a_t \leq c_t \leq b_t$ implies $c_t \rightarrow a$ (as $t \downarrow 0$) and continuity of f delivers $f(c_t) \rightarrow f(a)$, i.e.,

$$\int f d\mu_t \rightarrow \int f d\delta_a. \quad (71)$$

Else, for all $f \in C^+([0, 1])$ and all $t > 0$,

$$\frac{1}{b_t - a_t} \int_{a_t}^{b_t} f(q) dq = \frac{1}{b_t - a_t} \int f 1_{[a_t, b_t]} d\lambda$$

for some $c_t \in [a_t, b_t]$. Set $f_t = f 1_{[a_t, b_t]}$ for all $t > 0$. Notice that $\{f_t\}_{t>0}$ is a decreasing (as $t \downarrow 0$) net of usc functions on a compact metric space and its limit is $f 1_{[a, b]}$. A Monotone Convergence Theorem for nets of upper semicontinuous functions (along the lines of [?, Proposition 7.12]) delivers

$$\int f 1_{[a_t, b_t]} d\lambda \rightarrow \int f 1_{[a, b]} d\lambda$$

and

$$\int f d\mu_t = \frac{1}{b_t - a_t} \int f 1_{[a_t, b_t]} d\lambda \rightarrow \frac{1}{b - a} \int f 1_{[a, b]} d\lambda = \int f d\lambda_{[a, b]} \quad (72)$$

Equations (71) and (72) yield the desired convergence. ■

Proof of Lemma 45 It follows from Lemmas 58 and 59 below. ■

Lemma 58 *An uncertainty monetary net $\{h_t, \mu_t\}_{t \in (0, 1]}$ is a small uncertainty if and only if:*

- (i) $\lim_{t \downarrow 0} \varphi(t) = \lim_{t \downarrow 0} \psi(t) = 0$ when $\mu \neq \delta_p$ with $p \in \{0, 1\}$;
- (ii) $\lim_{t \downarrow 0} \varphi^2(t) \bar{q}_t = \lim_{t \downarrow 0} \psi(t) = 0$ when $\mu = \delta_0$;
- (iii) $\lim_{t \downarrow 0} \varphi(t) = \lim_{t \downarrow 0} \psi^2(t) (1 - \bar{q}_t) = 0$ when $\mu = \delta_1$.

Moreover, $\{h_t, \mu_t\}_{t \in (0, 1]}$ is a quadratic small uncertainty if and only if:

- (a) $\varphi(t) = O(t)$ and $\psi(t) = O(t)$ when $\mu \neq \delta_p$ with $p \in \{0, 1\}$;
- (b) $\varphi^2(t) \bar{q}_t = O(t^2)$ and $\psi(t) = O(t)$ when $\mu = \delta_0$;
- (c) $\varphi(t) = O(t)$ and $\psi^2(t) (1 - \bar{q}_t) = O(t^2)$ when $\mu = \delta_1$.

Proof It follows from Proposition 27 and its obvious uncertainty version. ■

Lemma 59 *A small uncertainty $\{h_t, \mu_t\}_{t \in (0, 1]}$ is controllable when $\mu \neq \delta_p$ with $p \in \{0, 1\}$.*

Proof Since $\bar{q}_t \in (0, 1)$, it holds

$$\frac{E_{\bar{q}_t}(h_t^2)}{\sum_{\omega \in \text{supp } \bar{q}_t} h_t^2(\omega)} = \frac{\varphi^2(t) \bar{q}_t + \psi^2(t) (1 - \bar{q}_t)}{\varphi^2(t) + \psi^2(t)} \leq 1 \quad \forall t \in (0, 1]$$

To show that $E_{\bar{q}_t}(h_t^2) \asymp \sum_{\omega \in \text{supp } \bar{q}_t} h_t^2(\omega)$ is enough to check that

$$\liminf_{t \downarrow 0} \frac{\varphi^2(t) \bar{q}_t + \psi^2(t) (1 - \bar{q}_t)}{\varphi^2(t) + \psi^2(t)} > 0. \quad (73)$$

Suppose $\mu \neq \delta_p$, with $p \in \{0, 1\}$, so that $\bar{q} \in (0, 1)$. Let $0 < k < \min\{\bar{q}, 1 - \bar{q}\}$. Then,

$$\frac{\varphi^2(t) \bar{q}_t + \psi^2(t) (1 - \bar{q}_t)}{\varphi^2(t) + \psi^2(t)} \geq \frac{\varphi^2(t) k + \psi^2(t) k}{\varphi^2(t) + \psi^2(t)} = k > 0$$

and so $E_{\bar{q}_t}(h_t^2) \asymp \sum_{\omega \in \text{supp } \bar{q}_t} h_t^2(\omega)$. ■

Lemma 60 Suppose $\mu = \delta_0$. Consider the following conditions on an uncertainty monetary net $\{h_t, \mu_t\}_{t \in (0, 1]}$:

- (i) $\varphi^2(t) \asymp \psi^2(t)$;
- (ii) $\{h_t, \mu_t\}_{t \in (0, 1]}$ is a controllable;
- (iii) $\varphi^2(t) = O(\psi^2(t))$.

Then,

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

If, in addition, $\lim_{t \downarrow 0} \varphi^2(t) / \psi^2(t) \in [-\infty, +\infty]$, then (i) and (ii) are equivalent.

(i) implies (ii) Suppose $\mu = \delta_0$, so that $\bar{q} = 0$. Suppose $\varphi^2(t) \asymp \psi^2(t)$, i.e., there is $k > 0$ such that eventually

$$\frac{1}{k} \psi^2(t) \leq \varphi^2(t) \leq k \psi^2(t).$$

Then,

$$\frac{\varphi^2(t) \bar{q}_t + \psi^2(t) (1 - \bar{q}_t)}{\varphi^2(t) + \psi^2(t)} = \frac{\bar{q}_t + \frac{\psi^2(t)}{\varphi^2(t)} (1 - \bar{q}_t)}{1 + \frac{\psi^2(t)}{\varphi^2(t)}} \geq \frac{\bar{q}_t + \frac{1 - \bar{q}_t}{k}}{1 + k} \rightarrow \frac{1}{1 + k} > 0$$

and so (73) holds.

(ii) implies (iii) Suppose there exists $k \in (0, 1)$ and $t_k \in (0, 1]$ such that

$$\frac{\varphi^2(t) \bar{q}_t + \psi^2(t) (1 - \bar{q}_t)}{\varphi^2(t) + \psi^2(t)} \geq k \quad (74)$$

for all $0 < t \leq t_k$. Since $\bar{q}_t \rightarrow 0$, wlog we can assume $\bar{q}_t < k$. Hence, (74) implies

$$\varphi^2(t) \bar{q}_t + \psi^2(t) (1 - \bar{q}_t) \geq k (\varphi^2(t) + \psi^2(t))$$

that is, $\psi^2(t) (1 - \bar{q}_t - k) \geq \varphi^2(t) (k - \bar{q}_t)$

$$\frac{\psi^2(t)}{\varphi^2(t)} \geq \frac{k - \bar{q}_t}{1 - \bar{q}_t - k} \rightarrow \frac{k}{1 - k}$$

Hence, $\varphi^2(t) = O(\psi^2(t))$.

(ii) implies (i) when

$$\lim_{t \downarrow 0} \frac{\varphi(t)}{\psi(t)} \in [-\infty, +\infty]. \quad (75)$$

By what just proved, $\varphi^2(t) = O(\psi^2(t))$. It remains to show that $\lim_{t \downarrow 0} \varphi^2(t)/\psi^2(t) > 0$. By (75), this is not the case if and only if $\varphi^2(t) = o(\psi^2(t))$. Hence, suppose, per contra, that $\varphi^2(t) = o(\psi^2(t))$. Then, $\varphi^2(t)\bar{q}_t = o(\psi^2(t)(1-\bar{q}_t))$ since $\bar{q}_t \rightarrow 0$, and so

$$\frac{\varphi^2(t)\bar{q}_t + \psi^2(t)(1-\bar{q}_t)}{\varphi^2(t) + \psi^2(t)} \sim \frac{\psi^2(t)(1-\bar{q}_t)}{\psi^2(t)} = 1 - \bar{q}_t \rightarrow 0$$

This contradicts (ii) and we conclude that $\varphi^2(t) \asymp \psi^2(t)$. ■

Proof of Lemma 41 We prove the “if,” the converse being trivial. Let $\mu \neq \delta_p$ with $p \in \{0, 1\}$. We want to show that $\text{supp } \bar{q} = \Omega$, i.e., $\bar{q} \in (0, 1)$. Suppose per contra that $\bar{q} = 0$, i.e., $\int_{[0,1]} q d\mu(q) = 0$. Since $q^n \leq q$ on $[0, 1]$ for all $n \geq 1$, it follows that $\int_{[0,1]} q^n d\mu(q) = 0$ for all $n \geq 1$. By the Weierstrass Approximation Theorem, this in turn implies $\int_{[0,1]} \xi(q) d\mu(q) = 0$ for all continuous functions $\xi : [0, 1] \rightarrow \mathbb{R}$. Hence, $\int_{[0,1]} \xi(q) d\mu(q) = \int_{[0,1]} \xi(q) d\delta_0(q)$ for all such ξ , which implies $\mu = \delta_0$. Suppose $\bar{q} = 1$, i.e., $\int_{[0,1]} (1-q) d\mu(q) = 0$. Let $T : [0, 1] \rightarrow [0, 1]$ be such that $T(q) = 1 - q$ for all $q \in [0, 1]$. By a change of variable, $\int_{[0,1]} p d(\mu \circ T)(p) = \int_{[0,1]} (1-q) d\mu(q) = 0$. By the previous argument, $\mu \circ T = \delta_0$, i.e., $\mu = \delta_1$. We conclude that $\bar{q} \in (0, 1)$, i.e., $\text{supp } \bar{q} = \Omega$. ■

Proof of Proposition 42 It is a special case of Proposition 47. ■

Proof of Proposition 43 It is a special case of proposition 61. Observe that, by (56),

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \bar{q}_t \gamma^2(t) + o(t^2) = 2\lambda_u(w) \bar{q}_t t^2 + o(t^2) = o(t^2)$$

since $\bar{q}_t t^2 = o(t^2)$ because $\bar{q}_t \rightarrow 0$. ■

Proof of Proposition 47 We first show that $\gamma^2(t) = O(\|h_t\|_{\bar{q}_t}^2)$. By Lemma 41, $\bar{q} \in (0, 1)$. Hence, it holds

$$0 \leq \frac{\varphi^2(t)}{\|h_t\|_{\bar{q}_t}^2} = \frac{\bar{q}_t \varphi^2(t)}{\bar{q}_t \varphi^2(t) + (1-\bar{q}_t) \psi^2(t)} \frac{1}{\bar{q}_t} \leq \frac{1}{\bar{q}_t} \rightarrow \frac{1}{\bar{q}}, \quad (76)$$

$$0 \leq \frac{\psi^2(t)}{\|h_t\|_{\bar{q}_t}^2} = \frac{(1-\bar{q}_t) \psi^2(t)}{\bar{q}_t \varphi^2(t) + (1-\bar{q}_t) \psi^2(t)} \frac{1}{(1-\bar{q}_t)} \leq \frac{1}{1-\bar{q}_t} \rightarrow \frac{1}{1-\bar{q}}, \quad (77)$$

Moreover, setting $\xi = \varphi \vee \psi$ and $\eta = \varphi \wedge \psi$, it holds:

$$\begin{aligned} 0 &\leq \frac{|\varphi(t)| |\psi(t)|}{\|h_t\|_{\bar{q}_t}^2} = \frac{|\varphi(t)| |\psi(t)|}{\bar{q}_t \varphi^2(t) + (1-\bar{q}_t) \psi^2(t)} \leq \frac{|\varphi(t)| |\psi(t)|}{\bar{q}_t (1-\bar{q}_t) \varphi^2(t) + (1-\bar{q}_t) \bar{q}_t \psi^2(t)} \\ &\leq \frac{1}{\bar{q}_t (1-\bar{q}_t)} \frac{|\varphi(t)| |\psi(t)|}{\varphi^2(t) + \psi^2(t)} \leq \frac{1}{\bar{q}_t (1-\bar{q}_t)} \frac{\xi^2(t)}{\xi^2(t) + \eta^2(t)} \\ &\leq \frac{1}{\bar{q}_t (1-\bar{q}_t)} \rightarrow \frac{1}{\bar{q}(1-\bar{q})}. \end{aligned} \quad (78)$$

Hence, there exists $a > 0$ such that, for each t small enough,

$$0 \leq \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \frac{\varphi^2(t) + \psi^2(t) + 2|\varphi(t)| |\psi(t)|}{\|h_t\|_{\bar{q}_t}^2} \leq 3a,$$

so that $\gamma^2(t) = O(\|h_t\|_{\bar{q}_t}^2)$. In turn, this implies $\gamma^2(t) o(1) = o(\|h_t\|_{\bar{q}_t}^2)$. Hence,

$$\begin{aligned}\pi_t(w + W_t) &= \frac{1}{2}\lambda_u(w)(\bar{q}_t - \vartheta_t)\gamma^2(t) + \frac{1}{2}\lambda_v(w)(\vartheta_t - \bar{q}_t^2)\gamma^2(t) + o(\|h_t\|_{\bar{q}_t}^2) \\ &= \frac{1}{2}\lambda_u(w)(\bar{q} - \vartheta + o(1))\gamma^2(t) + \frac{1}{2}\lambda_v(w)(\vartheta - \bar{q}^2 + o(1))\gamma^2(t) + o(\|h_t\|_{\bar{q}_t}^2) \\ &= \frac{1}{2}\lambda_u(w)(\bar{q} - \vartheta)\gamma^2(t) + \frac{1}{2}\lambda_v(w)(\vartheta - \bar{q}^2)\gamma^2(t) + o(\|h_t\|_{\bar{q}_t}^2),\end{aligned}$$

and so (49) holds. As to (i)-(iii), we have

$$\bar{q} - \vartheta = 0 \Leftrightarrow \int (q - q^2) d\mu(q) = 0 \Leftrightarrow \mu(q \in [0, 1] : q - q^2 = 0) = 1 \Leftrightarrow \mu(\{0, 1\}) = 1 \Leftrightarrow \text{supp } \mu \subseteq \{0, 1\}$$

while, setting $m = \int q d\mu(q)$, it holds:

$$\begin{aligned}\vartheta - \bar{q}^2 = 0 &\Leftrightarrow \int q^2 d\mu(q) - \left(\int q d\mu(q)\right)^2 = 0 \Leftrightarrow \int (q - m)^2 d\mu(q) = 0 \\ &\Leftrightarrow \mu(q \in [0, 1] : q = m) = 1 \Leftrightarrow \mu(\{m\}) = 1 \Leftrightarrow \mu = \delta_m.\end{aligned}$$

The rest is obvious. ■

Proof of Theorem 48 (i) Given any $0 \leq a \leq b \leq \infty$, since $\mu_t \xrightarrow{sq} \mu$, it holds:

$$a \leq \liminf_{t \downarrow 0} \frac{\Lambda_u(h_t, \mu_t)}{\|h_t\|_{\bar{q}_t}^2} \leq \limsup_{t \downarrow 0} \frac{\Lambda_u(h_t, \mu_t)}{\|h_t\|_{\bar{q}_t}^2} \leq b \Leftrightarrow a \leq (\bar{q} - \vartheta) \liminf_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq (\bar{q} - \vartheta) \limsup_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq b.$$

In particular, $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ if and only if $0 < a \leq b < \infty$, i.e., if and only if $\bar{q} - \vartheta > 0$ and

$$0 < \liminf_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \limsup_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} < \infty. \quad (79)$$

A similar argument shows that $\Lambda_v(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ if and only if $\vartheta - \bar{q}^2 > 0$ and (79) holds. Hence, both $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ and $\Lambda_v(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ if and only if $\bar{q}^2 < \vartheta < \bar{q}$ and $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$. In view of Proposition 47-(i), the desired result follows. Moreover, we have $\Lambda_v(h_t, \mu_t) \sim \|h_t\|_{\bar{q}_t}^2$ and $\Lambda_u(h_t, \mu_t) \sim \|h_t\|_{\bar{q}_t}^2$ when (79) becomes

$$\liminf_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} = \limsup_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \in \mathbb{R}_{++}, \quad (80)$$

that is, $\gamma(t) \sim \|h_t\|_{\bar{q}_t}$.

(ii) From point (i) we have $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ (resp., $\Lambda_u(h_t, \mu_t) \sim \|h_t\|_{\bar{q}_t}^2$) if and only if $\bar{q} - \vartheta > 0$ and (79) (resp., (80)) holds. That is, by Proposition 47-(iii), if and only if $\text{supp } \mu \neq \{0, 1\}$ and (79) (resp., (80)) holds. Moreover, from $\gamma(t) = O(\|h_t\|_{\bar{q}_t})$ it follows that there is $b > 0$ such that

$$\limsup_{t \downarrow 0} \frac{\Lambda_v(h_t, \mu_t)}{\|h_t\|_{\bar{q}_t}^2} = \limsup_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq b \limsup_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2).$$

Hence, $\limsup_{t \downarrow 0} \Lambda_v(h_t, \mu_t) / \|h_t\|_{\bar{q}_t}^2 = 0$ if and only if $\limsup_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) = 0$, that is, by Proposition 47-(ii), if and only if $\mu = \delta_p$ for some $p \in (0, 1)$. We conclude that $\Lambda_u(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ and $\Lambda_v(h_t, \mu_t) = o(\|h_t\|_{\bar{q}_t}^2)$ if and only if $\mu = \delta_p$ for some $p \in (0, 1)$ and $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.

(iii) From point (i) we have $\Lambda_v(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ (resp., $\Lambda_v(h_t, \mu_t) \sim \|h_t\|_{\bar{q}_t}^2$) holds if and only if $\vartheta - \bar{q}^2 > 0$ and (79) (resp., (80)) holds. That is, by Proposition 47-(ii), if and only if $\mu \neq \delta_p$ for some $p \in (0, 1)$ and (79) (resp., (80)) holds. Moreover, from $\gamma(t) = O(\|h_t\|_{\bar{q}_t})$ it follows that there is $b > 0$ such that

$$\limsup_{t \downarrow 0} \frac{\Lambda_2(h_t, \mu_t)}{\|h_t\|_{\bar{q}_t}^2} = \limsup_{t \downarrow 0} (\bar{q}_t - \vartheta_t) \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq b \limsup_{t \downarrow 0} (\bar{q}_t - \vartheta_t).$$

Hence, $\limsup_{t \downarrow 0} \Lambda_v(h_t, \mu_t) / \|h_t\|_{\bar{q}_t}^2 = 0$ if and only if $\limsup_{t \downarrow 0} (\bar{q}_t - \vartheta_t) = 0$, that is, by Proposition 47-(ii), if and only if $\text{supp } \mu = \{0, 1\}$. We conclude that $\Lambda_u(h_t, \mu_t) = o(\|h_t\|_{\bar{q}_t}^2)$ and $\Lambda_v(h_t, \mu_t) \asymp \|h_t\|_{\bar{q}_t}^2$ if and only if $\text{supp } \mu = \{0, 1\}$ and $\gamma(t) \asymp \|h_t\|_{\bar{q}_t}$.

(iv) It holds

$$\limsup_{t \downarrow 0} \frac{\Lambda_u(h_t, \mu_t)}{\|h_t\|_{\bar{q}_t}^2} = \limsup_{t \downarrow 0} (\bar{q}_t - \vartheta_t) \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \lim_{t \downarrow 0} (\bar{q}_t - \vartheta_t) \limsup_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2}$$

and

$$\limsup_{t \downarrow 0} \frac{\Lambda_v(h_t, \mu_t)}{\|h_t\|_{\bar{q}_t}^2} = \limsup_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \lim_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) \limsup_{t \downarrow 0} \frac{\gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2},$$

and so $\lim_{t \downarrow 0} \Lambda_u(h_t, \mu_t) / \|h_t\|_{\bar{q}_t}^2 = \lim_{t \downarrow 0} \Lambda_v(h_t, \mu_t) / \|h_t\|_{\bar{q}_t}^2 = 0$ if and only if either $\gamma(t) = o(\|h_t\|_{\bar{q}_t})$ or $\lim_{t \downarrow 0} (\bar{q}_t - \vartheta_t) = \lim_{t \downarrow 0} (\vartheta_t - \bar{q}_t^2) = 0$ (or both). But, the latter equality cannot hold since it requires that both (ii) and (iii) in Proposition 47 hold, which is not possible since $\mu \neq \delta_p$ with $p \in \{0, 1\}$. $\blacksquare$

Lemma 61 *Let $\{h_t, \mu_t\}_{t \in (0, 1]}$ be an uncertain monetary net, with $\mu_t \xrightarrow{sq} \delta_0$. If the net is a small uncertainty that admits an approximation (37), then*

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \bar{q}_t \gamma^2(t) + o(\|h_t\|_{\bar{q}_t}^2), \quad (81)$$

provided $\vartheta_t = O(\bar{q}_t^2)$. In particular, $\bar{q}_t \gamma(t) = o(\|h_t\|_{\bar{q}_t})$.

A similar result holds when $\mu_t \xrightarrow{sq} \delta_1$.

Proof Suppose $\bar{q}_t > 0$. By (76), (77), and (78) it holds:

$$\begin{aligned} 0 &\leq \frac{\bar{q}_t^2 \varphi^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \bar{q}_t^2 \frac{1}{\bar{q}_t} \rightarrow 0, \\ 0 &\leq \frac{\bar{q}_t^2 \psi^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \frac{\bar{q}_t^2}{1 - \bar{q}_t} \rightarrow 0, \\ 0 &\leq \frac{\bar{q}_t^2 |\varphi(t)| |\psi(t)|}{\|h_t\|_{\bar{q}_t}^2} \leq \frac{\bar{q}_t^2}{\bar{q}_t (1 - \bar{q}_t)} \rightarrow 0. \end{aligned}$$

The same inequalities trivially holds if $\bar{q}_t = 0$. Hence,

$$0 \leq \frac{\bar{q}_t^2 \gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \frac{\bar{q}_t^2 \varphi^2(t) + \bar{q}_t^2 \psi^2(t) + 2\bar{q}_t^2 |\varphi(t)| |\psi(t)|}{\|h_t\|_{\bar{q}_t}^2} \rightarrow 0,$$

so that $\bar{q}_t^2 \gamma^2(t) = o(\|h_t\|_{\bar{q}_t}^2)$. Hence,

$$\begin{aligned}\pi_t(w + W_t) &= \frac{1}{2} \lambda_u(w) (\bar{q}_t - \vartheta_t) \gamma^2(t) + \frac{1}{2} \lambda_v(w) (\vartheta_t - \bar{q}_t^2) \gamma^2(t) + o(\|h_t\|_{\bar{q}_t}^2) \\ &= \frac{1}{2} \lambda_u(w) \bar{q}_t \gamma^2(t) + o(\|h_t\|_{\bar{q}_t}^2)\end{aligned}$$

provided $\vartheta_t = O(\bar{q}_t^2)$. The rest is obvious. $\blacksquare$

Proof of Proposition 52 Since $\{h_t, \mu_t\}_{t \in (0,1]}$ is controllable and $\bar{q}_t \rightarrow 0$, by Lemma 45 $\varphi^2(t) = O(\psi^2(t))$, i.e., there is $k > 0$ such that $\varphi^2(t) \leq k\psi^2(t)$ for all t small enough. By (81), we need to show that $\bar{q}_t \gamma^2(t) = o(\|h_t\|_{\bar{q}_t}^2)$. It holds

$$\begin{aligned}\frac{\gamma^2(t)}{\psi^2(t)} &= \frac{\varphi^2(t) + \psi^2(t) - 2\varphi(t)\psi(t)}{\psi^2(t)} \leq k + 1 - 2\frac{\varphi(t)}{\psi(t)} \\ &\leq k + 1 + 2\sqrt{k} = (1 + \sqrt{k})^2\end{aligned}$$

Hence,

$$0 \leq \frac{\bar{q}_t^2 \gamma^2(t)}{\|h_t\|_{\bar{q}_t}^2} \leq \frac{\bar{q}_t (1 + \sqrt{k})^2 \psi^2(t)}{\bar{q}_t \varphi^2(t) + (1 - \bar{q}_t) \psi^2(t)} = \frac{\bar{q}_t (1 + \sqrt{k})^2}{\bar{q}_t \frac{\varphi^2(t)}{\psi^2(t)} + (1 - \bar{q}_t)} \rightarrow 0$$

because the denominator converges to 1 since $\varphi^2(t) = O(\psi^2(t))$. This proves that $\bar{q}_t \gamma^2(t) = o(\|h_t\|_{\bar{q}_t}^2)$, as desired. $\blacksquare$

Proof of Proposition 54 By the finite intersection property, $\bigcap_t \text{co}(\text{supp } \mu_t) \neq \emptyset$. In particular,

$\bigcap_t \text{co}(\text{supp } \mu_t) = \{p\}$ for some $p \in [0, 1]$. For, first observe that $\text{diam}(\text{supp } \mu_t) = \text{diam}(\text{co}(\text{supp } \mu_t))$.

If $p', p'' \in \bigcap_t \text{co}(\text{supp } \mu_t)$ with $p' \neq p''$, then $\text{diam}(\text{co}(\text{supp } \mu_t)) > |p'' - p'| > 0$, which contradicts the hypotheses.

For all $t > 0$, denote by a_t and b_t the minimum and maximum of $\text{supp } \mu_t$, so that $\text{co}(\text{supp } \mu_t) \subseteq [a_t, b_t]$. Since $a_t \leq p \leq b_t$ and $b_t - a_t = \text{diam}(\text{supp } \mu_t) \rightarrow 0$, we have $\lim_{t \downarrow 0} a_t = \lim_{t \downarrow 0} b_t = p$. For all $f \in C([0, 1])$ and all $t > 0$, there exists $c_t \in [a_t, b_t]$ such that

$$\int f d\mu_t = \int_{a_t}^{b_t} f d\mu_t = f(c_t).$$

Then $a_t \leq c_t \leq b_t$ implies $c_t \rightarrow p$ (as $t \downarrow 0$) and continuity of f delivers $f(c_t) \rightarrow f(p)$, that is,

$$\int f d\mu_t \rightarrow \int f d\delta_p.$$

Thus $\mu_t \xrightarrow{w} \delta_p$. $\blacksquare$

B Taylor

B.1 Preliminaries

Let $\Omega = \{1, 2, \dots, N\}$ be a finite state space $\Delta = \Delta(\Omega)$ the probability simplex. If μ is a Borel probability measure on Δ ,

$$\bar{\mu} = \int_{\Delta} q d\mu(q) \in \Delta$$

is called *barycenter* of μ .

Let $I \subseteq \mathbb{R}$ be a nonsingleton interval and consider two functions $u : I \rightarrow \mathbb{R}$ and $v : I \rightarrow \mathbb{R}$ that are continuous on I and thrice continuously differentiable on its interior, with $u', v' > 0$. Set

$$C(\mathbf{x}, \mu) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{\omega \in \Omega} u(x_{\omega}) q_{\omega} \right) \right) d\mu(q) \right) \quad \forall \mathbf{x} \in I^{\Omega}.$$

Notice that for each $\mathbf{x} \in I^{\Omega}$, $u(\mathbf{x}) = \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} \in u(I)^{\Omega}$ (and $u(I)$ is an interval since u is continuous), and

$$\begin{aligned} \langle u(\mathbf{x}), \cdot \rangle : \Delta &\rightarrow \mathbb{R} \\ q &\mapsto \sum_{\omega \in \Omega} u(x_{\omega}) q_{\omega} = \langle u(x), q \rangle \end{aligned}$$

is affine, hence it is continuous, bounded, and measurable, with range in $u(I)$. Therefore,

$$\begin{aligned} (v \circ u^{-1}) \circ \langle u(x), \cdot \rangle : \Delta &\rightarrow \mathbb{R} \\ q &\mapsto v(u^{-1}(\sum_{\omega \in \Omega} u(x_{\omega}) q_{\omega})) \end{aligned}$$

is continuous, bounded, and measurable too, with range in $v(I)$. Hence,

$$\int_{\Delta} v(u^{-1}(\langle u(x), q \rangle)) d\mu(q) \in v(I)$$

is well defined, and so is

$$v^{-1} \left(\int_{\Delta} v(u^{-1}(\langle u(x), q \rangle)) d\mu(q) \right) \in I.$$

The next two lemmas on derivatives will play a key role in the derivation of the Taylor approximation.

Lemma 62 *Let O be an open subset of a finite dimensional euclidean space, (S, Σ) be a measurable space, and $f : O \times S \rightarrow \mathbb{R}$ be a function with the following properties:*

- (a) *for each $\mathbf{x} \in O$, $s \mapsto f(\mathbf{x}, s)$ is Σ -measurable;*
- (b) *for each $s \in S$, $\mathbf{x} \mapsto f(\mathbf{x}, s)$ is k -times continuously differentiable on O ;*
- (c) *the functions f and $\partial_{\alpha} f$ are bounded (hence uniformly bounded by M) on $O \times S$ for every multi-index α such that $|\alpha| \leq k$.*

Then, for each probability measure P on Σ :

- (i) *the function defined on O by*

$$F_P(\mathbf{x}) = \int_S f(\mathbf{x}, s) dP(s)$$

is k -times continuously differentiable;

- (ii) *the functions $s \mapsto \partial_{\alpha} f(\mathbf{x}, s)$ are Σ -measurable for all $\mathbf{x} \in O$, with*

$$\partial_{\alpha} F_P(\mathbf{x}) = \int_S \partial_{\alpha} f(\mathbf{x}, s) dP(s) \tag{82}$$

for all $\mathbf{x} \in O$ and α such that $|\alpha| \leq k$.

Moreover, as $\mathbf{y}$ ranges in O , Q ranges in the set of all probability measures on Σ , β ranges in the set of all multi-indexes of length not greater than k , $\sup_{\mathbf{y}, Q, \beta} |\partial_\beta F_Q(\mathbf{y})| \leq M$.

Proof. First observe that, by (a) and (c), for each $\mathbf{x} \in O$, $f(\mathbf{x}, \cdot)$ is bounded and measurable, then F_P is well defined on O for every P in the set $\Delta^\sigma(S, \Sigma)$ of all probability measures on Σ .

Let $k = 1$, and arbitrarily choose a multi-index α with $|\alpha| \leq 1$. All components of α except one, say the j -th, are zero, and $\alpha_j = 1$. Denote by $\mathbf{e}^j$ the j -th element of the canonical basis of the euclidean space.

For each $\mathbf{x} \in O$, there is $\eta = \eta_{\mathbf{x}} > 0$ such that $\mathbf{x} + t\mathbf{e}^j \in O$ for all $t \in (-\eta, \eta)$. For each vanishing sequence $\{t_n\}_{n \in \mathbb{N}} \in (-\eta, \eta) \setminus \{0\}$ of real numbers

$$\partial_j f(\mathbf{x}, s) = \lim_{n \rightarrow \infty} \frac{f(\mathbf{x} + t_n \mathbf{e}^j, s) - f(\mathbf{x}, s)}{t_n}$$

for all $s \in S$. Thus, for each $\mathbf{x} \in O$, $\partial_j f(\mathbf{x}, \cdot)$ is the statewise limit of a sequence $\{t_n^{-1} [f(\mathbf{x} + t_n \mathbf{e}^j, \cdot) - f(\mathbf{x}, \cdot)]\}_{n \in \mathbb{N}}$ of measurable functions, and it is measurable.

For each $\mathbf{x} \in O$, $t \in (-\eta_{\mathbf{x}}, \eta_{\mathbf{x}}) \setminus \{0\}$, and $s \in S$, by the Mean Value Theorem on the segment $[x_j, x_j + t]$ (if $t > 0$, or $[x_j + t, x_j]$ if $t < 0$), there is $\lambda \in (0, 1)$ such that

$$\frac{|f(\mathbf{x} + t\mathbf{e}^j, s) - f(\mathbf{x}, s)|}{|t|} = |\partial_j f(\mathbf{x} + \lambda t\mathbf{e}^j, s)| \leq M.$$

Then, for each $\mathbf{x} \in O$, and each vanishing sequence $\{t_n\}_{n \in \mathbb{N}} \in (-\eta_{\mathbf{x}}, \eta_{\mathbf{x}}) \setminus \{0\}$, the sequence of functions $\{t_n^{-1} [f(\mathbf{x} + t_n \mathbf{e}^j, \cdot) - f(\mathbf{x}, \cdot)]\}_{n \in \mathbb{N}}$ is uniformly bounded by M . The Dominated Convergence Theorem implies

$$\begin{aligned} \int \partial_j f(\mathbf{x}, s) dP(s) &= \int \lim_{n \rightarrow \infty} \frac{f(\mathbf{x} + t_n \mathbf{e}^j, s) - f(\mathbf{x}, s)}{t_n} dP(s) \\ \text{DCT} \quad &= \lim_{n \rightarrow \infty} \int \frac{f(\mathbf{x} + t_n \mathbf{e}^j, s) - f(\mathbf{x}, s)}{t_n} dP(s) \\ &= \lim_{n \rightarrow \infty} \frac{\int f(\mathbf{x} + t_n \mathbf{e}^j, s) dP(s) - \int f(\mathbf{x}, s) dP(s)}{t_n} \\ &= \lim_{n \rightarrow \infty} \frac{F_P(\mathbf{x} + t_n \mathbf{e}^j) - F_P(\mathbf{x})}{t_n} \end{aligned}$$

for every $P \in \Delta^\sigma(S, \Sigma)$. Since this is true for all vanishing sequences $\{t_n\}_{n \in \mathbb{N}} \in (-\eta_{\mathbf{x}}, \eta_{\mathbf{x}}) \setminus \{0\}$, F_P has partial derivative at $\mathbf{x}$, in direction j , given by

$$\partial_j F_P(\mathbf{x}) = \int \partial_j f(\mathbf{x}, s) dP(s). \quad (83)$$

Summing up, $\partial_\alpha f(\mathbf{x}, \cdot)$ is measurable (and bounded) for all $\mathbf{x} \in O$, and, for each $P \in \Delta^\sigma(S, \Sigma)$, the function $F_P : O \rightarrow \mathbb{R}$ has α -th partial derivative given by (82) at every $\mathbf{x} \in O$. Moreover,

$$\sup_{x \in O, P \in \Delta^\sigma(S, \Sigma)} |\partial_\alpha F_P(\mathbf{x})| = \sup_{x \in O, P \in \Delta^\sigma(S, \Sigma)} \left| \int \partial_\alpha f(\mathbf{x}, s) dP(s) \right| \leq \sup_{x \in O, P \in \Delta^\sigma(S, \Sigma)} \int |\partial_j f(\mathbf{x}, s)| dP(s) \leq M$$

and the generality of α implies $\sup_{x \in O, P \in \Delta^\sigma(S, \Sigma), |\alpha| \leq 1} |\partial_\alpha F_P(\mathbf{x})| \leq M$.

Let $\mathbf{x}_n \rightarrow \mathbf{x}$ in O . For each $s \in S$, by (b),

$$\lim_{n \rightarrow \infty} \partial_\alpha f(\mathbf{x}_n, s) = \partial_\alpha f(\mathbf{x}, s)$$

then the sequence of measurable functions $\{\partial_\alpha f(\mathbf{x}_n, \cdot)\}_{n \in \mathbb{N}}$ statewise converges to $\partial_\alpha f(\mathbf{x}, \cdot)$ and, by (c), it is uniformly bounded, another application of the Dominated Convergence Theorem yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \partial_\alpha F_P(\mathbf{x}_n) &= \lim_{n \rightarrow \infty} \int \partial_\alpha f(\mathbf{x}_n, s) dP(s) \\ \text{DCT} \quad &= \int \partial_\alpha f(\mathbf{x}, s) dP(s) = \partial_\alpha F_P(\mathbf{x}). \end{aligned}$$

This shows the continuity of $\partial_\alpha F_P$.

Assume the result is true for $k \in \mathbb{N}$. Next we show it holds for $k + 1$. The induction hypothesis guarantees that points (i) and (ii) hold verbatim for all the derivatives of order not greater than k and $\sup_{x \in O, P \in \Delta^\sigma(S, \Sigma), |\alpha| \leq k} |\partial_\alpha F_P(\mathbf{x})| \leq M$. Let $|\beta| = k + 1$ and j be the first non-zero component of β , then $\beta = \alpha + \mathbf{e}^j$ where α is a multi-index of length k . Set

$$g(\mathbf{x}, s) = \partial_\alpha f(\mathbf{x}, s) \quad \forall (\mathbf{x}, s) \in O \times S.$$

By the induction hypothesis, and the assumptions on f :

- for each $\mathbf{x} \in O$, $s \mapsto g(\mathbf{x}, s) = \partial_\alpha f(\mathbf{x}, s)$ is measurable;
- for each $s \in S$, $\mathbf{x} \mapsto g(\mathbf{x}, s)$ is 1-time continuously differentiable on O , since $\mathbf{x} \mapsto f(\mathbf{x}, s)$ is $k + 1$ -times continuously differentiable on O ;
- the function $g = \partial_\alpha f$ and all its first partial derivatives are uniformly bounded by M on $O \times S$.

Then (by the initial step), for each probability measure P on Σ , the function defined on O by

$$G_P(\mathbf{x}) = \int g(\mathbf{x}, s) dP(s) = \partial_\alpha F_P(\mathbf{x})$$

is 1-time continuously differentiable and the function $s \mapsto \partial_j g(\mathbf{x}, s) = \partial_\beta f(\mathbf{x}, s)$ is measurable for all $\mathbf{x} \in O$, with

$$\partial_j G_P(\mathbf{x}) = \int \partial_j g(\mathbf{x}, s) dP(s) \quad \text{i.e.} \quad \partial_\beta F_P(\mathbf{x}) = \int \partial_\beta f(\mathbf{x}, s) dP(s)$$

for all $\mathbf{x} \in O$. Moreover, again by the initial step,

$$\sup_{x \in O, P \in \Delta^\sigma(S, \Sigma)} |\partial_\beta F_P(\mathbf{x})| = \sup_{x \in O, P \in \Delta^\sigma(S, \Sigma)} |\partial_j G_P(\mathbf{x})| \leq M$$

which concludes the proof, thanks to the generality of β . ■

Set $\phi = v \circ u^{-1} : u(I) \rightarrow v(I)$ and $\psi = v^{-1} : v(I) \rightarrow I$, clearly they are strictly increasing and continuous. Set $W = \text{int } I$.

Lemma 63 *The functions $\phi : u(W) \rightarrow v(W)$ and $\psi : v(W) \rightarrow W$ are thrice continuously differentiable.*

In particular, for each $w \in W$,

$$\phi'(u(w)) = \frac{v'(w)}{u'(w)}, \quad \phi''(u(w)) = \frac{v''(w)}{u'(w)^2} - v'(w) \frac{u''(w)}{u'(w)^3}, \quad (85a)$$

$$\psi'(\phi(u(w))) = \frac{1}{v'(w)}, \quad \psi''(\phi(u(w))) = -\frac{v''(w)}{v'(w)^3}, \quad (85b)$$

and there exist $\varepsilon > 0$ such that $[w - \varepsilon, w + \varepsilon] \subseteq W$ and $M > 1$ such that the absolute values of u , v , ϕ , and ψ – as well as their first, second, and third derivatives – are bounded by M on $[w - \varepsilon, w + \varepsilon]$, $u([w - \varepsilon, w + \varepsilon])$, and $v([w - \varepsilon, w + \varepsilon])$, respectively.

Proof. Since u and v are strictly increasing and continuous, then $u(\text{int } I) = \text{int } u(I)$ and $v(\text{int } I) = \text{int } v(I)$, that is

$$u(W) = \text{int } u(I) \text{ and } v(W) = \text{int } v(I).$$

Moreover, $u, v \in \mathcal{C}^3(W)$ and $u', v' > 0$ imply $u^{-1} \in \mathcal{C}^3(u(W))$ and $(u^{-1})' > 0$, $\phi = v \circ u^{-1} \in \mathcal{C}^3(u(W))$ and $\phi' > 0$, $\psi = v^{-1} \in \mathcal{C}^3(v(W))$ and $\psi' > 0$. The relations (85a) and (85b) descend from the usual calculus of the derivatives of inverse functions and composite functions. The final part descends from $\mathcal{C}^3$ differentiability and the Weierstrass Theorem. ■

B.2 Approximation

Theorem 64 Let $\mathbf{x} \in (-\varepsilon, \varepsilon)^\Omega$, then

$$\left| C(w + \mathbf{x}, \mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2} \lambda_u(w) \sigma_{\bar{\mu}}^2(\mathbf{x}) - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \sigma_{\mu}^2(\langle \mathbf{x}, \cdot \rangle) \right] \right| \leq L |\mathbf{x}|^3 \quad (86)$$

for all Borel probability measures μ on Δ , where ε and M are as in Lemma 63 and

$$L = \frac{125M^{13} + 75M^9 + 5M^5}{6} |\Omega|^{3/2}.$$

Before entering the proof's details notice that

$$\begin{aligned} & -\frac{1}{2} \lambda_u(w) \sigma_{\bar{\mu}}^2(\mathbf{x}) - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \sigma_{\mu}^2(\langle \mathbf{x}, \cdot \rangle) \\ &= -\frac{1}{2} \lambda_u(w) \mathbf{x}^\top [\sigma_{\bar{\mu}}(\mathbf{e}^k, \mathbf{e}^j)] \mathbf{x} - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \mathbf{x}^\top [\sigma_{\mu}(\langle \mathbf{e}^k, \cdot \rangle, \langle \mathbf{e}^j, \cdot \rangle)] \mathbf{x} \\ &= \frac{1}{2} \mathbf{x}^\top [-\lambda_u(w) \sigma_{\bar{\mu}}(\mathbf{e}^k, \mathbf{e}^j) - (\lambda_v(w) - \lambda_u(w)) \sigma_{\mu}(\langle \mathbf{e}^k, \cdot \rangle, \langle \mathbf{e}^j, \cdot \rangle)] \mathbf{x} \\ &= \frac{1}{2} \mathbf{x}^\top [(\lambda_u(w) - \lambda_v(w)) \sigma_{\mu}(\langle \mathbf{e}^k, \cdot \rangle, \langle \mathbf{e}^j, \cdot \rangle) - \lambda_u(w) \sigma_{\bar{\mu}}(\mathbf{e}^k, \mathbf{e}^j)] \mathbf{x} = \mathfrak{X} \end{aligned}$$

and

$$\begin{aligned} \sigma_{\mu}(\langle \mathbf{e}^k, \cdot \rangle, \langle \mathbf{e}^j, \cdot \rangle) &= \int_{\Delta} \langle \mathbf{e}^k, q \rangle \langle \mathbf{e}^j, q \rangle d\mu(q) - \left(\int_{\Delta} \langle \mathbf{e}^k, q \rangle d\mu(q) \right) \left(\int_{\Delta} \langle \mathbf{e}^j, q \rangle d\mu(q) \right) \\ &= \int_{\Delta} q_k q_j d\mu(q) - \bar{\mu}_k \bar{\mu}_j \\ \sigma_{\bar{\mu}}(\mathbf{e}^k, \mathbf{e}^j) &= \sum_{i=1}^N \mathbf{e}_i^k \mathbf{e}_i^j \bar{\mu}_i - \left(\sum_{i=1}^N \mathbf{e}_i^k \bar{\mu}_i \right) \left(\sum_{i=1}^N \mathbf{e}_i^j \bar{\mu}_i \right) = \begin{cases} -\bar{\mu}_k \bar{\mu}_j & k \neq j \\ \bar{\mu}_j - \bar{\mu}_k \bar{\mu}_j & k = j \end{cases} = \delta_{kj} \bar{\mu}_j - \bar{\mu}_k \bar{\mu}_j \end{aligned}$$

finally

$$\begin{aligned} \mathfrak{X} &= \frac{1}{2} \mathbf{x}^\top \left[(\lambda_u(w) - \lambda_v(w)) \left(\int_{\Delta} q_k q_j d\mu(q) - \bar{\mu}_k \bar{\mu}_j \right) - \lambda_u(w) (\delta_{kj} \bar{\mu}_j - \bar{\mu}_k \bar{\mu}_j) \right] \mathbf{x} \\ &= \frac{1}{2} \mathbf{x}^\top \left[\lambda_u(w) \int_{\Delta} q_k q_j d\mu(q) - \lambda_u(w) \bar{\mu}_k \bar{\mu}_j - \lambda_v(w) \int_{\Delta} q_k q_j d\mu(q) + \lambda_v(w) \bar{\mu}_k \bar{\mu}_j - \lambda_u(w) \delta_{kj} \bar{\mu}_j + \lambda_u(w) \bar{\mu}_k \bar{\mu}_j \right] \mathbf{x} \\ &= \frac{1}{2} \mathbf{x}^\top \left[(\lambda_u(w) - \lambda_v(w)) \int_{\Delta} q_k q_j d\mu(q) + \lambda_v(w) \bar{\mu}_k \bar{\mu}_j - \lambda_u(w) \delta_{kj} \bar{\mu}_j \right] \mathbf{x} \end{aligned}$$

Proof. Notice that, for all $\mathbf{x} \in (-\varepsilon, \varepsilon)^\Omega = O$, $w\mathbf{1} + \mathbf{x} \in (w - \varepsilon, w + \varepsilon)^\Omega \subseteq I^\Omega$.

Define

$$\begin{aligned} g : O \times \Delta &\rightarrow \mathbb{R} \\ (\mathbf{x}, q) &\mapsto \langle u(w + \mathbf{x}), q \rangle = \sum_{i=1}^N u(w + x_i) q_i \in (u(w - \varepsilon), u(w + \varepsilon)) \end{aligned}$$

next we show that g satisfies assumptions (a), (b), (c) of Lemma 62 for 3-times continuously differentiable functions.

- (a) For each $\mathbf{x} \in O$, $q \mapsto g(\mathbf{x}, q)$ is continuous, bounded, and measurable.
- (b) For each $q \in \Omega$, $\mathbf{x} \mapsto g(\mathbf{x}, q)$ is thrice continuously differentiable on O . In fact, given $q \in \Delta$, for all $\mathbf{x} \in O$ and all $j, k, l \in \{1, 2, \dots, N\}$

$$\begin{aligned}\partial_j g(\mathbf{x}, q) &= q_j u'(w + x_j) \\ \partial_{kj} g(\mathbf{x}, q) &= \begin{cases} 0 & k \neq j \\ q_j u''(w + x_j) & k = j \end{cases} = q_j u''(w + x_j) \delta_{kj} \\ \partial_{lkj} g(\mathbf{x}, q) &= \begin{cases} 0 & k \neq j \\ 0 & l \neq k = j \\ q_j u'''(w + x_j) & l = k = j \end{cases} = q_j u'''(w + x_j) \delta_{lkj}\end{aligned}$$

and the functions defined by the above equations (for fixed q and j, k, l) are continuous on O .

- (c) The functions g , $\partial_j g$, $\partial_{kj} g$, and $\partial_{lkj} g$ are bounded on $O \times \Omega$ for all $j, k, l \in \{1, 2, \dots, N\}$; in fact, given $j, k, l \in \{1, 2, \dots, N\}$, for all $(\mathbf{x}, q) \in O \times \Delta$ (choosing M like in Lemma 63)

$$\begin{aligned}|g(\mathbf{x}, q)| &= \left| \sum_{i=1}^N u(w + x_i) q_i \right| \leq \sum_{i=1}^N |u(w + x_i)| q_i \leq M \\ |\partial_j g(\mathbf{x}, q)| &= |q_j u'(w + x_j)| \leq M \\ |\partial_{kj} g(\mathbf{x}, q)| &= |q_j u''(w + x_j) \delta_{kj}| \leq M \\ |\partial_{lkj} g(\mathbf{x}, q)| &= |q_j u'''(w + x_j) \delta_{lkj}| \leq M.\end{aligned}$$

Consider

$$\begin{aligned}f &= \phi \circ g : O \times \Delta \rightarrow \mathbb{R} \\ (\mathbf{x}, q) &\mapsto \phi \circ g(\mathbf{x}, q) \in (v(w - \varepsilon), v(w + \varepsilon))\end{aligned}$$

next we show that f satisfies assumptions (a), (b), (c) of Lemma 62 for 3-times continuously differentiable functions.

- (a) For each $\mathbf{x} \in O$, $q \mapsto \phi \circ g(\mathbf{x}, q)$ is continuous, bounded, and measurable.
- (b) For each $q \in \Omega$, $\mathbf{x} \mapsto \phi \circ g(\mathbf{x}, q)$ is thrice continuously differentiable on O ; in fact, given $q \in \Delta$, for all $\mathbf{x} \in O$ and all $j, k, l \in \{1, 2, \dots, N\}$

$$\begin{aligned}\partial_j (\phi \circ g)(\mathbf{x}, q) &= \phi'(g(\mathbf{x}, q)) \partial_j g(\mathbf{x}, q) \\ \partial_{kj} (\phi \circ g)(\mathbf{x}, q) &= \phi''(g(\mathbf{x}, q)) \partial_k g(\mathbf{x}, q) \partial_j g(\mathbf{x}, q) + \phi'(g(\mathbf{x}, q)) \partial_{kj} g(\mathbf{x}, q) \\ \partial_{lkj} (\phi \circ g)(\mathbf{x}, q) &= \phi'''(g(\mathbf{x}, q)) \partial_l g(\mathbf{x}, q) \partial_k g(\mathbf{x}, q) \partial_j g(\mathbf{x}, q) + \phi''(g(\mathbf{x}, q)) (\partial_{lk} g(\mathbf{x}, q) \partial_j g(\mathbf{x}, q) + \partial_k g(\mathbf{x}, q) \partial_{lj} g(\mathbf{x}, q)) \\ &\quad + \phi''(g(\mathbf{x}, q)) \partial_l g(\mathbf{x}, q) \partial_{kj} g(\mathbf{x}, q) + \phi'(g(\mathbf{x}, q)) \partial_{lkj} g(\mathbf{x}, q) \\ &= \phi'''(g(\mathbf{x}, q)) \partial_l g(\mathbf{x}, q) \partial_k g(\mathbf{x}, q) \partial_j g(\mathbf{x}, q) + \phi'(g(\mathbf{x}, q)) \partial_{lkj} g(\mathbf{x}, q) \\ &\quad + \phi''(g(\mathbf{x}, q)) [\partial_j g(\mathbf{x}, q) \partial_{lk} g(\mathbf{x}, q) + \partial_k g(\mathbf{x}, q) \partial_{lj} g(\mathbf{x}, q) + \partial_l g(\mathbf{x}, q) \partial_{kj} g(\mathbf{x}, q)]\end{aligned}$$

and the functions defined by the above equations (for fixed q and j, k, l) are continuous on O .

- (c) The functions $\phi \circ g$, $\partial_j (\phi \circ g)$, $\partial_{kj} (\phi \circ g)$, and $\partial_{lkj} (\phi \circ g)$ are bounded by $5M^4$ on $O \times \Omega$ for all $j, k, l \in \{1, 2, \dots, N\}$; in fact, given $j, k, l \in \{1, 2, \dots, N\}$, for all $(\mathbf{x}, q) \in O \times \Delta$ (choosing M

like in Lemma 63 and observing $g(\mathbf{x}, q) \in (u(w - \varepsilon), u(w + \varepsilon))$

$$|(\phi \circ g)(\mathbf{x}, q)| = |\phi(g(\mathbf{x}, q))| \leq M \leq 5M^4$$

$$|\partial_j(\phi \circ g)(\mathbf{x}, q)| = |\phi'(g(\mathbf{x}, q))| |\partial_j g(\mathbf{x}, q)| \leq M^2 \leq 5M^4$$

$$|\partial_{kj}(\phi \circ g)(\mathbf{x}, q)| \leq |\phi''(g(\mathbf{x}, q))| \partial_k g(\mathbf{x}, q) \partial_j g(\mathbf{x}, q)| + |\phi'(g(\mathbf{x}, q))| \partial_{kj} g(\mathbf{x}, q)| \leq M^3 + M^2 \leq 5M^4$$

$$\begin{aligned} |\partial_{lkj}(\phi \circ g)(\mathbf{x}, q)| &\leq |\phi'''(g(\mathbf{x}, q))| \partial_l g(\mathbf{x}, q) \partial_k g(\mathbf{x}, q) \partial_j g(\mathbf{x}, q)| + |\phi'(g(\mathbf{x}, q))| \partial_{lkj} g(\mathbf{x}, q)| \\ &\quad + |\phi''(g(\mathbf{x}, q))| [|\partial_j g(\mathbf{x}, q) \partial_{lk} g(\mathbf{x}, q)| + |\partial_k g(\mathbf{x}, q) \partial_{lj} g(\mathbf{x}, q)| + |\partial_l g(\mathbf{x}, q) \partial_{kj} g(\mathbf{x}, q)|] \\ &\leq M^4 + M^2 + 3M^3 \leq 5M^4. \end{aligned}$$

Then (by Lemma 62), for each Borel probability measure μ on Δ the function defined on O by

$$F_\mu(\mathbf{x}) = F(\mathbf{x}) = \int f(\mathbf{x}, q) d\mu(q) = \int_\Delta \phi \left(\sum_{i=1}^N u(w + x_i) q_i \right) d\mu(q)$$

is thrice continuously differentiable; the functions $q \mapsto \partial_\alpha f(\mathbf{x}, q)$ are measurable for all $\mathbf{x} \in O$, with

$$\partial_\alpha F(\mathbf{x}) = \int \partial_\alpha f(\mathbf{x}, q) d\mu(q)$$

for all $\mathbf{x} \in O$ and α such that $|\alpha| \leq 3$; $\sup_{\mathbf{x} \in O, \mu \in \Delta^\sigma(\Delta, \mathcal{B}), |\alpha| \leq 3} |\partial_\alpha F_\mu(\mathbf{x})| \leq 5M^4$.

Finally, for all $\mathbf{x} \in O$ and all $q \in \Delta$, $g(\mathbf{x}, q) \in (u(w - \varepsilon), u(w + \varepsilon))$ implies $f(\mathbf{x}, q) = \phi(g(\mathbf{x}, q)) \in v(u^{-1}((u(w - \varepsilon), u(w + \varepsilon)))) = (v(w - \varepsilon), v(w + \varepsilon))$ and $F(\mathbf{x}) \in (v(w - \varepsilon), v(w + \varepsilon))$.²⁰ Thus

$$c_\mu(\mathbf{x}) = \psi \circ F_\mu(\mathbf{x}) \in (w - \varepsilon, w + \varepsilon) \quad \forall \mathbf{x} \in O$$

is well defined and thrice continuously differentiable on $O = (-\varepsilon, \varepsilon)^N$.

Next we explicitly compute its derivatives, using repeatedly the relations obtained above as well as those provided by Lemma 63. As we did for F we just write c rather than c_μ .

For all $\mathbf{x} \in O$ and all $j, k, l \in \{1, 2, \dots, N\}$

$$\begin{aligned} \partial_j c(\mathbf{x}) &= \partial_j(\psi \circ F)(\mathbf{x}) = \psi'(F(\mathbf{x})) \partial_j F(\mathbf{x}) \\ \partial_{kj} c(\mathbf{x}) &= \partial_{kj}(\psi \circ F)(\mathbf{x}) = \psi''(F(\mathbf{x})) \partial_k F(\mathbf{x}) \partial_j F(\mathbf{x}) + \psi'(F(\mathbf{x})) \partial_{kj} F(\mathbf{x}) \\ \partial_{lkj} c(\mathbf{x}) &= \partial_{lkj}(\psi \circ F)(\mathbf{x}) = \psi'''(F(\mathbf{x})) \partial_l F(\mathbf{x}) \partial_k F(\mathbf{x}) \partial_j F(\mathbf{x}) + \psi'(F(\mathbf{x})) \partial_{lkj} F(\mathbf{x}) \\ &\quad + \psi''(F(\mathbf{x})) [\partial_j F(\mathbf{x}) \partial_{lk} F(\mathbf{x}) + \partial_k F(\mathbf{x}) \partial_{lj} F(\mathbf{x}) + \partial_l F(\mathbf{x}) \partial_{kj} F(\mathbf{x})] \end{aligned}$$

in particular

$$|c(\mathbf{x})| = |\psi(F(\mathbf{x}))| \leq M \leq 205M^{13}$$

$$|\partial_j c(\mathbf{x})| = |\psi'(F(\mathbf{x}))| |\partial_j F(\mathbf{x})| \leq 5M^5 \leq 205M^{13}$$

$$|\partial_{kj} c(\mathbf{x})| \leq |\psi''(F(\mathbf{x}))| \partial_k F(\mathbf{x}) \partial_j F(\mathbf{x})| + |\psi'(F(\mathbf{x}))| \partial_{kj} F(\mathbf{x})| \leq 25M^9 + 5M^5 \leq 205M^{13}$$

$$\begin{aligned} |\partial_{lkj} c(\mathbf{x})| &\leq |\psi'''(F(\mathbf{x}))| \partial_l F(\mathbf{x}) \partial_k F(\mathbf{x}) \partial_j F(\mathbf{x})| + |\psi'(F(\mathbf{x}))| \partial_{lkj} F(\mathbf{x})| \\ &\quad + |\psi''(F(\mathbf{x}))| [|\partial_j F(\mathbf{x}) \partial_{lk} F(\mathbf{x})| + |\partial_k F(\mathbf{x}) \partial_{lj} F(\mathbf{x})| + |\partial_l F(\mathbf{x}) \partial_{kj} F(\mathbf{x})|] \\ &\leq 125M^{13} + 5M^5 + 75M^9 \leq 205M^{13} \end{aligned}$$

²⁰Clearly, $F(\mathbf{x}) \in [v(w - \varepsilon), v(w + \varepsilon)]$, if $F(\mathbf{x}) = v(w - \varepsilon)$, then $\int (f(\mathbf{x}, q) - v(w - \varepsilon)) d\mu(q) = 0$ and $f(\mathbf{x}, q) = v(w - \varepsilon)$ for μ almost all $q \in \Delta$, a contradiction. A similar contradiction derives from assuming $F(\mathbf{x}) = v(w + \varepsilon)$.

by Taylor's inequality (see, e.g., [4, p. 95])

$$\begin{aligned} \left| c(\mathbf{x}) - \left[c(\mathbf{0}) + \nabla c(\mathbf{0}) \mathbf{x} + \frac{1}{2} \mathbf{x}^\top \nabla^2 c(\mathbf{0}) \mathbf{x} \right] \right| &\leq \frac{125M^{13} + 75M^9 + 5M^5}{6} \left(\sum_{i=1}^N |x_i| \right)^3 \\ &\leq \frac{125M^{13} + 75M^9 + 5M^5}{6} N^{3/2} |\mathbf{x}|^3 \\ &= L |\mathbf{x}|^3 \end{aligned} \quad (88)$$

Notice that L does not depend on μ .

Since

$$c(\mathbf{x}) = v^{-1} \left(\int_{\Delta} v \left(u^{-1} \left(\sum_{i=1}^N u(w + x_i) q_i \right) \right) d\mu(q) \right) = C(w + \mathbf{x}, \mu) \quad \forall \mathbf{x} \in O,$$

it only remains to explicitly evaluate $c(\mathbf{0})$, $\nabla c(\mathbf{0}) \mathbf{x}$, and $\mathbf{x}^\top \nabla^2 c(\mathbf{0}) \mathbf{x}$.

First,

$$\begin{aligned} g(\mathbf{0}, q) &= u(w) \quad \forall q \in \Delta \\ F(\mathbf{0}) &= \int \phi(g(\mathbf{0}, q)) d\mu(q) = \phi(u(w)) \end{aligned}$$

then

$$c(\mathbf{0}) = v^{-1}(\phi(u(w))) = w.$$

Second, for all $j \in \{1, 2, \dots, N\}$,

$$\begin{aligned} \partial_j g(\mathbf{0}, q) &= q_j u'(w) \quad \forall q \in \Delta \\ \partial_j F(\mathbf{0}) &= \int \partial_j \phi(g(\mathbf{0}, q)) d\mu(q) = \int \phi'(g(\mathbf{0}, q)) \partial_j g(\mathbf{0}, q) d\mu(q) \\ &= \int \phi'(u(w)) q_j u'(w) d\mu(q) = \phi'(u(w)) u'(w) \bar{\mu}_j \\ &= v'(w) \bar{\mu}_j \end{aligned}$$

then

$$\partial_j c(\mathbf{0}) = \psi'(F(\mathbf{0})) \partial_j F(\mathbf{0}) = \psi'(\phi(u(w))) v'(w) \bar{\mu}_j = \bar{\mu}_j.$$

Finally, for all $j, k \in \{1, 2, \dots, N\}$,

$$\begin{aligned} \partial_{kj} g(\mathbf{0}, q) &= q_j u''(w) \delta_{kj} \quad \forall q \in \Delta \\ \partial_{kj} F(\mathbf{0}) &= \int \partial_{kj} \phi(g(\mathbf{0}, q)) d\mu(q) = \int [\phi''(g(\mathbf{0}, q)) \partial_k g(\mathbf{0}, q) \partial_j g(\mathbf{0}, q) + \phi'(g(\mathbf{0}, q)) \partial_{kj} g(\mathbf{0}, q)] d\mu(q) \\ &= \int [\phi''(u(w)) q_k u'(w) q_j u'(w) + \phi'(u(w)) q_j u''(w) \delta_{kj}] d\mu(q) \\ &= \phi''(u(w)) u'(w)^2 \int q_k q_j d\mu(q) + \phi'(u(w)) u''(w) \delta_{kj} \int q_j d\mu(q) \\ &= \phi''(u(w)) u'(w)^2 \int q_k q_j d\mu(q) + \phi'(u(w)) u''(w) \delta_{kj} \bar{\mu}_j \end{aligned}$$

then

$$\begin{aligned}
\partial_{kj}c(\mathbf{0}) &= \psi''(F(\mathbf{0})) \partial_k F(\mathbf{0}) \partial_j F(\mathbf{0}) + \psi'(F(\mathbf{0})) \partial_{kj} F(\mathbf{0}) \\
&= \psi''(\phi(u(w))) v'(w) \bar{\mu}_k v'(w) \bar{\mu}_j + \psi'(\phi(u(w))) \left(\phi''(u(w)) u'(w)^2 \int q_k q_j d\mu(q) + \phi'(u(w)) u''(w) \delta_{kj} \bar{\mu}_j \right) \\
&= -\frac{v''(w)}{v'(w)^3} v'(w)^2 \bar{\mu}_k \bar{\mu}_j + \frac{1}{v'(w)} \left(\left(\frac{v''(w)}{u'(w)^2} - v'(w) \frac{u''(w)}{u'(w)^3} \right) u'(w)^2 \int q_k q_j d\mu(q) + \frac{v'(w)}{u'(w)} u''(w) \delta_{kj} \bar{\mu}_j \right) \\
&= -\frac{v''(w)}{v'(w)} \bar{\mu}_k \bar{\mu}_j + \left(\frac{v''(w)}{v'(w)} - \frac{u''(w)}{u'(w)} \right) \int q_k q_j d\mu(q) + \frac{u''(w)}{u'(w)} \delta_{kj} \bar{\mu}_j \\
&= (\lambda_u(w) - \lambda_v(w)) \int q_k q_j d\mu(q) + \lambda_v(w) \bar{\mu}_k \bar{\mu}_j - \lambda_u(w) \delta_{kj} \bar{\mu}_j
\end{aligned}$$

as wanted. In fact

$$\left| c(\mathbf{x}) - \left[c(\mathbf{0}) + \nabla c(\mathbf{0}) \mathbf{x} + \frac{1}{2} \mathbf{x}^\top \nabla^2 c(\mathbf{0}) \mathbf{x} \right] \right| \leq L |\mathbf{x}|^3$$

then amounts to

$$\left| C(w + \mathbf{x}, \mu) - \left[w + \langle \mathbf{x}, \bar{\mu} \rangle + \frac{1}{2} \mathbf{x}^\top \left[(\lambda_u(w) - \lambda_v(w)) \int_{\Delta} q_k q_j d\mu(q) + \lambda_v(w) \bar{\mu}_k \bar{\mu}_j - \lambda_u(w) \delta_{kj} \bar{\mu}_j \right] \mathbf{x} \right] \right| \leq L |\mathbf{x}|^3$$

that is (86). ■

B.3 Other Approximations

For all $\mathbf{x} \in (-\varepsilon, \varepsilon)^\Omega$ and each Borel probability measure μ on Δ , set

$$R_2(\mathbf{x}, \mu) = C(w + \mathbf{x}, \mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2} \lambda_u(w) \sigma_{\bar{\mu}}^2(\mathbf{x}) - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \sigma_{\bar{\mu}}^2(\langle \mathbf{x}, \cdot \rangle) \right].$$

Equation (88) in the proof of Theorem 64 shows that

$$|R_2(\mathbf{x}, \mu)| \leq \frac{125M^{13} + 75M^9 + 5M^5}{6} \left(\sum_{\omega \in \Omega} |x_\omega| \right)^3 = \frac{L}{|\Omega|^{3/2}} \left(\sum_{\omega \in \Omega} |x_\omega| \right)^3. \quad (89)$$

Corollary 65 *Let $\mathbf{x} \in (-\varepsilon, \varepsilon)^\Omega$, then for each Borel probability measure μ on Δ ,*

$$|R_2(\mathbf{x}, \mu)| \leq \frac{L}{\bar{m}^{3/2}} \|\mathbf{x}\|_{2, \bar{\mu}}^3 \quad (90)$$

where $\bar{m} = \min \{\bar{\mu}_\omega : \omega \in \text{supp } \bar{\mu}\}$, and

$$|R_2(\mathbf{x}, \mu)| \leq L |\Omega|^{3/2} \|\mathbf{x}\|_{\infty, \bar{\mu}}^3. \quad (91)$$

The proof builds on the following fact.

Proposition 66 *Let $\mathbf{y}, \mathbf{z} \in (-\varepsilon, \varepsilon)^\Omega$:*

1. $\mathbf{y}|_{\text{supp } \bar{\mu}} = \mathbf{z}|_{\text{supp } \bar{\mu}} \Leftrightarrow \mathbf{y} = \mathbf{z} \text{ } \bar{\mu}\text{-a.e. on } \Omega$;
2. $\mathbf{y}|_{\text{supp } \bar{\mu}} = \mathbf{z}|_{\text{supp } \bar{\mu}} \Rightarrow E_{\bar{\mu}}(\mathbf{y}) = E_{\bar{\mu}}(\mathbf{z})$ and $\sigma_{\bar{\mu}}^2(\mathbf{y}) = \sigma_{\bar{\mu}}^2(\mathbf{z})$;
3. $\mathbf{y}|_{\text{supp } \bar{\mu}} = \mathbf{z}|_{\text{supp } \bar{\mu}} \Rightarrow \langle \mathbf{y}, \cdot \rangle = \langle \mathbf{z}, \cdot \rangle \text{ } \mu\text{-a.e. on } \Delta$;
4. $\mathbf{y}|_{\text{supp } \bar{\mu}} = \mathbf{z}|_{\text{supp } \bar{\mu}} \Rightarrow \sigma_{\bar{\mu}}^2(\langle \mathbf{y}, \cdot \rangle) = \sigma_{\bar{\mu}}^2(\langle \mathbf{z}, \cdot \rangle)$ and $C(w + \mathbf{y}, \mu) = C(w + \mathbf{z}, \mu)$.

Proof. 1. and 2. are trivial.

3. For each $\omega \in \Omega$, $\bar{\mu}_\omega = 0$ is equivalent to $\int_\Delta \langle \mathbf{e}^\omega, q \rangle d\mu(q) = 0$, which in turn is equivalent to

$$\langle \mathbf{e}^\omega, \cdot \rangle = 0 \quad \mu\text{-a.e. on } \Delta.$$

Thus, $\langle \mathbf{y}, \cdot \rangle = \sum_{\omega \in \Omega} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle = \sum_{\omega \in \text{supp } \bar{\mu}} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle + \sum_{\omega \in \Omega \setminus \text{supp } \bar{\mu}} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle \stackrel{\mu\text{-a.e.}}{=} \sum_{\omega \in \text{supp } \bar{\mu}} y_\omega \langle \mathbf{e}^\omega, \cdot \rangle$
 $= \sum_{\omega \in \text{supp } \bar{\mu}} z_\omega \langle \mathbf{e}^\omega, \cdot \rangle \stackrel{\mu\text{-a.e.}}{=} \sum_{\omega \in \text{supp } \bar{\mu}} z_\omega \langle \mathbf{e}^\omega, \cdot \rangle + \sum_{\omega \in \Omega \setminus \text{supp } \bar{\mu}} z_\omega \langle \mathbf{e}^\omega, \cdot \rangle = \langle \mathbf{z}, \cdot \rangle.$

4. The equality $\sigma_\mu^2(\langle \mathbf{y}, \cdot \rangle) = \sigma_\mu^2(\langle \mathbf{z}, \cdot \rangle)$ follows from 3. while, setting $u(w + \mathbf{x}) = [u(w + x_\omega)]_{\omega \in \Omega}$ for $\mathbf{x} = \mathbf{y}, \mathbf{z}$,

$$\begin{aligned} \mathbf{y}|_{\text{supp } \bar{\mu}} = \mathbf{z}|_{\text{supp } \bar{\mu}} &\Rightarrow u(w + \mathbf{y})|_{\text{supp } \bar{\mu}} = u(w + \mathbf{z})|_{\text{supp } \bar{\mu}} \\ &\Rightarrow \langle u(w + \mathbf{y}), \cdot \rangle = \langle u(w + \mathbf{z}), \cdot \rangle \quad \mu\text{-a.e. on } \Delta \\ &\Rightarrow v(u^{-1}(\langle u(w + \mathbf{y}), \cdot \rangle)) = v(u^{-1}(\langle u(w + \mathbf{z}), \cdot \rangle)) \quad \mu\text{-a.e. on } \Delta \\ &\Rightarrow \int_\Delta v(u^{-1}(\langle u(w + \mathbf{y}), q \rangle)) d\mu(q) = \int_\Delta v(u^{-1}(\langle u(w + \mathbf{z}), q \rangle)) d\mu(q) \\ &\Rightarrow C(w + \mathbf{y}, \mu) = C(w + \mathbf{z}, \mu) \end{aligned}$$

where the second implication, again follows from 3. ■

Proof of Corollary 65. For each $\mathbf{x} \in (-\varepsilon, \varepsilon)^\Omega$, the vector $\mathbf{x}^{\bar{\mu}} = \mathbf{x}|_{\text{supp } \bar{\mu}} \mathbf{0}$ (which coincides with $\mathbf{x}$ on $\text{supp } \bar{\mu}$ and with $\mathbf{0}$ on $\Omega \setminus \text{supp } \bar{\mu}$) also belongs to $(-\varepsilon, \varepsilon)^\Omega$, and obviously $\mathbf{x}$ and $\mathbf{x}^{\bar{\mu}}$ coincide on $\text{supp } \bar{\mu}$. Thus Proposition 66 and (89) deliver

$$\begin{aligned} |R_2(\mathbf{x}, \mu)| &= \left| C(w + \mathbf{x}, \mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}) - \frac{1}{2} \lambda_u(w) \sigma_{\bar{\mu}}^2(\mathbf{x}) - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \sigma_\mu^2(\langle \mathbf{x}, \cdot \rangle) \right] \right| \\ &= \left| C(w + \mathbf{x}^{\bar{\mu}}, \mu) - \left[w + E_{\bar{\mu}}(\mathbf{x}^{\bar{\mu}}) - \frac{1}{2} \lambda_u(w) \sigma_{\bar{\mu}}^2(\mathbf{x}^{\bar{\mu}}) - \frac{1}{2} (\lambda_v(w) - \lambda_u(w)) \sigma_\mu^2(\langle \mathbf{x}^{\bar{\mu}}, \cdot \rangle) \right] \right| \\ &\leq \frac{L}{|\Omega|^{3/2}} \left(\sum_{\omega \in \Omega} |x_\omega^{\bar{\mu}}| \right)^3 = \frac{L}{|\Omega|^{3/2}} \left(\sum_{\omega \in \text{supp } \bar{\mu}} |x_\omega| \right)^3. \end{aligned}$$

In particular,

- setting $\bar{m} = \min \{\bar{\mu}_\omega : \omega \in \text{supp } \bar{\mu}\}$,

$$\sum_{\omega \in \text{supp } \bar{\mu}} |x_\omega| \leq |\text{supp } \bar{\mu}|^{1/2} \sqrt{\sum_{\omega \in \text{supp } \bar{\mu}} x_\omega^2} \leq |\Omega|^{1/2} \sqrt{\sum_{\omega \in \text{supp } \bar{\mu}} \frac{\bar{\mu}_\omega}{\bar{m}} x_\omega^2} \leq \frac{|\Omega|^{1/2}}{\bar{m}^{1/2}} \|\mathbf{x}\|_{2, \bar{\mu}}$$

thus

$$|R_2(\mathbf{x}, \mu)| \leq \frac{L}{|\Omega|^{3/2}} \frac{|\Omega|^{3/2}}{\bar{m}^{3/2}} \|\mathbf{x}\|_{2, \bar{\mu}}^3$$

which is (90);

- analogously

$$\sum_{\omega \in \text{supp } \bar{\mu}} |x_\omega| \leq \sum_{\omega \in \text{supp } \bar{\mu}} \max_{\omega' \in \text{supp } \bar{\mu}} |x_{\omega'}| = \|\mathbf{x}\|_{\infty, \bar{\mu}} |\text{supp } \bar{\mu}| \leq |\Omega| \|\mathbf{x}\|_{\infty, \bar{\mu}}$$

thus

$$|R_2(\mathbf{x}, \mu)| \leq \frac{L}{|\Omega|^{3/2}} |\Omega|^3 \|\mathbf{x}\|_{\infty, \bar{\mu}}^3$$

which is (91).

As wanted. ■

Now assume there is a net $(\mathbf{x}_t, \mu_t)_{t>0}$ such that $\|\mathbf{x}_t\|_{\infty, \bar{\mu}_t} \rightarrow 0$, then, by (91),

$$\frac{|R_2(\mathbf{x}_t, \mu_t)|}{\|\mathbf{x}_t\|_{\infty, \bar{\mu}_t}^3} \leq L |\Omega|^{3/2}$$

and $|R_2(\mathbf{x}_t, \mu_t)| = O(\|\mathbf{x}_t\|_{\infty, \bar{\mu}_t}^3) = o(\|\mathbf{x}_t\|_{\infty, \bar{\mu}_t}^2)$. It is important to observe that the same argument does not apply to (90) in fact, in this case, $\|\mathbf{x}_t\|_{2, \bar{\mu}_t} \rightarrow 0$ and (90) deliver,

$$\frac{|R_2(\mathbf{x}_t, \mu_t)|}{\|\mathbf{x}_t\|_{2, \bar{\mu}_t}^3} \leq \frac{L}{\bar{m}_t^{3/2}}$$

which does not allow to conclude $|R_2(\mathbf{x}_t, \mu_t)| = O(\|\mathbf{x}_t\|_{\infty, \bar{\mu}_t}^3)$ unless $\bar{m}_t^{3/2}$ is bounded away from 0.

C Outline of the multinomial case

Consider the multinomial case, where $\Omega = \{1, \dots, n\}$ is a general finite state space. Let $h_t : \Omega \rightarrow \mathbb{R}$ be such that, for each $t > 0$,

$$h_t(\omega) = \begin{cases} \varphi_1(t) & \text{if } \omega = 1 \\ \dots & \dots \\ \varphi_n(t) & \text{if } \omega = n \end{cases}, \quad (92)$$

where $\varphi_i : (0, 1] \rightarrow \mathbb{R}$, for $i = 1, \dots, n$, are real-valued functions defined on $(0, 1]$. Here, the prior μ_t can be viewed as a Borel probability measure on the simplex $\Delta_{n-2} = \{x \in \mathbb{R}_+^{n-1} : \sum_{i=1}^{n-1} x_i = 1\}$ of $\mathbb{R}^{n-1}$.

We need some further notation to write the quadratic approximations (14) and (15) in a way suitable for our purposes. Given $h_t : \Omega \rightarrow \mathbb{R}$, let $\Gamma(t) \in \mathbb{R}^{n-1}$ be given by

$$\Gamma_i(t) = h_t(i) - h_t(n), \quad \forall i = 1, \dots, n-1.$$

We denote by $\Gamma'(t)$ its $(n-1) \times 1$ transpose. If $n = 2$, then the vector $\Gamma(t)$ reduces to the scalar $\gamma(t)$ in (47).

Set $\bar{q}_t = (\bar{q}_t(1), \dots, \bar{q}_t(n-1)) \in \Delta_{n-2}$, where $\bar{q}_t(i) = \int q d\mu_t(q)$ for $i = 1, \dots, n-1$. Consider the $(n-1) \times (n-1)$ matrices:

$$\bar{Q}_t = \bar{q}_t I = \begin{bmatrix} \bar{q}_t(1) & 0 & \dots & 0 \\ 0 & \bar{q}_t(2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \bar{q}_t(n-1) \end{bmatrix}$$

and

$$\bar{Q}_t^2 = \bar{Q}_t \bar{Q}_t = \begin{bmatrix} \bar{q}_t^2(1) & \bar{q}_t(1)\bar{q}_t(2) & \dots & \bar{q}_t(1)\bar{q}_t(n-1) \\ \bar{q}_t(2)\bar{q}_t(1) & \bar{q}_t^2(2) & \dots & \bar{q}_t(2)\bar{q}_t(n-1) \\ \dots & \dots & \dots & \dots \\ \bar{q}_t(n-1)\bar{q}_t(1) & \dots & \dots & \bar{q}_t^2(n-1) \end{bmatrix}$$

If $n = 2$, then the matrices $\bar{Q}_t$ and $\bar{Q}_t^2$ reduce to the scalars $\bar{q}_t$ and $\bar{q}_t^2$ of Sections 7 and 8.

Set $\vartheta_t(i, j) = \int q(i) q(j) d\mu_t(q)$ and $\vartheta_t(i) = \vartheta_t(i, i) = \int q^2(i) d\mu_t(q)$. Consider the $(n-1) \times (n-1)$ matrix:

$$\Theta_t = \begin{bmatrix} \vartheta_t(1) & \vartheta_t(1, 2) & \dots & \vartheta_t(1, n-1) \\ \vartheta_t(2, 1) & \vartheta_t(2) & \dots & \vartheta_t(2, n-1) \\ \dots & \dots & \dots & \dots \\ \vartheta_t(n-1, 1) & \dots & \dots & \vartheta_t(n-1) \end{bmatrix}$$

If $n = 2$, then the matrix Θ_t reduces to the scalar ϑ_t of Sections 7 and 8.

Some algebra, which is detailed in the Appendix, shows that

$$\sigma_{\bar{q}_t}^2(h_t) = \Gamma_t(\bar{Q}_t - \bar{Q}_t^2) \Gamma'_t \quad (93)$$

and

$$\sigma_{\mu_t}^2(E(h_t)) = \Gamma_t(\Theta_t - \bar{Q}_t^2) \Gamma'_t. \quad (94)$$

Hence, for all $w \in \mathbb{R}$, (14) implies

$$\pi_t(w + h_t) = \frac{1}{2} \lambda_u(w) \Gamma_t(\bar{Q}_t - \Theta_t) \Gamma'_t + \frac{1}{2} \lambda_v(w) \Gamma'_t(\Theta_t - \bar{Q}_t^2) \Gamma_t + o(\|h_t\|^2).$$

If $n = 2$, then this quadratic approximation reduces to (48).

Here,

$$\Lambda_u(t) = \Gamma_t(\bar{Q}_t - \Theta_t) \Gamma'_t$$

and

$$\Lambda_v(t) = \Gamma'_t(\Theta_t - \bar{Q}_t^2) \Gamma_t.$$

The limit behavior, as t goes to zero, of $\Lambda_u(t)$ and $\Lambda_v(t)$ determines whether risk and model uncertainty attitudes have a first or second order effect, or whether they are instead negligible. We omit the details.

Derivation of Eq. (94)²¹

$$\begin{aligned} \sigma_{\mu_t}^2(E(h_t)) &= \int \left(\sum_{i=1}^n \varphi_i(t) q(i) \right)^2 d\mu_t(q) - \left(\int \sum_{i=1}^n \varphi_i(t) q(i) d\mu_t(q) \right)^2 \\ &= \int \left(\sum_{i=1}^n \varphi_i^2(t) q^2(i) + 2 \sum_{1 \leq i < j \leq 1} \varphi_i(t) \varphi_j(t) q(i) q(j) \right) d\mu_t(q) - \left(\sum_{i=1}^n \varphi_i(t) \bar{q}_t(i) \right)^2 \\ &= \sum_{i=1}^n \varphi_i^2(t) \vartheta_t(i) + 2 \sum_{1 \leq i < j \leq 1} \varphi_i(t) \varphi_j(t) \vartheta_t(i, j) - \sum_{i=1}^n \varphi_i^2(t) \bar{q}_t^2(i) - 2 \sum_{1 \leq i < j \leq 1} \varphi_i(t) \varphi_j(t) \bar{q}_t(i) \bar{q}_t(j) \\ &= \sum_{i=1}^n \varphi_i^2(t) (\vartheta_t(i) - \bar{q}_t^2(i)) + 2 \sum_{1 \leq i < j \leq n} \varphi_i(t) \varphi_j(t) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\ &= \sum_{i=1}^{n-1} \varphi_i^2(t) (\vartheta_t(i) - \bar{q}_t^2(i)) + 2 \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \varphi_j(t) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\ &\quad + \varphi_n^2(t) (\vartheta_t(n) - \bar{q}_t^2(n)) - 2 \sum_{j=1}^{n-1} \varphi_n(t) \varphi_j(t) (\vartheta_t(n, j) - \bar{q}_t(n) \bar{q}_t(j)) \\ &= \sum_{i=1}^{n-1} \varphi_i^2(t) (\vartheta_t(i) - \bar{q}_t^2(i)) + 2 \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \varphi_j(t) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\ &\quad + \varphi_n^2(t) \left(1 + \sum_{i=1}^{n-1} \vartheta_t(i) - 2 \sum_{1 \leq i < j \leq n-1} \vartheta_t(i, j) - \left(1 - \sum_{i=1}^{n-1} \bar{q}_t(i) \right)^2 \right) \\ &\quad - 2 \sum_{j=1}^{n-1} \varphi_n(t) \varphi_j(t) \left(\bar{q}_t(j) - \sum_{i=1}^{n-1} \vartheta_t(i, j) - \left(1 - \sum_{i=1}^{n-1} \bar{q}_t(i) \right) \bar{q}_t(j) \right) \end{aligned}$$

²¹ A similar argument holds for (93).

$$\begin{aligned}
&= \sum_{i=1}^{n-1} \varphi_i^2(t) (\vartheta_t(i) - \bar{q}_t^2(i)) + 2 \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \varphi_j(t) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\
&\quad + \varphi_n^2(t) \left(\sum_{i=1}^{n-1} \vartheta_t(i) - 2 \sum_{1 \leq i < j \leq n-1} \vartheta_t(i, j) - \sum_{i=1}^{n-1} \bar{q}_t(i)^2 \right) \\
&\quad - 2 \sum_{j=1}^{n-1} \varphi_n(t) \varphi_j(t) \left(- \sum_{i=1}^{n-1} \vartheta_t(i, j) + \sum_{i=1}^{n-1} \bar{q}_t(i) \bar{q}_t(j) \right) \\
&= \sum_{i=1}^{n-1} (\varphi_i^2(t) + \varphi_n^2(t)) (\vartheta_t(i) - \bar{q}_t^2(i)) + 2 \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \varphi_j(t) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\
&\quad - 2 \varphi_n^2(t) \sum_{1 \leq i < j \leq n-1} \vartheta_t(i, j) + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} \varphi_n(t) \varphi_j(t) \vartheta_t(i, j) - 2 \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} \varphi_n(t) \varphi_j(t) \bar{q}_t(i) \bar{q}_t(j) \\
&= \sum_{i=1}^{n-1} (\varphi_i^2(t) + \varphi_n^2(t)) (\vartheta_t(i) - \bar{q}_t^2(i)) + 2 \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \varphi_j(t) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\
&\quad - 2 \varphi_n^2(t) \sum_{1 \leq i < j \leq n-1} \vartheta_t(i, j) + 2 \varphi_n(t) \left(\sum_{1 \leq i < j \leq n-1} \varphi_j(t) \vartheta_t(i, j) + \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \vartheta_t(i, j) + \sum_{i=1}^{n-1} \varphi_i(t) \vartheta_t(i) \right) \\
&\quad - 2 \varphi_n(t) \left(\sum_{1 \leq i < j \leq n-1} \varphi_j(t) \bar{q}_t(i) \bar{q}_t(j) + \sum_{1 \leq i < j \leq n-1} \varphi_i(t) \bar{q}_t(i) \bar{q}_t(j) + \sum_{i=1}^{n-1} \varphi_i(t) \bar{q}_t^2(i) \right) \\
&= \sum_{i=1}^{n-1} (\varphi_i^2(t) + \varphi_n^2(t) - 2 \varphi_i(t) \varphi_n(t)) (\vartheta_t(i) - \bar{q}_t^2(i)) \\
&\quad - 2 \sum_{1 \leq i < j \leq n-1} (\varphi_i(t) \varphi_j(t) - \varphi_n^2(t) - 2 \varphi_n(t) (\varphi_i(t) + \varphi_j(t))) (\vartheta_t(i, j) - \bar{q}_t(i) \bar{q}_t(j)) \\
&= \sum_{i=1}^{n-1} (\varphi_i(t) - \varphi_n(t))^2 (\vartheta_t(i) - \bar{q}_t^2(i)) = \Gamma_t (\Theta_t - \bar{Q}_t^2) \Gamma_t'
\end{aligned}$$

as desired. ■

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