Inequality and Asset Prices*

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February 15, 2012

Abstract

What is the relationship between wealth inequality and asset prices? We study this question in a dynamic two-agent economy with incomplete markets. Agents face correlated labor-income risk, but there is no aggregate risk. The only asset is a Lucas tree, which is traded subject to a no-short-selling constraint. We find that asset prices are increasing in wealth inequality. The asset price is highest when the poor agent hits the no-short-selling constraint and exits the asset market. Since the asset supply of the impoverished agent dries up while the rich agent’s demand stays high, there is a surge in the asset price at this point. Furthermore, asset-price volatility is increasing in inequality. Analogous results are obtained in an economy with a short-term bond and in a production economy with capital.

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1 Introduction

Over the last three decades, both income and wealth inequality have risen in the U.S. and in other rich countries (for evidence, see Atkinson, Piketty & Saez (2011) for income inequality and Lindert (2000) and Kennickell (2009) for wealth inequality). Ever larger fractions of aggregate wealth are held by a small share of the population, and the earning’s share of the rich has been increasing dramatically over the past 30 years. What are the implications for asset prices and asset returns when markets are dominated by a group of rich individuals? More specifically, this paper asks the following question: what is the relationship between wealth inequality and asset prices?

If rich individuals have investment needs and strategies that are different from those of the average investor, this should have an impact on asset prices. For example, rich investors’ income is more correlated with capital returns than the income of the average investor. This means that rich investors will place more emphasis in their investment decision on states of the world in which capital performs poorly than the average investor (since their marginal utility of consumption is higher in these states).1 Note that a mechanism along these lines also opens up the possibility that government policy affects asset-market outcomes through redistributive policies. If the government passes tax legislation that favors the wealthy, this would exacerbate the potential effects of rising inequality on asset prices.

The first obvious approach to answer the question about the relationship between inequality and asset prices is to study empirically how asset markets have responded to changes in inequality in the past. However, there is a severe drawback to this approach: essentially, there are very few data points available. Inequality (in both income and wealth) were at high levels in the U.S. from the turn of the century until the Great Depression. From the 1930s on, inequality started to drop significantly and stayed at relatively low levels until the 1980s, when it started its surge to current levels. Since inequality is extremely slow-moving, we are essentially left with three data points over the period where reliable data are available. Furthermore, other variables that also impact asset markets may have co-moved with inequality, making it difficult – if not impossible – to identify the

1 Other stories for why rich investors’ behavior could differ from the average investor are that rich investors have different risk aversion (as in Dumas (1989)) or different discount factors than poorer investors (as in Krusell & Smith (1997)). However, our framework does not consider these possibilities and focuses on the effects that arise when preferences are uniform across agents.
contribution of inequality to changes in asset prices.\textsuperscript{2} \textsuperscript{3}

In light of these severe drawbacks to the empirical approach, we opt for a theoretical approach. We study a simple two-agent incomplete-markets model with a single asset that generates joint dynamics in the wealth distribution and asset prices.

1.1 Model overview

There are two groups of agents who face idiosyncratic labor-income risk in the form of a two-state Markov chain. Agents’ labor-income shocks are perfectly negatively correlated, so that there is no aggregate risk in the economy. We deliberately abstract away from aggregate risk in order to focus the analysis on inequality. Markets are incomplete. In order to smooth consumption agents have a single asset at their disposal, namely, a Lucas tree with a constant dividend (which may be seen as a perpetual bond). There is a no-short-selling constraint on the asset, which occasionally binds for one group of agents.

In addition, we study two economies in which the Lucas tree is replaced by 1) a short-term bond and 2) physical capital in a production economy. Again, there are borrowing limits that occasionally bind. The qualitative results from the Lucas-tree economy also obtain in these modified settings.

1.2 Results

As expected, in all settings agents use the asset as a buffer stock to insure against income shocks. They accumulate the asset in times when they have high labor income, and they decumulate it when labor income is low. When faced with a very long unproductive spell, agents ultimately hit the liquidity constraint and (endogenously) cease to participate in asset markets. At this point, consumption inequality as well as wealth inequality reach their maximal levels.

Asset prices are higher, and expected returns are lower, when the wealth distribution becomes more unequal. Asset prices reach their maximal levels when the income-poor agent ceases to participate in asset markets, in which case only rich

\textsuperscript{2}Examples for such confounding variables are aggregate consumption growth and political conditions – note that the era of low inequality coincides almost perfectly with the Cold War.

\textsuperscript{3}A stark pattern in the data is that stock markets were most volatile in pre-Great-Depression era and after 1980, which coincides with times of high inequality. Our model is successful in explaining this fact, but because of the low number of essential observations we are cautious to interpret this as evidence favoring our model.
agents participate in asset markets. At this point rich agents’ asset demand is high since they want to insure against the case that their income drops again. This high demand drives up the asset price and thus lowers its return. The high asset price from this extreme situation feeds back into regions where both groups of agents still participate in asset markets but where the wealth distribution is skewed: since there is a chance that the extreme state is reached and the asset can be sold off at a high price, the current valuation for the asset is high for both agents.

Large drops in asset prices occur when the wealth-rich agent receives a bad income shock. In this situation, the wealth-poor agent enters the market again and buys shares of the tree, whereas the wealth-rich agent sells them off. The likelihood that the economy enters the extreme state of the poor agent being constrained drops, and with it the expectation of high asset prices in the future. The large drops in prices in situations of high wealth inequality (or large increases in price, in the case of income changing in the reverse direction) lead to the second main result: asset-price volatility is increasing in wealth inequality.

Using a continuous-time framework allows us to obtain sharp characterizations of asset prices and consumption policies at the boundaries of the state space where one group of agents becomes constrained. We find that the elasticity of the asset price with respect to the rich agent’s wealth share goes to infinity when approaching the constraint. In the economy with the short-term bond, we can show that a discrete downward jump in the bond yield occurs at the moment when the poor agent exits the market.

1.3 Literature

Scheinkman & Weiss (1986) is the paper that is most similar to ours, but in their framework the income-poor agent has no income at all. This income process together with an Inada condition on utility makes the household cling to the asset when income-poor, so that in equilibrium the liquidity constraint is never reached. Scheinkman & Weiss (1986) find that asset returns are increasing in the income-rich agent’s asset share, which is not the case in our setting – recall that they are increasing in inequality, so they are inverse-U-shaped in an agent’s wealth share. This difference stems from another assumption they make: They put a labor decision into the model and assume the disutility of labor to be linear. This utility function essentially fixes the productive (i.e. income-rich) agent’s consumption at a constant level and so simplifies the analysis.

Dumas (1989) studies an economy with two agents who have different risk aversion. However, in his economy there is only aggregate risk and no idiosyn-
cratic risk. The welfare theorems hold and the market allocation can be backed out from a planner’s problem, which is not the case in our setting. Long spells of positive productivity shocks make the less risk-averse agent’s asset share increase, however the more risk-averse agent has a higher asset share after long spells of bad luck.

There are also several computational papers studying economies inhabited by two types of agents: Telmer (1993), Lucas (1994) and Heaton & Lucas (1996). Haan (1996) studies settings with larger (finite) numbers of agents. All these papers have in common that they do not find an exact solution of the model but recur to numerical approximating techniques. The literature focuses on global properties of equilibrium, such as the average equality premium, the average risk-free rate etc. While the models are capable – and indeed likely – to produce effects like ours, the authors do not mention these. This is probably because of the large dimension of the state space and the different focus of their papers. Our framework has the advantage that it is extremely simple and can so highlight the pricing differences that occur across the state space and at the constraint.

Another area where incomplete-markets models with a small number of (usually two) agents is used is the international-finance and trade literature. Stepanchuk & Tsyrennikov (2011), for example, study a two-country setting with one type of stock in each country and a risk-free bond. Again, the state space is vastly larger in this model than it is in ours. Our more analytic approach may help to shed light on the behavior of asset prices in regions where constraints start to bind in models such as Stepanchuk & Tsyrennikov (2011). Furthermore, it can indicate when it is important to consider discontinuities in pricing functions in the computational algorithms used to solve such models.

Finally, an entire industry has evolved that uses heterogeneous-agents economies with a continuum of agents to explain asset prices Krusell & Smith (1997), Gomes & Michaelides (2008), Guvenen (2009), etc. The key difference to our model is that shocks to individual agents are independent across agents in these models, whereas they are correlated in our model. Independence of idiosyncratic shocks makes changes in the relative wealth distribution wash out, which leads to Krusell & Smith (1998) approximate-aggregation result: prices can be forecast very accurately using only the mean asset holdings of all agents. Our economy is a stark counterexample to approximate aggregation: the wealth share held by the rich agent (the measure of inequality) is indeed very important for predicting future asset returns.
2 Setting: A Lucas-tree economy

Following Kehoe & Levine (2001), we write down the simplest possible economy with idiosyncratic risk; there is no aggregate risk.

Two classes of agents (1 and 2) of the same measure inhabit the economy. There is one asset (a Lucas tree) in the economy, which is in fixed supply 1. We will come to the case of the bond and of physical capital in later sections. The asset yields a constant dividend stream $0 \leq r < 1$. If $r = 0$, we may interpret the asset as fiat money. The asset’s price is denoted by $P_t$. Agent 1’s labor income $y$ follows a two-state Markov process with switching hazard $\eta > 0$ between the states $0 < y_l \leq y_h \leq 1$. The aggregate endowment of the economy is fixed at 1 (i.e. there is no aggregate uncertainty), so agent 2’s labor income is given by $1 - r - y$. Time is continuous: $t \in [0, T]$. Of course we are especially interested in the limiting case where $T \to \infty$ or $T = \infty$ (there might be equilibria for $T = \infty$ that are not the limit of a finite economy). Both agents have standard preferences over consumption:

$$E_0 \int_0^T e^{-\rho t} u(c_t) dt,$$

where $u' > 0$ and $u'' < 0$.

2.1 The agent’s problem

The budget constraint at $t$, given the inherited asset position from $t - \Delta t$, is

$$c_t \Delta t + P_t a_t = y_t \Delta t + r a_{t-\Delta t} + P_t a_{t-\Delta t}.$$

Dividing by $\Delta t$ and taking limits as $\Delta t \to 0$, we obtain

$$c_t + P_t \dot{a}_t = y_t + r a_t,$$

where we have denoted the drift in the asset position by $\dot{a}_t = \lim_{\Delta t \to 0} (a_t - a_{t-\Delta t}) / \Delta t$.

Now, let the aggregate state be given by $t, y_t, A_t \in [0, 1]$, where $A_t$ denotes the asset position of a typical type-1 agent. Let the individual agent’s state be $a_t$.\(^4\)

\(^4\)Note that we have to give the individual the possibility to deviate with his position $a_t$ from the aggregate position of his group $A_t$. Think big-$K$, little-$K$ here. If we wrote a problem where the individual controls the aggregate state $A_t$, then this would give the individual control over prices. We are interested in the competitive setting where this is not the case; the case with two non-atomistic players with market power is potentially interesting but beyond the scope of this paper.
Then the (individual) agent’s problem is to choose a consumption function \( c(\cdot) \) contingent on the state \((t, y, A, a)\) in order to

\[
\max_{c(\cdot)} \quad E_0 \int_0^T e^{-\rho t} u[c(t, y_t, A_t, a_t)]
\]

\[
\text{s.t.} \quad \dot{a}_t = \frac{y_t + ra_t - c(t, y_t, A_t, a_t)}{P(t, y_t, A_t)}
\]

\[
c(t, y, A, 0) \leq y \quad \text{for all } t \in [0, T], \text{ for all } A \in [0, 1]
\]

given

\[
\dot{A}_t = d^A(t, y_t, A_t)dt
\]

\[
P(t, y, A),
\]

where the first constraint is the law of motion for the individual agent’s asset and the second constraint is the liquidity constraint that limits the agent’s consumption to his flow income when his asset position is zero. The agent takes as given the law of motion for the aggregate state \( A \), which is summarized by a drift function \( d^A(\cdot) \). She also takes as given the pricing function \( P(\cdot) \).

The HJB for the value function \( V(t, y, A, a) \) is

\[
-V_t + \rho V = \max_c \left\{ u(c) + \frac{y + ra - c}{P} V_a \right\} + \eta V_y + d^A V_A,
\]

where we introduce the “discrete derivative” \( V_y \equiv V(y', \cdot) - V(y, \cdot) \) in which \( y \) denotes the current state of income and \( y' \) is the other income state (to which one might jump).

The first-order condition is

\[
P u_{c}(c) = V_a.
\]

It asserts that the marginal value of keeping the asset \( V_a \) has to equal the value of selling a marginal unit at price \( P_t \) and consuming it.

It is now convenient to introduce the infinitesimal generator, which is a partial differential operator that tells us the expected growth of any smooth function \( f(\cdot) \) defined on the state space \((t, y, A, a)\):

\[
\mathcal{A} f = \lim_{\Delta t \to 0} E_t \left[ \frac{f(t + \Delta t, y_{t+\Delta t}, A_{t+\Delta t}, a_{t+\Delta t}) - f(t, y_t, A_t, a_t)}{\Delta t} \right] = f_t + \eta f_y + d^A f_A + \frac{y + ra - c^*}{P} f_a
\]
where \( c^* \) is the optimal consumption rule for agent 1.

Using the infinitesimal generator, we can re-state the HJB as follows:

\[
\mathcal{A}V = \rho V - u(c^*).
\]

Taking derivatives of the HJB with respect to \( a \) and using the first-order condition, we find the Euler equation

\[-V_{at} + \rho V_a = \frac{r}{P}V_a + \frac{y + ra - c}{P}V_{aa} + \eta V_{ya} + d^AV_{aA}.
\]

At this step it is important to note that \( P_a = 0 \), i.e. the individual agent cannot influence prices! (This is the important part of the big-\( K \)-little-\( k \) trick... ) Using the infinitesimal generator, we can re-write the first-order condition as

\[
\mathcal{A}V_a = (V_a)_t + \eta(V_a)_y + d^A(V_a)_A + \frac{y + ra - c}{P}(V_a)_a = \left( \rho - \frac{r}{P} \right) V_a.
\]

Using the FOC, we obtain the standard Euler equation,

\[
\frac{\mathcal{A}[Pu_c(c)]}{Pu_c(c)} = \left( \rho - \frac{r}{P} \right),
\]

It says that the agent’s marginal valuation of the asset \( Pu_c(c) \) follows a martingale when we adjust for discounting \( \rho \) and the dividend stream \( r \). Precisely, it states that the percentage growth rate of the marginal valuation of the asset grows at rate \( \rho \) minus the assets dividend-price ratio. By symmetry, this Euler equation must hold for the consumption plans of both type-1 and type-2 agents.

For the case of complete markets, agents enjoy perfect insurance and thus \( c \) is constant. Then the above Euler equation tells us that the marginal valuation of the asset must be constant and the price is given by \( P_{cm} = r/\rho \).

### 2.2 Equilibrium

In equilibrium (big-\( K \),little-\( k \)), the privately chosen consumption must equal aggregate consumption of the respective group. Mathematically, this means that

\[c(t, y, A, A) = C(t, y, A),\]

where \( C(\cdot) \) is the consumption pattern associated with the aggregate law of motion \( d^A \): The single agent always follows his group and thus \( a_t = A_t \) always.
So the marginal valuation of the asset given the marginal utility implied by the aggregate consumption pattern $C$ must satisfy the martingale property established above. Using $C = c$ in the Euler equation gives us

$$\frac{A[P_{u_c}(C)]}{Pu_c(C)} = \eta \left[ \frac{P'u_{c}(C')}{Pu_c(C)} - 1 \right] + \frac{P_t + P_A d_A}{P} + u_{cc}(C) \left[ \frac{C_t + C_A d_A}{u_c(C)} \right] = \rho - \frac{r}{P}$$

The percentage change in the valuation of the asset for a typical agent 1 is composed of two terms: The first term captures what happens if there is a reversal in $y$ and both $P$ and $C'$ change discretely. The second group of terms captures what happens when no reversal happens. The percentage change under this scenario is just the sum of the percentage change in the asset price and the percentage change in marginal utility. The percentage change in marginal utility, in turn, is given by the coefficient of absolute risk aversion times the time change in consumption.\(^5\)

Note that we are following an approach here that is similar to the \textit{primal approach} in the Ramsey problem, or the \textit{first-order approach} in moral-hazard problems: We impose on the equilibrium that the allocation fulfill the agents’ first-order conditions. However, note that the Euler equation is only a \textit{necessary} for optimality, but not \textit{sufficient}. We will have to check sufficiency later, which should not be too problematic since the problem is well-behaved and concave.\(^6\)

Of course, the Euler equation also has to hold for all agents of type 2. Let us denote variables pertaining to agent 2 by tildes: $\tilde{C}$ for consumption etc. Asset market clearing requires that assets sold by group 1 are all bought by group 2, i.e. $\dot{A} + \tilde{\dot{A}}$. Walras’ Law then tells us that the other market, the one for the consumption good, must be automatically in equilibrium when taking into account the information from the agents’ budget constraints: We obtain $C + \tilde{C} = 1$, which is the resource constraint for this economy.

\(^5\) . . . or equal to the coefficient of relative risk aversion times the percentage change in consumption (multiply by $C/C$ to see this) – but we will not be able to exploit this, unlike in the case of absolute risk aversion.

\(^6\) A complication here is that we have to check if it is profitable for the agent to choose long-term deviations from his group (note that the relevant space here to apply the one-shot-deviation principle are all nodes in the space $(t, y, A, a)$, not the space $(t, y, A)$). This is maybe what Kehoe & Levine (2001) mean what they say that the state space is large in this kind of problems.
Using $\tilde{C} = 1 - C$ in agent 2’s Euler equation, we obtain

$$\frac{\mathcal{A}[Pu_c(1 - C)]}{Pu_c(1 - C)} = \eta \left[ \frac{Pu_c(1 - C)}{Pu_c(1 - C)} - 1 \right] + \frac{P + P_A d^A}{P} - \frac{u_{cc}(1 - C)}{u_c(1 - C)} [C_t + C_A d^A] = \rho - \frac{r}{P}. $$

### 2.3 Constant absolute risk aversion

Now, use the preferences $u(c) = -\frac{1}{\lambda} e^{-\lambda c}$ which have the property of constant absolute risk aversion: $u_{cc}(c)/u_c(c) = -\lambda$. This gives us the opportunity to exploit the fact that the last terms in agent 1 and agent 2’s Euler equations are of the same magnitude when $u_{cc}/u_c = \text{const}$ – note that $\frac{d}{dt} C = C_t + C_A d^A = -\frac{d}{dt} \tilde{C}$.

Add both Euler equations and divide by 2 to obtain an expression that tells us about the evolution of prices in equilibrium:

$$\frac{P_t + P_A d^A}{P} \equiv \hat{\rho} = \rho - \frac{r}{P} - \eta \left[ \frac{P'}{P} \left( \frac{1}{2} \frac{u_c(C')}{u_c(C)} + \frac{1}{2} \frac{u_c(1 - C')}{u_c(1 - C)} \right) - 1 \right]. \quad (2)$$

We have defined $\hat{\rho} = \frac{d}{dt} \rho$ on the left-hand side as the percentage change in prices over a short interval of time under the scenario that no reversal in $y$ occurs. Also, we introduce the following term: We will call the ratio $u_{cc}(C')/u_c(C) = e^{\lambda(C - C')}$ “insurance motive”: The higher this ratio, the more does a reversal in $y$ hurt the agent in terms of marginal utility. We see that average consumption risk is playing the role that the representative agent’s consumption risk played in the representative-agent economy (see section 2.9): The higher average consumption risk, the lower $\hat{\rho}$, i.e. the lower is the assets returns in “normal times” (i.e. when the reversal does not occur).

We can also see from this equation how inequality influences asset prices. Using the fact that $\exp(\lambda \Delta C)$ is a convex function in $\Delta C$ and $\exp(\lambda 0) = 1$, we see that the term $\frac{1}{2} [u_c(C')/u_c(C) + u_c(1 - C')/u_c(1 - C)]$ is always larger than 1; indeed, the higher the consumption jump $\Delta C = C' - C$ is for the agents, the higher the average insurance motive. So consumption inequality will depress the asset return below the level of a representative-agent economy without aggregate consumption risk $\rho - r/P$, see again section 2.9.7

7Note that in PDE language, we are following characteristic lines in $(t, A)$-space, always in the scenario where the reversal in income $y$ does not happen.
Now, subtract the two Euler equations from each other and divide by 2 to obtain
\[
\dot{C} = [C_t + C_A d^A] = \frac{\eta}{2\lambda} \frac{P'(C') - u_c(1-C')}{u_c(1-C)}.
\] (3)

So we see that the evolution of consumption in normal times is governed by the difference in the insurance motive between the two agents. If type 1’s insurance motive is larger than unity (i.e. she fears a reversal), then type 2’s insurance motive must be lower than unity by the aggregate resource constraint. So in this case, we have \( \dot{C} > 0 \), i.e. type 1’s consumption rises in normal times. Intuitively, if she becomes worse off upon a reversal, she must become better off if no reversal occurs. Once the reversal happens, by the same token the dynamics of the consumption distribution must reverse: His consumption will now trend upwards in normal times.

### 2.4 Constrained case

The only situation in which consumption rates are constrained by the no-short-selling limit is at the boundaries of the state space, i.e. at \( a = 0 \) for her. In equilibrium, we will have that the asset-poor agent is consuming less than her income at \( a = 0 \): It is optimal to save for the case that income drops to the low level. When income is low, however, the agent is constrained: The entire income is consumed and no savings take place. For consumption, this means that \( C(t, 0, y_l) = y_l \) and \( C_t(t, 0, y_l) = 0 \) for all \( t \). Agent 2’s Euler equation must hold with equality and \( d^A = 0 \), so we find that asset prices obey
\[
\frac{P_t(t, 0, y_l)}{P(t, 0, y_l)} = \rho - \frac{r}{P(t, 0, y_l)} - \eta \left[ \frac{P(t, 0, y_h)u_c(C(t, 0, y_h))}{P(t, 0, y_l)u_c(y_h + r)} - 1 \right].
\] (4)

In a mirror-symmetric way, for the case where agent 2 is constrained we find from agent 1’s Euler equation
\[
\frac{P_t(t, 1, y_h)}{P(t, 1, y_h)} = \rho - \frac{r}{P(t, 1, y_h)} - \eta \left[ \frac{P(t, 1, y_l)u_c(C(t, 1, y_h))}{P(t, 1, y_h)u_c(y_h + r)} - 1 \right].
\] (5)

### 2.5 Solution strategy

We start with the time-dependent case and let the economy end at some \( T > 0 \). Consider the situation at \( T - \Delta t \), where \( \delta T \) is small. The asset will be worthless at \( T \), so agents should just eat their endowment and the fruits that fall from the
tree, i.e. \(C(T - \Delta t, a, y) = ra + y\). So the tree must be worth exactly the number of fruits that fall from it in the time interval \([T - \Delta t, T]\), i.e. we can approximate \(P(T - \Delta t, a, y) = r \Delta t\). Now we can use standard PDE solution techniques on a grid to solve from \(T - \Delta t\) backwards: The pricing function is updated backward in time using (2), and the consumption function is updated using (3). At the boundaries of the state space \(A \in \{0, 1\}\) we enforce the no-short-selling limits and update prices using equations (4) and (5).

### 2.6 Results

The following figure shows the stationary equilibrium for CARA utility \(u(c) = -\lambda \exp(-\lambda c)\), parameterized by \(\rho = 0.04\), \(\lambda = 2\), \(\eta = 0.1\), \(w_l = 0.3\), \(w_h = 0.4\) and \(r = 0.3\).

![Figure 1: Asset economy](image-url)
The upper panel shows the pricing function. The solid red line depicts the price as a function of agent 1’s asset position \( A \) when agent 1 is income-poor. The solid blue line depicts the price when agent 1’s income is high. The dashed black line gives the benchmark return on the tree from the complete-markets economy.

The middle panel shows the return on the asset. Over a time interval \( \Delta t \) the return is \( r \Delta t \) (dividends) plus the price increment \( P_{t+\Delta t} - P_t \) divided by the buying price \( P_t \). So the expected return of the asset for a short time interval, normalized by time, is

\[
R = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t \left[ \frac{r \Delta t + P_{t+\Delta t} - P_t}{P_t} \right] = r + \frac{AP}{P}.
\]

We see that the return is inverse-U-shaped in \( a \) and lowest when one agent is both income- and asset-rich. Also, it is easy to see that the volatility of asset prices and asset returns is highest when \( a \) goes to zero or one.

### 2.7 Pricing formula

The results in figure 1 show that the asset price is increasing in wealth inequality: The further the \( A \) moves away from zero (the equitable wealth distribution), the higher asset prices. Furthermore, the asset price is even higher if the asset-rich agent is also income rich. We will now develop some intuition for this result using standard insights from asset pricing.

First, notice that whenever both agents participate in the asset market, we can think of both of them pricing the asset. The discretized Euler equation for agent 1 reads

\[
P(A, y) = (1 - \rho \Delta t) \left[ (1 - \eta \Delta t) \left[ r \Delta t + P_{t+\Delta t} \frac{u_c(C_{t+\Delta t})}{u_c(C_t)} \right] + \eta \Delta t (r \Delta t + P'_{t+\Delta t}) \frac{u_c(C'_{t+\Delta t})}{u_c(C_t)} \right] + o(\Delta t).
\]

It says that the price of the asset at \( t \) can be decomposed into the dividend stream \( r \Delta t \) over a short time interval plus the discounted value of the asset after the time interval.\(^8\) For the discounting, we have to take into account marginal-utility ratios. If the agent is worse off in a future state, he will value payoffs in this state

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\(^8\)One can take limits of equation (6) as \( \Delta t \to 0 \) to see that it is equivalent to the Euler equation (1).
higher. Note that in a situation where both agents are unconstrained, one of the utility ratios will be larger and the other one will be lower than one. If an agent is income-rich, he is saving and experiences steady small gains in consumption if no reversal occurs. If a reversal occurs, however, there is a large sudden drop in consumption, and payoffs in this scenario are highly valued.

In the situation where agent 1 is constrained, however, only agent 2 is active in the asset market and effectively prices the asset. Agent 2’s Euler equation then gives us the following pricing equation:

$$P(0, y_l) = (1 - \rho \Delta t) \left[ (1 - \eta \Delta t)(r \Delta t + P(0, y_l)) + \eta \Delta t(r \Delta t + P(0, y_l)) \frac{u_c(1 - C(0, y_l))}{u_c(1 - C(0, y_l))} \right] + o(\Delta t).$$

Since consumption does not change when agent 1 stays with the low income, the marginal-utility ratio is unity for the case where no reversal occurs. In the case of a reversal, the rich agent experiences a drop in consumption. The marginal-utility ratio is then positive, i.e. the rich agent values insurance for this bad state. Since times can only get worse for the rich agent in this situation, discount factors are large. This manifests itself in asset returns being lowest in this situation, as the second panel of figure 1 shows. The discount factor of the poor agent would be low, but is not relevant for pricing since the agent is not participating in asset markets. Finally, note that we can obtain the symmetric case where agent 2 is constrained by setting $A = 1$ is equation (6) and noting that $C_{t+\Delta t} = C_t$ in the case that no reversal occurs.

In order to think of the asset price as a sum of discounted future dividends, we can recursively substitute in for future prices $P_{t+\Delta t}$ and $P'_{t+\Delta t}$ using equations (6) and (7). Since discount rates are especially high (i.e. the pricing agent appears especially patient) in the situation when the income-rich agent owns all assets, future dividends are discounted less in situations where it is likely that the constraint is hit soon.
Formally, we define the following pricing kernel for this economy:

\[ \Lambda_t = e^{-\rho t} u_c(C_t) \xi^{k_t}, \]

where \( \xi = \frac{u_c(1 - C(0, y_h))}{u_c(1 - C(0, y_l))} \frac{u_c(C(0, y_l))}{u_c(C(0, y_h))} > 1 \)

and \( k_t \) is the number of times that a reversal has occurred until \( t \) when the economy was at state \((A, y) = (0, y_l)\). This pricing kernel uses marginal utility of agent 1 almost always; only in the situation where agent 2 is income-rich and owns all assets do we have to make adjustments in the form of the term in \( \xi > 1 \). With respect to the usual representative-agent pricing kernel, the difference is in the term in \( \xi \). Each time that the agent ceases to participate in the market, the kernel is adjusted upward. This is because the other agent is in a favorable situation and places a high value on future payoffs.

Using the above pricing kernel we then guess the following pricing equation:

\[ \Lambda_t P_t = E_t \int_t^T \Lambda_s r ds. \] (8)

We will now derive a difference equation for the random variable \( H_t = \Lambda_t P_t \) to show that the above pricing equation indeed holds.

Using the law of iterated expectations, we can write

\[ \Lambda_t P_t = \Lambda_t r \Delta t + E_t \left[ E_{t+\Delta t} \int_{t+\Delta t}^T \Lambda_s r ds \right] + o(\Delta t). \]

This gives us the difference equation \( H_t = \Lambda_t r \Delta t + E_t[H_{t+\Delta t}] \), which we can see as a recursive way of calculating \( B_t \) backward in time from the terminal condition \( H_T = 0 \) (note that \( P_T = 0 \)). We will now show that at each step, \( H_{t+\Delta t} = \Lambda_{t+\Delta t} P_{t+\Delta t} \) implies \( H_t = \Lambda_t P_t \), which together with the terminal condition will prove the assertion that \( H_t = \Lambda_t P_t \) for all \( t \leq T \).

Using the pricing equation at \( t + \Delta t \), re-arranging and dividing by \( e^{-\rho t} \xi^{k_t} \Delta t \) we obtain

\[ E_t \left[ \frac{e^{-\rho \Delta t} \xi^{k_{t+\Delta t}-k_t} u_c(C_{t+\Delta t}) P_{t+\Delta t} - u_c(C_t) P_t}{\Delta t} \right] = -r u_c(C_t) + o(\Delta t). \] (9)
We will now show that equation (9) is implied by the Euler equation of the agent that is pricing the asset at t.

Whenever $A > 0$ or $y = y_h$, $\Delta t$ may always be chosen small enough such that $k_{t+\Delta t} = k_t$ with probability $1 - o(\Delta t)$. The above then implies that

$$-\rho u_c(C_t)P_t + \mathcal{A}[u_c(C_t)P_t] = -ru_c(C_t),$$

which is agent 1’s Euler equation (1).

In the situation $(A, y) = (0, y_l)$, $k_t$ goes up by one with probability $\eta\Delta t$ over a short period of time. The correction factor $\xi$ then makes us use agent 2’s Euler equation, as we will see now:

$$e^{-\rho\Delta t}\left[(1 - \eta\Delta t)u_c(C)P + \eta\Delta t\xi u_c(C')P'\right] - u_c(C)P = -ru_c(C) + o(\Delta t)$$

Now multiply by $u_c(1 - C)/u_c(C)$ and use the definition of $\xi$ to see that this is equivalent to agent 2’s Euler equation:

$$\mathcal{A}[u_c(1 - C)P] = \rho u_c(1 - C)P - u_c(1 - C)r.$$

We see that in each case, we can go the steps backwards and obtain equation (9) from the Euler equation of the pricing agent, which completes the argument that (8) is a valid pricing formula.

### 2.8 Stationary equilibria: Bubbles and fiat money

So far, we have restricted ourselves to study equilibria that emerge as the limit of a finite-horizon economy. Figure 1 showed equilibrium policies and prices for the case $r > 0$. In the case where $r = 0$, the asset has the interpretation of fiat money. When using the above solution algorithm, we obviously end up with the non-monetary equilibrium where money is not valued (i.e. $P(t, a, y) = 0$). The asset is not valued in the final period $T - \Delta t$, and this feeds back forever since there are no dividends from it. Consumption equals the autarkic levels (i.e. $C(t, a, y) = y$), i.e. there is no insurance whatsoever.

However, there is clearly the possibility of stationary equilibria that do not arise as the limit of a finite-horizon economy. There could be bubble equilibria in which the asset price is higher than the value of its discounted dividends. For the case of money, this is the monetary equilibrium in which money has a positive price.
Any stationary equilibrium is characterized by taking the limiting case in equations (2) and (3) where time derivatives are set to zero: $P_t = 0$ and $C_t = 0$. Then the PDEs collapse to a system of four ODEs for the functions $P_l, P_h, C_l, C_h : [0, 0.5] \rightarrow \mathbb{R}^+$ in the variable $a$ (where $l$ is for $y = y_l$ and $h$ for $y = y_h$). The ODEs are given by:

$$P_A d^A = \rho P - r - \eta \left[ P' \left( \frac{1}{2} \frac{u_c(C')}{u_c(C)} + \frac{1}{2} \frac{u_c(1 - C')}{u_c(1 - C)} \right) - P \right] \quad (10)$$

$$C_A d^A = \frac{\eta}{2\lambda} \frac{P'}{P} \left[ \frac{u_c(C'')}{u_c(C)} - \frac{u_c(1 - C'')}{u_c(1 - C)} \right] \quad (11)$$

The initial conditions at $A = 0$ are $C_l(0) = y_l, C_h(0) = C_{h0}, P_l(0) = P_{l0}$ and $P_h(0) = P_{h0}$. We have to guess two variables (e.g. prices $(P_{l0}, P_{h0})$) and can then back out the third (e.g. $C_{h0}$) from agent 2’s Euler equation at $(A = 0, y = y_l)$, which is given in (4):

$$C_{h0} = 1 - \frac{y_h + r}{\lambda} \ln \left( \frac{\eta P_{h0}}{\rho P_{l0} - r} \right).$$

Symmetry of equilibrium imposes the following two terminal conditions at $A = 0.5$:

$$C_l(0.5) = 1 - C_h(0.5),$$

$$P_l(0.5) = P_h(0.5).$$

So we have to find two initial values to fulfill two conditions, which makes us expect that there is generically a finite number of solutions/equilibria.

The system of ODEs (10) and (11) is difficult to solve because the functions $C_l$ and $P_l$ have infinite slope as $A$ approaches zero (and so do $C_h$ and $P_h$ as $A$ approaches one). To see this, note that the right-hand side of both (10) and (11) approaches a constant as $A \rightarrow 0$ (since consumption and price functions are assumed continuous). However, the drift $d_A = y_l + rA - C_l$ converges to zero since $C_l \rightarrow y_l$ by continuity of $C_l$. This implies that the derivatives of $C_l$ and $P_l$ must converge to plus or minus infinity in order for the ODEs to hold. This feature can clearly be seen in figure 1: Small movements in $A$ lead to ever larger increases in $P_l$ as $A \rightarrow C$. As for $C_l$, its slope in $A$ goes to minus infinity.

At first, it seems strange that the pricing function can have infinite slope. Why do agents not demand more of the asset if small decreases in the aggregate state (which always occur when $y = y_l$) lead to large increases in asset prices, which
suggests that returns to the asset should be high? The key is that asset prices do not change dramatically over time since the drift $d_A$ converges to zero as $A \to 0$. In fact, since $\dot{P} = P_A d_A$, equation (10) tells us that $\dot{P}$ converges to a constant as $A \to 0$. So asset returns are not going to infinity as $P_A \to \infty$.

As for $C_l$, we also see that $\dot{C} = C_A d_A$ goes to a constant as $A \to 0$. This behavior is in line with the standard consumption-savings model in continuous time: The consumption function is continuous and is equal to the endowment $y_l$ in the case that the constraint binds. So the drift $d_A$ converges to zero, which implies an infinite slope in the consumption function in $A$ since $\dot{C}$ approaches a constant.

Since $d_A \to 0$, it is not clear that the economy will ever reach the constraint – it might just approach it at an ever slower rate but never get there. However, again reading equation (11) with the equality $\dot{C} = C_A d_A$ in mind tells us that the constraint must be reached in finite time. Since the right-hand side of (11) converges to a constant, $\dot{C}$ converges to a negative constant and $C_l$ must reach $C_l(0) = y_l$ at some point in time.

2.8.1 Computing stationary equilibria

In order to consider all possible cases, we will re-parameterize the initial conditions for the system of ODEs. We know that $C_{l0} = y_l < C_{h0} < y_h$, so we introduce a parameter $\gamma_0 = \in (0, 1)$ such that $C_{h0} = y_l + \gamma_0(y_h - y_l)$. For prices, we know that they always must lie above the complete-markets level $r/\rho$. So we introduce a parameter $\pi_0 \in (0, 1)$ and set $P_{h0} = \frac{r}{\rho} - \ln \pi_0$ so that $P_{h0}$ may take on any value between the complete-markets level and infinity. $P_{l0}$ can then be backed out from agent 2’s Euler equation (4), in which we set the time derivative of the function $P$ to zero since we are looking for a stationary equilibrium.

We now summarize the following recursive procedure of mapping $(\gamma_0, \pi_0)$ to prices and consumption at $A = 0$:

$$C_{l,0} = y_l,$$
$$C_{h,0} = y_l + \gamma_0(y_h - y_l),$$
$$P_{h,0} = \frac{r}{\rho} - \ln \pi_0$$
$$P_{l,0} = \frac{r + \eta \frac{u_c(1-C_{h0})}{u_c(y_h+r)} P_{h0}}{\rho + \eta}$$

From these initial conditions we then have to solve the system of four ODEs given
in (10) and (11) and check if the two terminal conditions at \( A = \frac{1}{2} \) are satisfied. To do this, we vary the parameters \((\gamma_0, \pi_0)\) on the unit square, which exhausts all possible initial conditions.

However, since \( P_l \) and \( C_l \) have infinite slope at \( A = 0 \) standard methods fail. We circumvent this by making time \( t \) the independent variable instead of \( A \). We will move backward in time in small steps \( \Delta t \) in time for the low-income scenario by:

\[
\begin{align*}
A_{t-\Delta t} &= A_t - d^A \Delta t, \\
C_{l,t-\Delta t} &= C_{l,t} - \dot{C}_l \Delta t, \\
P_{l,t-\Delta t} &= P_{l,t} - \dot{P}_l \Delta t.
\end{align*}
\]

So the increments will be very small (initially even zero) in \( A \), which allows for a good approximation at the crucial point \( A = 0 \). In order to solve backward the functions \((C_l, P_l)\), we also need to obtain the values of \((C_h, P_h)\) at \( A_{t+\Delta t} \). In order to do this, we have to go forward in time in the high-income scenario. The time increment \( \Delta t_h \) that makes the increment \( \Delta A_h = d^A_h \Delta t_h \) the same as the increment \( \Delta A_l = -d^A_l \Delta t \) in the low-income scenario. This yields:

\[
\Delta t_h = -\frac{d^A_l}{d^A_h} \Delta t.
\]

In the first iterations \( \Delta t_h \) is very small; indeed \( \Delta t_h = 0 \) in the first step. We then calculate

\[
\begin{align*}
C_h(A_{t-\Delta t}) &= C_h(A_t) + \dot{C}_h \Delta t_h, \\
P_h(A_{t-\Delta t}) &= P_h(A_t) + \dot{P}_h \Delta t_h.
\end{align*}
\]

We iterate on this algorithm until \( A_{t+\Delta t} > \frac{1}{2} \) and then check the terminal conditions. It may also happen that \( \dot{A}_t \) becomes non-negative, which also means there is no solution for the given initial conditions.

### 2.9 Benchmark: Representative-agent economy

For comparison to the two-case case, consider a model where the only source of income are dividends \( y \) from a tree, which follow a two-state Markov process with transition rate \( \eta \). The Euler equation for the representative agent is

\[
\frac{\mathcal{A}[P u_c(C)]}{P u_c(C)} = \frac{P_t}{P} + \eta \left[ \frac{P(t, y') u_c(y')}{P(t, y) u_c(y)} - 1 \right] = \rho - \frac{y}{P},
\]
where we have used the equilibrium condition $C = y$, which implies $C(t, y) = 0$
So we get
\[ \dot{p} = \frac{P_t}{P} = \rho - \frac{y}{P} - \eta \left[ \frac{P(t, y')u_c(y')}{P(t, y)u_c(y)} - 1 \right] \]
For the case without (aggregate) risk, set $y' = y$ and $P' = P$ and the bracket on
the right-hand side disappears.

2.10 Comparison to a Bewley economy

We will now compare the situation in our setting to a standard Bewley setting. Consider an incomplete-markets economy with a large number of ex-ante identical agents that face uncorrelated income risk. Specifically, assume that there is a measure-one continuum of identical agents whose labor income switches between $y_h$ and $y_l$ at hazard rate $\eta$. Unlike in the 2-agent setting before, assume that the labor-income processes are uncorrelated across agents. The only asset in the economy is again a Lucas tree with constant dividend $r$, so aggregate resources are constant over time and equal to $y = y_l + y_h + r$ by the law of large numbers. As before, agents are not allowed to go short in the asset. We are looking for a stationary equilibrium with a constant asset price $P$.

The agent’s budget constraint is
\[ P\dot{a}_t = ra_t + y_t - c_t. \]
For a given price level $P$ and dividend $r$, the consumer chooses consumption and savings to maximize expected discounted utility. In order to bring the model into the form of the standard Bewley model, define the helper variable $\tilde{a}_t \equiv Pa_t$ – note that we are keeping $P$ fixed while solving the agent’s problem. $\tilde{a}_t$ denotes real asset holdings, i.e. asset holdings measured in terms of the consumption good. In the case of money $\tilde{a}_t$ are real balances. Re-writing the budget constraint in terms of $\tilde{a}_t$ yields
\[ \dot{\tilde{a}}_t = \frac{r}{P} \tilde{a}_t + y_t - c_t. \]
Under this budget constraint we now have a standard Bewley problem in which a saver has access to one asset $\tilde{a}$ with a constant real rate of return $r/P$; for the case of money this return is zero.\footnote{Recall that we restrict attention to equilibria in which the price level is stationary. Of course there may be other equilibria in which the price level increases or decreases and so the return on money differs from zero, see the discussion in chapter 17 of Ljungqvist & Sargent (2004).} This means we can apply standard methods to obtain
the saver’s optimal policy functions for real asset holdings \( \tilde{a}_t \) given the return \( r/P \). As is well-known, \( r/P < \rho \) implies that there is a unique stationary distribution of asset holdings. If, however, \( r/P \geq \rho \) then agents “save away to infinity” and there exists no stationary distribution of asset holdings in the economy. We define \( P(r) = r/\rho \) as the lower bound on the price level that ensures that a stationary asset distribution exists.

As is well-known, we start off the economy at the ergodic asset distributions so that total real asset holdings in the economy remain constant over time. We denote total real asset holdings \( \tilde{A} \) in the economy as a function of the return by writing

\[
\tilde{A}(r/P) \equiv \int_0^1 \tilde{a}_i(r/P) di
\]

where \( \tilde{a}_i(r/P) \) are agent \( i \)'s real asset holding given the return \( r/P \) at time zero. It is well-known that \( \tilde{A}(r/P) \) is a continuous, strictly increasing function in \( r/P \) with the property that \( \lim_{r/P \to \rho} \tilde{A}(r/P) = \infty \). Since the borrowing limit is zero, we also know that \( \tilde{A}(0) > 0 \). To see this, note that even if the return to the asset is zero (the case of fiat money) some agents hold positive amounts in the tree. This is due to the precautionary-savings motive: agents with the high income realization build a buffer stock in order to save for the day when their income drops. This is the reason why money is held and valued in the standard Bewley model.

In equilibrium, we must have that aggregate demand for real asset holdings \( \tilde{A}(r/P) \) equals its supply. The supply of real assets at price level \( P \) is given by \( P \) where there is one tree in the economy, which in terms of goods is worth \( P \).

Let us first determine the equilibrium in the case of fiat money. The price level is determined by the money-market-clearing condition

\[
\tilde{A}(0) = MP,
\]

where the money supply \( M = 1 \) since we assumed that the asset is in unit supply. This equation pins down the price level in the economy as \( P = \tilde{A}(0) > 0 \). The situation is illustrated in figure 2. The money demand schedule \( \tilde{A}(0) \) is a straight vertical line; for any price level the return of money is zero and so the demand for real balances is invariant in \( P \). The supply of real balances is given by the 45-degree line \( P \). The unique intersection between the demand and supply schedule marks the monetary equilibrium.\(^{11}\)

---

\(^{10}\)Note that the asset price \( P \) denotes the quantity of goods that can be obtained when selling one unit of the asset. This is the inverse of how the price is usually denoted in monetary models: How much money do we need to buy one unit of the good.

\(^{11}\)For a discussion of money in Bewley models see chapter 17 in Ljungqvist & Sargent (2004).
There is also an equilibrium in which money is not valued. If $P = 0$, then autarky (i.e. $c_{t,i} = y_{t,i}$) is an equilibrium where money holdings of individual agents are indeterminate. Agents cannot use money to transfer resources to time and thus autarky is the only feasible consumption policy. Any money holdings are optimal since the stuff is useless.

Clearly, the monetary equilibrium constitutes a bubble: money is valued above its fundamental value, which is zero since there is no dividend to it. The fundamental equilibrium is the one with $P = 0$.

We will now show that for $r > 0$ the economy has a unique equilibrium which approaches the monetary equilibrium as $r$ approaches zero from above. Asset-market clearing implies

$$\tilde{A}(r/P) = P.$$  

As we saw before, $\tilde{A}(r/P)$ goes to infinity as $P$ approaches $P(r) = r/\rho$ from above. As we let $P$ become large, the return $r/P$ to the asset vanishes and real asset demand approaches $\tilde{A}(0)$. Furthermore, $\tilde{A}$ must be continuous and decreasing in $P$ since total asset holdings are continuous and increasing in $r/P$. The dashed hyperbola $\tilde{A}(r/P)$ in figure 2 shows real asset demand for a given $r > 0$. As for money, there must be a unique intersection of the real-asset-demand and the real-asset-supply schedule. There cannot be any equilibrium for $P \leq \bar{P}(r/P)$ since asset demand is infinite in this region. Thus, there is a unique equilibrium for the case $r > 0$. It is clear from the graph that in this equilibrium, both the price
level $P$ and real asset holdings $\tilde{A}$ are above the levels in the monetary equilibrium of the fiat-money economy.

Furthermore, note that for any fixed $P$, real asset demand $\tilde{A}(r/P)$ approaches $\tilde{A}(0)$ as we let $r \to 0$. This clearly implies that the equilibrium price level and real asset holdings converge to the monetary equilibrium as $r \to 0$.

This seems to suggest that there must be a bubble in the price of the asset for small $r$ – after all, dividends become arbitrarily small while the price of the asset approaches a positive constant. Also, the limiting equilibrium constitutes a bubble. However, we will now see that the equilibrium for $r > 0$ is actually not a bubble. Indeed, observe that when discounting future dividends by the equilibrium interest rate $r/P$, we obtain

$$\int_0^\infty e^{-\frac{r}{P}t} r dt = \frac{rP}{r} = P.$$ 

So the asset price equals the fundamental value of the asset and there is no bubble. This is possible since the interest rate and thus discounting go to zero in lockstep with the dividend. For the limiting case of money the above expression contains a division zero by zero and thus ceases to have meaning.

So interestingly, the price level $P_{eq}(r)$ and real asset holdings $\tilde{A}_{eq}(r)$ associated with the unique fundamental equilibrium for $r > 0$ converge to the bubble equilibrium (and not the fundamental equilibrium) of the fiat-money economy as we let $r \searrow 0$.\(^{(12)}\)

### 3 The economy with a bond

Consider the same setting as before, but now replace the tree with a bond as the single asset in the economy. Every agent can buy bonds $b_t \geq -\bar{B}$ at $t$, where $\bar{B} \geq 0$ is an exogenous borrowing limit. The bond costs $(1 - q_t \Delta t)$ at $t$ and yields one unit of consumption at $t + \Delta t$. So $q_t$ is the bond yield. The budget constraint at $t$ is

$$c_t \Delta t + b_t (1 - q_t \Delta t) = (\leq) y_t \Delta t + b_{t-\Delta t},$$

where $b_{t-\Delta t}$ is the asset position the agent has inherited from the previous period. Dividing by $\Delta t$ and taking limits $\Delta t \to 0$, we find

$$c_t + a_{b,t} = y_t + q_t b_t,$$

\(^{(12)}\)We thank Viktor Tsyrennikov and Ludo Visschers for helpful discussions concerning the Bewley economy.
where \( a_b = \lim_{\Delta t \to 0} (b_t - b_{t-\Delta t})/\Delta t \) denotes the drift in bonds.\(^{13}\) We see that there cannot be jumps in the position of the bond, if not consumption would have a mass point and cannot be smooth.

The Euler equation is easily derived as

\[
\frac{Au_c(c)}{u_c(c)} = \rho - q,
\]

which is of course entirely standard. In equilibrium, we look for a consumption function \( C(t, B, y) \), where \( B \in [-\bar{B}, \bar{B}] \) is the asset position of agent 1. \( C(\cdot) \) must be continuous in \( (t, B) \) for each \( y \). (Clearly, we need a continuous consumption function: If there was a jump in the function, then the growth rate of marginal utility is infinite when this jump is crossed, which makes it clearly desirable to re-allocate consumption between before and after the jump.)

Bond-market clearing requires that \( C + \tilde{C} = 1 \) (We re-normalize the total endowment to 1, i.e. \( y_l + y_h = 1 \)). The two Euler equations are

\[
\frac{u_c(c)}{u_c(C)} [C_t + (y + rB - C)C_B] + \eta \left[ \frac{u_c(C')}{u_c(C)} - 1 \right] = \rho - q
\]

\[
\frac{u_c(1 - C)}{u_c(1 - C)} [-C_t - (y + rB - C)C_B] + \eta \left[ \frac{u_c(1 - C')}{u_c(1 - C)} - 1 \right] = \rho - q
\]

Again, assuming CARA preferences simplifies things considerably, since \( u_{cc}/u_c = -\lambda \). Then, adding both equations and dividing by two yields

\[
\frac{1}{2} \frac{u_c(C')}{u_c(C)} + \frac{1}{2} \frac{u_c(1 - C')}{u_c(1 - C)} - 1 = \frac{\rho - q}{\eta}, \tag{12}
\]

so the average insurance motive in the economy determines the bond yield: The more the average agent wants to insure, the lower the yield. Again, since \( e^{\lambda \Delta C} \) is strictly convex, higher consumption risk will translate into lower yields for the asset; the increase in demand for the asset by the agent with the downward risk is stronger that the decrease in demand for the agent with the upward risk for this utility function. We can solve for the bond yield:

\[
q = \rho - \eta \left( \frac{1}{2} \frac{u_c(C')}{u_c(C)} + \frac{1}{2} \frac{u_c(1 - C')}{u_c(1 - C)} - 1 \right)
\]

\(^{13}\)Note that we can derive the same equation when starting from

\[
c_t \Delta t + b_t = (\leq) y_t \Delta t + (1 + q_t \Delta t)b_{t-\Delta t},
\]

although here the yield from the bond is stochastic, i.e. it depends on the state of \( y \) that realizes in the next period. In continuous time this doesn’t seem to matter!!

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Again, it is instructive to think about the complete-markets case: if agents are fully insured and consumption is constant across time and states, then the insurance motives are one and the bond yield must equal $\rho$. The function $e^{\lambda \Delta C}$ tells us how bond yields react to uncertainty. The higher $\Delta C$, the larger the average insurance motive and the lower the bond yield. Increasing $\lambda$ (the coefficient of absolute risk aversion) has the same effect. For a fixed $\Delta C$, a higher $\lambda$ means lower bond yields since agents are more risk-averse and the average insurance motive is higher.

Subtracting the agents’ Euler equations from each other and dividing by 2 yields

$$\dot{C} = C_t + (y + rB - C)C_B = \frac{\eta}{2\lambda} \left[ \frac{u_c(C')}{u_c(C)} - \frac{u_c(1 - C')}{u_c(1 - C)} \right] \tag{13}$$

Again, the difference in insurance motives determines the trend in consumption for normal times.

### 3.1 Constrained case: Sudden drop in bond yield

Let us now consider the situation where the first agent is constrained, i.e. $B = -\bar{B}$. We will focus on an equilibrium where the income-rich agent is saving, the income-poor agent is dissaving and the income-poor agent is constrained when hitting the bound on bonds. We will now see that this gives rise to a jump in the bond yield when the income-poor agent becomes borrowing-constrained. When at the constraint, only the income-rich agent prices the asset; from this agent’s Euler equation we find

$$q_0 = \rho - \eta \left( \frac{u_c(1 - C_h)}{u_c(1 - C_l)} - 1 \right).$$

In the moment before the income-poor agent hits the constraint, both agents participate in the market and the Euler equations give us

$$q_{\lim} = \rho - \eta \left( \frac{1}{2} \frac{u_c(C_h)}{u_c(C_l)} \frac{u_c(1 - C_h)}{u_c(1 - C_l)} - 1 \right).$$

Since the income-rich agent’s insurance motive is larger than the income-poor agent’s, the average insurance motive of agents pricing the asset drops as the income-poor agent becomes constrained. We see that $q_0 < q_{\lim}$ if $C_h > C_l$, which will be the case in our equilibrium.
How can this discontinuity in bond yields be consistent with optimizing behavior and continuity of the consumption functions? Recall that $C_l$ must be continuous in $B$, otherwise marginal utility would not grow at the optimal rate in the moment before the income-poor agent becomes constrained. We will now see from the poor agent’s budget constraint how a continuous consumption stream can be obtained:

$$y_l - q_0 B = C_l = y_l - q_{lim} B - \dot{B}_{lim}.$$  

The first equality represents the budget constraint when the agent is at the constraint. The second equality uses the budget constraint in the moment before reaching the constraint and imposes that the consumption function must be continuous. Since $q_0 < q_{lim}$, it must be that $\dot{B}_{lim} < 0$, where the inequality is strict (whereas of course $\dot{B}_0 = 0$ once the agent is constrained). The agent’s interest payments drop in the moment that he becomes constrained because the interest rate tanks. This is depicted in the lower panel of figure 3, which shows the elements of the poor agent’s budget constraint before and after hitting the budget constraint. The agent rationally foresees the decrease in interest payments and maintains a smooth consumption stream by selling off positive quantities of the bond before reaching the constraint. This is shown in the upper panel of figure 3. The proceeds from these sales break away when becoming constrained, exactly off-setting the decrease in interest payment.

For the rich agent, the picture is as follows. Before the constraint is reached, high interest rates make it optimal to save and accumulate more bonds. Once the constraint is reached, equilibrium in the bond market requires that the agent does not further increase her bond position. This is ensured by the interest rate dropping just enough to make maintaining the same bond position optimal.

Figure 4 illustrates the reason for the sudden drop in the bond yield in terms of demand and supply. The rich agent’s demand for the bond is decreasing in the bond price $1 - q$. The cheaper the bond, the more this agent desires to save. In the situation before the constraint is hit, the poor agent’s supply of the bond is as depicted by the straight increasing line in figure 4. When prices are high, the poor agent supplies new bonds, i.e. $\dot{B} < 0$. The unconstrained equilibrium is such that the rich agent accumulates more bonds, which are issued by the poor agent at price $1 - q_{lim}$.

Once the poor agent hits the constraint, he cannot issue any new bonds any more. The supply function becomes inelastic at this point, which is depicted by the kinked upward-sloping line in figure 4. The price $1 - q_{lim}$ does not clear the market any more because demand exceeds supply. The price has to increase to
1 - q_0 (i.e. the bond yield has to drop) in order to bring demand for new bonds to zero.

### 3.2 Solution strategy

We will solve for the equilibrium by backward-iterating on the consumption functions. For this, we first derive how the consumption function changes in time in a non-stationary solution.

The first agent’s consumption at \( B = -\bar{B} \) equals labor income minus interest payments,

\[
C(t, -\bar{B}, y_l) = y_l - q(t, -\bar{B}, y_l)\bar{B},
\]

from which we can back out the bond yield as function of consumption:

\[
q(t, -\bar{B}, y_l) = \frac{y_l - C(t, -\bar{B}, y_l)}{\bar{B}}. \tag{14}
\]

We can now use the second agent’s Euler equation (which has to hold with equal-
Figure 4: Bond demand and supply at the constraint

ity, since he is not constrained) to find that

\[
\dot{C}(t, -\bar{B}, y_1) = C_t(t, -\bar{B}, y_1) = \frac{\rho}{\lambda} - \frac{y_t - C(t, -\bar{B}, y_1)}{\lambda \bar{B}} - \frac{\eta}{\lambda} \left( \frac{u_c(1 - \tilde{C})}{u_c(1 - C)} - 1 \right),
\]

where we have used the fact that the economy stays put at \( B = -\bar{B} \), i.e. \( a_B = y_t - r\bar{B} - C = 0 \) and so the term in \( C_B \) is zero.

When the second agent is constrained, we analogously find

\[
q(t, \bar{B}, y_h) = \frac{C(t, \bar{B}, y_h) - y_h}{\bar{B}},
\]

\[
\dot{C}(t, \bar{B}, y_h) = C_t(t, \bar{B}, y_h) = -\frac{\rho}{\lambda} + \frac{C(t, \bar{B}, y_h) - y_h}{\lambda \bar{B}} + \frac{\eta}{\lambda} \left( \frac{u_c(\tilde{C})}{u_c(C)} - 1 \right).
\]

Note that we can autonomously solve the PDE (or system of ODEs, in the infinite-horizon case) for consumption given by equations (13) and (15) and then back out the bond yield from this using (12) and (14). Economically, the restriction that both agents’ marginal utilities must grow at the same rate (plus continuity of consumption) is strong enough so that prices are not needed to solve for the allocation. Note that this is not the case in the economy with the risky asset, since asset returns are state-contingent there.
The following would be the solution strategy for solving the system of two differential equations for \( C(\cdot, 1) \) and \( C(\cdot, 2) \) in the infinite-horizon/stationary case:

We guess that the first agent is constrained at \((0, y_l)\) and make a guess for \( C(-\bar{B}, y_l) \) that lies below \( y_l \) (interest payments have to be positive) and above zero. Then we can back out the bond yield as

\[
q(-\bar{B}, y_l) = \frac{y_l - C(-\bar{B}, y_l)}{B}
\]

Then the rich agent’s Euler equation pins down consumption at \( B = -\bar{B} \) and \( y = y_h \):

\[
C(-\bar{B}, y_h) = C_l(-\bar{B}) + \frac{1}{\lambda} \ln \left( 1 + \frac{\rho - q(-\bar{B}, y_l)}{\eta} \right)
\]

The system of ODEs is then

\[
C_B = \frac{\eta}{2\lambda(y + qB - C)} \left( e^{-\lambda(\tilde{C} - C)} - e^{\lambda(\tilde{C} - C)} \right)
\]

Note that it is important here that the laws of motion \( a_B = y + qB - C \) stay away from zero.

The following terminal condition at \( B = 0 \) on the two ODEs is imposed by symmetry of the equilibrium:

\[
C(0, \bar{y}) = 1 - C(0, \bar{y}).
\]

Note that this implies that both consumption functions, \( C(B, y_l) \) and \( C(B, y_h) \), have the same slope at 0:

\[
C_B(0, y_l) = \frac{\eta(e^{-\lambda\Delta C} - e^{\lambda\Delta C})}{2\lambda(y_l - C_l)} = \frac{\eta(e^{\lambda\Delta C} - e^{-\lambda\Delta C})}{2\lambda(1 - y_h - 1 + \tilde{C}_h)} = C_B(0, y_h).
\]

So if we solve an analogous system of ODEs from \( \bar{B} \) back to zero, the solutions connect at \( B = 0 \) and are differentiable at this point.

### 3.3 Results

The following figure shows the results for the stationary case with CARA utility and parameters \( \rho = 0.04, \lambda = 1, \eta = 0.1 \) and \( y_l = 0.4 \) (which means that \( y_h = 0.6 \)).
We see that the bond yield jumps down when the income-poor agent hits the borrowing constraint. He then ceases to be a net supplier of assets (in the sense that $\dot{B} = 0$) and prices have to adjust to make the net demand of the income-rich agents for the bond zero. Since these are keen to lend in order to insure against worse times, the yield of the asset drops.

### 3.4 Phase diagrams

Can we use phase diagrams to tell us something about the properties of equilibrium? First, we have to note that it will not be useful to work with time $t$ as the independent variable because the economy is stochastic. Unlike in a deterministic economy, the equilibrium functions are not smooth functions in time but jump when the exogenous state $w$ jumps. The only hope is thus to use the asset position $B$ as the independent variable, as Scheinkman & Weiss do in their analysis.

However, the first problem with this approach is that the two ODEs for $C_l$ and $C_h$ are not autonomous in $B$: $dC_l/dB$ and $dC_h/dB$ still depend on $B$ (through $\dot{B}$), and not only on $C_l$ and $C_h$ themselves. So there is no hope that we can make use of the theorems characterizing steady states of autonomous ODEs.\textsuperscript{14}

\textsuperscript{14}Even if we make the system autonomous by introducing a new independent variable $s$ and specifying $B(s) = s$, this is of little use: Since we have $dB/ds = 1 > 0$, there cannot be any
However, we can still follow Scheinkman & Weiss and make some inference in a $C_l - C_h$-diagram when just considering the signs of $dC_l/dB$ and $dC_h/dB$. Let us assume that in equilibrium the income-rich agent always saves ($\dot{B}_h > 0$) and the income-poor agent always dissaves ($\dot{B}_l < 0$), at least on the interior of the state space. This essentially means that we restrict the analysis on a subset $S$ of the full $(B, C_l, C_h)$-space and on the trajectories that stay within this subset $S$. If we make this restriction, we can proceed in the same way as Scheinkman & Weiss. The signs of $dC_l/dB$ and $dC_h/dB$ now depend solely on the sign of the function

$$\Gamma(C, C') = \frac{u_c(C')}{u_c(C)} - \frac{u_c(1 - C')}{u_c(1 - C)},$$

which is the difference in the insurance motives. We have $\Gamma(C, C') = 0$ iff $C = C'$. So both consumption functions stay constant in $B$ if and only if $C_l = C_h$ at the initial condition. Whenever $C_h > C_l$, then both $dC_l/dB$ and $dC_h/dB$ are strictly positive. Also, we will always stay in the region where $C_h > C_l$ since the trajectory cannot cross a steady state $C_h = C_l$ (trajectories cannot cross). This is the economically reasonable case: Consumption functions are increasing in income and in assets. The economically unreasonable case arises when $C_h < C_l$. We will then always stay in the region where $C_h < C_l$ and both $dC_l/dB$ and $dC_h/dB$ are negative throughout.

Analysis using this two-dimensional diagram becomes more complicated if we try to allow arbitrary signs of $\dot{B}$. The signs of $dC_l/dB$ and $dC_h/dB$ then depend on both the sign of $\Gamma$ and the signs of $\dot{B}_l$ and $\dot{B}_h$. However, the locus where $\dot{B}_l = 0$ (or $\dot{B}_h = 0$) in the $C_l - C_h$-plane changes when $B$ changes. This makes it hard to analyze trajectories (although it might still be possible).

4 Capital accumulation

Consider now an economy with standard capital accumulation and a Cobb-Douglas production function. Capital is the only asset, and capital holdings cannot be negative for any of the two agents. The labor endowment $z_t$ follows a two-state Markov chain just as $y_t$ did before (however now, wages are endogenous since they depend on the capital-labor ratio).

The law of motion for an agent’s capital holdings $k_t$ is

$$\frac{dk_t}{dt} = (r_t - \delta)k_t + z_tw_t - c_t.$$

steady state in this three-dimensional system.
For an unconstrained agent (i.e. \( k_t > 0 \)), the consumption process \( C \) must fulfill the Euler equation
\[
\frac{\partial u_c(C)}{u_c(C)} = \eta \left[ \frac{u_c(C')}{u_c(C)} - 1 \right] + \frac{u_{cc}(C)}{u_c(C)} \left[ C_t + C_k + C_{\tilde{k}} \right] = \rho + \delta - r,
\]
where \( r \) is a function of the aggregate capital stock \( K_t = k_t + \tilde{k}_t \).

Now, let us assume CARA preferences as before. Then, when both agents participate in asset markets, adding the two Euler equations yields
\[
\dot{C} + \dot{\tilde{C}} = \rho - \delta - \rho + \eta \left[ \frac{1}{2} \frac{u_c(C')}{u_c(C)} + \frac{1}{2} \frac{u_c(\tilde{C}')}{u_c(\tilde{C})} - 1 \right].
\]
The higher \( r \) (i.e. the lower the capital stock \( K \)), the higher consumption growth. Since consumption is increasing in \( K \), this means that capital is accumulated when its returns are high. Also, if the average insurance motive in the economy is high, then consumption growth and capital accumulation are high (unless a reversal happens).

Subtracting the two Euler equations yields
\[
\dot{C} - \dot{\tilde{C}} = \eta \lambda \left[ \frac{u_c(C')}{u_c(C)} - \frac{u_c(\tilde{C}')}{u_c(\tilde{C})} \right].
\]
So as before, the difference in the two insurance motives is linked to consumption growth in normal times. If agent 1 is in the good income state (and thus fears a reversal and has a high insurance motive), then agent 2 would welcome a reversal and has thus a low insurance motive. This means agent 1 is saving and his consumption trends upward (in normal times), whereas agent 2 is dissaving and his consumption trends downward relative to agent 1 in normal times.

### 4.1 P-K space

The laws of motion for \( P \) and \( K \) are
\[
\dot{K} = (r - \delta)K + w - C - \tilde{C} = Y - \delta K - C - \tilde{C}
\]
\[
\dot{P} = \frac{(z - P)w}{K} + \frac{P\tilde{C}}{K} - \frac{(1 - P)C}{K}
\]

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So if \( z > P \) (i.e. agent 1’s share of labor income is higher than his share of capital income), then \( P \) trends upward (when taking consumption decisions out of the picture). As for consumption, things are straightforward: When agent 1’s consumption is high, then his asset share tends to decrease; when agent 2’s consumption is high, this tends to tilt the asset distribution in agent 1’s favor.

### 4.2 The constrained case

When agent 2 is constrained (i.e. \( \tilde{k} = 0 \)), then only agents of type 1 price the asset. We have \( \tilde{C} = (1 - z)w \) and

\[
\dot{C} = \frac{r - \delta - \rho}{\lambda} + \frac{\eta}{\lambda} \left[ \frac{u_c(C')}{u_c(C)} - 1 \right].
\]

So since agent 1 has consumption risk in this situation, any stationary point (i.e. \( \dot{C} = 0 \)) must have an interest rate lower than in the representative-agent case. In general, this insurance motive increases consumption growth, which means higher savings.

Another noteworthy feature is the point when \( \tilde{k} \) reaches zero. Note that in this moment, the average insurance motive of participants in the asset market jumps upward, which implies that average consumption growth of asset-market participants jumps upward (since \( \bar{K} \) and thus \( r \) is continuous in time). So the trend of capital returns changes since the savings rate changes.

### 4.3 Results

Figure 6 shows the phase diagram for the capital economy. We have left out points where agents save in both income states (lower left corner) and points where both agents dissave in both states (upper right corner); these regions must be left under any shock history and so cannot be part of the ergodic set.

If agent 1 has \( k_1 = 17 \) units of capital and agent 2 has zero, then the economy will stay at this point if agent 1 is in her productive state (the red arrow becomes a dot here). Once the reversal happens, agent 1 starts to sell off capital and agent 2 starts to accumulate it. Agent 1 is dissaving faster than agent 2 is accumulating, so the total capital stock in the economy is decreasing (the blue arrow is not pointing diagonally left/up, but has a slightly flatter angle).

If agent 2 stays productive for some time, the wealth distribution becomes more equal, with total capital slightly decreasing. If another reversal happens and
agent 1 becomes productive again, the same happens in the opposite direction. However, the total capital stock never falls below a level of $K = 13$ or so. If agent 1 has a long productive spell, then he will again end up with the entire capital stock, but at a lower level (say $k_1 = 14$). From this point on, only agent 1 is saving, and the economy moves towards the point $(k_1 = 17, k_2 = 0)$ again, from which we started.

4.4 Benchmark case: Capital accumulation with a representative agent

Straightforward calculation yield that the Euler equation for the representative agent is

\[
\frac{\dot{A}u_c(C)}{u_c(C)} = \frac{u_{cc}(C)}{u_c(C)} \dot{C} = \rho + \delta - r,
\]

or for CARA preferences

\[
\dot{C} = \frac{r - \delta - \rho}{\lambda}.
\]

This gives rise to the usual convergence to a steady state: When $K$ is low, $r$ is high and consumption growth is high. Since $C$ is increasing in $K$, this means
the agent is saving until reaching the steady state where \( r = \delta + \rho \). The agent is
dissaving and consumption is decreasing when we start at a capital stock above
the steady-state level.

In the steady state, we have

\[
\begin{align*}
    r_{ss} &= \rho + \delta \\
    K_{ss} &= \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}
\end{align*}
\]

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