Stochastic Sorting*

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Abstract

We develop a general model of matching markets under uncertainty. In the model, agents’ payoff-relevant characteristics are realized after matching takes place, and matches are formed based on ex-ante attributes that are noisy signals of the true characteristics. We derive conditions under which there is positive or negative assortative matching, and determine the properties of the distributions of ex-post attributes of matched partners. The conditions for sorting relate properties of the match payoff function with the stochastic order imposed on the conditional distributions of the agents’ characteristics given their ex-ante attributes. We analyze both the transferable utility case and a class of risk sharing problems with nontransferable utility. Finally, we provide conditions under which the properties of the match payoff function and its degree of complementarity can be identified from observed match data.


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1 Introduction

In many economic environments, matching occurs with limited information. In the labor market for example, firms hire workers whose education level is known, but their true productivity is revealed only after a significant period of time has passed. Likewise, in the marriage market, the attractiveness of the partners at the age of marriage embodies a lot of uncertainty. For instance, the promising young consultant may fail to realize all of her earnings potential, and the attractive young football player may get out of shape and look less attractive later on. Finally, information aggregation within teams and partnerships (e.g. Marshak and Radner (1972)) often rests on the premise that collaborators have ex ante different abilities that result in differential signal informativeness about uncertain events. But it is the ex post realized signals which are the relevant inputs for making decisions once the team is formed.

This paper extends the standard Beckerian matching model to allow for stochastic characteristics (henceforth ex post types). Agents are heterogeneous, but their payoff-relevant ex post types are unknown at the match formation stage. As a result, matching takes place based on the current observable characteristics of the partners (henceforth ex ante types), which are noisy predictors of their ex post types. This setting affords a useful reinterpretation of the standard matching problem: each agent can be identified with a probability distribution of the ex post type given the agent’s ex ante type, and thus the allocation problem can be thought of as one of matching distributions. For example, if the ex ante type is education and the ex post type is productivity, then the distribution of productivity of those with an MBA from Harvard is different from that of those with a degree from a non top business school. Hence, if sorting is based on education, matched partners are pairs of distributions over productivities.

An important feature of our model is that, even if there is assortative matching based on ex ante types, there will be equilibrium mismatches in ex post types. Agents are matched together but the actual realization of types implies the allocation is not optimal ex post. Matches are formed between pairs that contradict the equilibrium properties of the deterministic Beckerian model. This is an appealing feature of the model since mismatch is a universal characteristic of the data, taken from any application. Very often, this is dealt with by assuming there is some additive measurement error, drawn from a particular distribution. While we do not per se think of the stochastic outcomes as measurement error, they can be interpreted as such.

In this framework, we derive sufficient conditions on the conditional distributions mentioned above as well as on the properties of the match payoff function that induce monotone matching (positive or negative assortative matching), without making parametric assumptions on the distribution. Intuitively,
the results differ depending on the stochastic order assumed for the conditional distributions of ex post types. We focus on the two textbook stochastic orders: first-order stochastic dominance (FOSD) and mean preserving spread (MPS). In each case, we derive suitable complementarity properties of the match payoff function that engender monotone matching.

In the transferable utility case, we find that under FOSD, if the match payoff function is supermodular, then there is positive assortative matching (PAM) in the ex ante types (i.e., higher ex ante types match with each other). In turn, the distribution of matched partners’ ex post types is logsupermodular, and thus a stochastic form of positive sorting ensues: higher ex-post types match together on average. The opposite result, negative assortative matching (NAM), obtains if the match value is submodular and FOSD is assumed. Observe that this result is closely related to the standard sorting result in Becker (1973), and actually contains it as a special case: deterministic types are ordered and therefore trivially satisfy FOSD. Hence, if the distribution of ex post types are degenerate for all values of ex ante types, Becker’s result ensues. Regarding MPS, we find that there is PAM provided the cross-partial derivative of the match payoff function with respect to ex post types is supermodular, and NAM if it this derivative is submodular in ex post types.

In the nontransferable utility case, we focus on a representative class of risk sharing problems. We analyze matching of strictly risk averse agents whose the objective to share risks. Following the work by Ackerberg and Botticini (2002) on landowners in early Renaissance Tuscany, it has long been recognized that an important motive for pairwise matching is risk sharing. Consequently, Serfes (2006), Schulhofer-Wohl (2006), Chiappori and Reny (2005) and Legros and Newman (2007) have modeled matching in the presence of risk aversion. A feature that makes this problem challenging is that, when agents are risk averse, in general the Pareto frontier of the match is nonlinear (utility cannot be transferred at a constant rate) and more complex conditions are needed to ensure monotone matching (see Legros and Newman (2007)). Whereas the literature so far has been concerned with the allocation problem of agents with different preferences, we analyze the distinctive case of matching agents with different conditional distributions of ex post types, i.e. they differ in their risky endowments.

Under risk sharing, we can pin down optimal sorting patterns in terms of the properties of the utility function and the stochastic order imposed on the distributions. When preferences satisfy Constant Relative Risk Aversion (CRRA) and the distributions are ordered by FOSD, then a sufficient condition for PAM is that the match payoff function exhibit sufficient complementarities in ex post types. When distributions are, in addition, members of the location family and the match payoff is modular, then the allocation always satisfies NAM. That is, in the absence of payoff complementarities and when the
riskiness of the distributions is constant, value is maximized by matching riskier with less risky types.

For general risk averse preferences, when one side of the matching market is risk neutral and the other side strictly risk averse, then independently of the stochastic order imposed on the distributions there is always PAM whenever the match value is supermodular. Likewise, when all agents exhibit Constant Absolute Risk Aversion (CARA) preferences, there is PAM if the match value is supermodular and no sorting when the match value is modular. This establishes that risk aversion is not sufficient for sorting: there must be an income effect (which is absent under CARA).

This setup is well-suited for identifying the technological features of a matching market. Unlike the deterministic matching model, there is variation in the observed match outcomes, because mismatch occurs in equilibrium. As a result, once we know the technological characteristics of the match payoff function, we can quantify the extent of the complementarities between different matched types. This is important because the complementarities determine the extent of the welfare loss from mismatch.

Our work is related to a vast literature on matching. There are matching models that deal with mismatch in different ways. First, models of random search with two-sided heterogeneity (e.g., Shimer and Smith (2000)) have equilibrium mismatch. This is due to the inability to meet new trading partners fast enough, and to the random and undirected nature of meetings between agents. Given the opportunity cost of delay, agents are willing to accept a less than perfect partner. The implication of these models is that while there is mismatch, the model predicts a sharply delineated range of matches: there is complete mismatch above the reservation type, and none at all below. This is often hard to reconcile with observed data. Chade (2006) gets beyond this by considering a matching model with search frictions and noisy types, albeit in a strictly nontransferable utility setting. The nature of mismatch in our model, as well as the notion of stochastic sorting, borrows from that paper (but we allow for transfers). Second, mismatch may be due to unobserved heterogeneity. For example, when types are multidimensional and at least one dimension is not observed to the econometrician, then the observed outcome appears like mismatch (see for example Choo and Siow (2006), Galichon and Salanié (2010) and Lindenlaub (2013)). Intuitively, from the agents’ viewpoint there is no mismatch since they observe the entire bundle of the characteristics of the partners.

Finally, there are a number of interesting issues that we have not analyzed, but that are nonetheless very promising for future research. First, we do not allow for the possibility of rematching. Upon the realization the mismatch of their ex post types, agents would typically mutually prefer to form new matches if allowed to do so. As a result, the existing realized matches would be unstable. When such rematching is completely frictionless, the problem trivially reduces to a sequence of Beckerian static
matching models. If instead, there is some cost, then only those for whom the mismatch is sufficiently big will be willing to incur the cost. This is akin to the search model of Eeckhout and Kircher (2011) who use the mismatch to identify complementarities. Second, we assume there is complete information: uncertainty is symmetric as no agent knows anyone’s ex post type. Under asymmetric information and an incentive to signal emerges, which would be interesting to analyze.

2 The Model Setup

Consider two populations (workers and firms, or men and women) each of measure one, in $\mathcal{X}$ and $\mathcal{Y}$. We distinguish between the ex ante type and the ex post realization. The ex ante types are $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and the ex post types are denoted by $\omega$ and $\sigma$. index a distribution. What matters for the value of a match is the ex post realized types $\omega, \sigma$. The match value is denoted by $h(\omega, \sigma)$. We can think of $x, y$ as education, and of $\omega, \sigma$ as income (in the marriage market) or productivity (in the labor market).

There are two sets of distributions. Those of the ex ante types denoted by $\Gamma(x)$ and $\Psi(y)$, as well as those of the ex post types $F(\omega|x)$ and $G(\sigma|y)$. The latter are contingent on the ex ante type.

The matching pattern is based on the ex ante types $x, y$ only, taking into account the distribution of ex post, realized types $\omega, \sigma$. Given one-to-one matching with transfers, the equilibrium notions of stability, CORE and competitive equilibrium are equivalent, each generating the same allocation. Denote by $\mu(x)$ the equilibrium allocation. If $\mu(x)$ is monotone, there is positive assortative matching provided $\mu$ is increasing, and negative assortative matching provided $\mu(x)$ is decreasing. Under PAM, from market clearing, the allocation must satisfy $\Gamma(x) = \Psi(\mu(x))$ for all $x$. Under NAM, it satisfies $\Gamma(x) = 1 - \Psi(\mu(x))$ for all $x$.

We investigate how the sorting pattern of ex ante types $x$ and $y$ relates to the properties of the distributional orders of $F$ and $G$ as well as the match value technology $v(\omega, \sigma)$. One can of course pick many orders, but two natural candidates are first order stochastic dominance (FOSD) and mean-preserving spread (MPS).

**Definition 1** Consider the following orders:

1. First Order Stochastic Dominance: $F \succeq_{FSD} G \iff F(z) \leq G(z), \forall z$ or $F_x \leq 0$ or $\int u(x)dF(x) \geq \int u(x)dG(x), \forall u$ increasing.

2. Second Order Stochastic Dominance: $F \succeq_{SSD} G \iff \int_0^z F(z)dz \geq \int_0^z G(z)dz$ or $\int_0^x F_x(z|x)dz \leq 0$ or $\int u(x)dF(x) \geq \int u(x)dG(x), \forall u$ increasing and concave.
3. Mean Preserving Spread: SOSD and \( E[\omega|x] = E[\sigma|y] \)

It is immediate that SOSD implies FOSD.

FOSD directly generalizes the standard matching model à la Becker to the case of stochastic types. Under FOSD, better types \( x \) have distributions that are on average better. In Becker, the type distribution is Dirac and therefore increasing types \( x \) and \( y \) trivially satisfy FOSD.

MPS instead does not capture any notion of better distributions. In particular, by imposing that all types have a distribution with a common mean, there is no dominance in the first-order sense. What the MPS captures is a notion of increasing variability in the outcomes, and therefore of increasing risk.

3 Transferable Utility

We analyze the baseline case with transferable utility. Agents are risk neutral, and the moreover they are indifferent between ex ante or ex post transfers, as long as the ex ante transfer is equal to the expected ex post transfer. We therefore focus on the ex ante transfer.

Denote by \( V(x,y) \) the ex ante value of a match between a type \( x \) and \( y \):

\[
V(x,y) = \int_\omega \int_\sigma f(\omega|x)g(\sigma|y)h(\omega,\sigma)d\omega d\sigma
\]

(1)

To analyze the sorting pattern, we do not need to calculate the wages. It is sufficient to verify the properties of the match value \( V(x,y) \). In particular, it is sufficient to check whether \( V \) is supermodular.

3.1 Dominance and Complementarities

We analyze two orders (FOSD and MPS) and attempt to find the properties on \( h(\omega,\sigma) \) that guarantee sorting. The following Proposition summarizes the result:

**Proposition 1** Let \( F,G \) be ordered by FOSD, then \( h(\omega,\sigma) \) is supermodular (sub–) \( \iff \) PAM (NAM).

Let \( F,G \) be ordered by Mean Preserving Spread. Then \( h_{\omega\sigma}(\omega,\sigma) \) is supermodular (sub–) \( \iff \) PAM (NAM).

**Proof.** In Appendix. □
The proof of the first part consists in twice integrating by parts the ex ante value of the match. This gives rise to the expression

\[ V(x, y) = \int_\sigma F(\omega|x)h(\omega, \sigma)d\omega - \int_\omega F(\omega|x)h(\omega, \sigma)d\omega + \int_\sigma \int_\sigma F(\omega|x)G(\sigma|y)h(\omega, \sigma)d\sigma d\omega \]

and the cross partial is then positive provided since the first two terms only depend on one of the variables \(x\) or \(y\):

\[ V_{xy} = \int_\sigma \int_\sigma F(\omega|x)G(\sigma|y)h(\omega, \sigma)d\sigma d\omega > 0 \]

which is satisfied whenever \(h(\omega, \sigma)\) since by FOSD \(F_x < 0\) and \(F_y < 0\). The result for the MPS in the distribution of types is obtained by integrating by parts four times. Now we obtain an expression for the value \(V(x, y)\), where several terms depend on \(\int_\omega F(\omega|x)d\omega\) which is independent of \(x\), and likewise, \(\int_\sigma G(\sigma|y)d\sigma\) is independent of \(y\). Then \(V(x, y)\) is supermodular in \((x, y)\) if and only if

\[ V_{xy} = \int_\sigma \left( \int_\sigma G(\sigma|y)d\sigma \right) \int_\omega \left( \int_\omega F(\omega|x)d\omega \right) h(\omega, \sigma)d\sigma. \]

By definition of a mean preserving spread, \(\int_\omega \left( \int_\sigma G(\sigma|y)d\sigma \right) \int_\omega \left( \int_\omega F(\omega|x)d\omega \right)\) is supermodular. Therefore, \(V(x, y)\) is supermodular whenever \(h(\omega, \sigma)\) is positive.

Supermodularity \((f_{xy} > 0\) for a differentiable function \(f\)) is sometimes interpreted as monotonicity of order two. For a multivariate function, it must be monotonic in each of any two dimensions. Likewise, supermodularity of \(f_{xy}\) (i.e. \(f_{xxyy}\)) is convexity of order two.

So far, we only allowed for the distributions \(f\) and \(g\) to be independent. It may well be the case that the ex post realizations of matched pairs are dependent. Let the joint distribution be expressed as \(k(\omega, \sigma|x, y) \neq f(\omega|x)g(\sigma|y)\). Then

\[ V(x, y) = \int_\omega \int_\sigma k(\omega, \sigma|x, y)h(\omega, \sigma)d\omega d\sigma. \]

A result by Karlin and Rinott (1980) establishes the following:

**Proposition 2 (Karlin and Rinott)** If \(k\) is logsupermodular in \((\omega, \sigma|x, y)\) and \(h(\omega, \sigma)\) is logsupermodular in \((\omega, \sigma)\), then \(V(x, y)\) is logsupermodular and as a result there is PAM.
The requirements are of course much stronger: $k$ logsupermodular is much stronger than FOSD and so is logsupermodularity of $h$ compared to supermodularity.

4 Risk Sharing

Now consider a general set up where the match value created consists in risk sharing. On the day of marriage, a potentially matched pair faces uncertainty about the future realizations of each other types. Given risk averse preferences, we assume that they commit to efficiently share risk ex post, once the types are revealed. We do not model the possibility for rematching (divorce).

Risk sharing and non-linear preferences imply that the pairwise Pareto frontier of the matched pair is non linear. As a consequence, the results from the former section under transferable utility are not applicable. Types $x$ and $y$ match ex ante and commit to ex post efficient risk sharing. Ex post, the pair generates joint output $h(\omega, \sigma)$, and ex ante, before the realization of the types, agents choose a partner taking into account and commit to the ex post efficient division of the surplus. Upon the realization of the types $\omega, \sigma$, the risk averse agents maximize the joint utility of individual consumption, i.e. they choose consumption $c_x, c_y$ to maximize $u(c_x) + u(c_y)$ where $c_x + c_y = h(\omega, \sigma)$.

The match value that embodies the optimal risk sharing problem is denoted by $\Phi(x, y, v)$ and represents the utility a woman of type $y$ obtains from a match with a man of type $x$ when she leaves him utility $v$. It can be written as:

$$\Phi(x, y, v) = \max_{c_x, c_y} \int_{\omega} \int_{\sigma} f(\omega|x)g(\sigma|y)u(c_y, v)d\omega d\sigma$$

s.t. $c_x + c_y = h(\omega, \sigma)$

$$\int_{\omega} \int_{\sigma} f(\omega|x)g(\sigma|y)u(c_x, v)d\omega d\sigma \geq v.$$ 

The maximization problem in the constraint can be written as (where $c$ is shorthand for $c_x$):

$$u'(h(\omega, \sigma) - c) = \lambda u'(c),$$

where $\lambda$ is the Langrangian multiplier. This gives an optimal consumption level $c^*(\omega + \sigma, \lambda)$. From the constraint, we then obtain the equilibrium value for $\lambda^*$ by solving:

$$\int_{\omega} \int_{\sigma} f(\omega|x)g(\sigma|y)u(h(\omega, \sigma), \lambda)d\omega d\sigma = v.$$
The task is to transform the objective such that it incorporates the Langrangian, and then check the supermodularity of the transformed problem. In particular, given the first-order condition is satisfied $\Phi_x + \Phi_v \partial v / \partial x = 0$, we calculate the total cross-partial derivative $d^2\Phi / dx dy$ which can be written as (after substituting for $\partial v / \partial x$ from the first order condition):

$$
\frac{d^2}{dxdy} \Phi = \Phi_{xy} + \Phi_{vy} \frac{\partial v}{\partial x} = \Phi_{xy} - \frac{\Phi_x}{\Phi_v} \Phi_{vy},
$$

There will be Positive Assortative Matching in ex ante types $x, y$ provided $\Phi_{xy} > \frac{\Phi_x}{\Phi_v} \Phi_{vy}$.

The question we aim to address here is whether we can find a tight relation between preferences, technology and the order on the distribution.

### 4.1 Constant Relative Risk Aversion (CRRA)

CRRA preferences can be represented by

$$
u(c) = \frac{c^\alpha}{\alpha}, \quad \alpha \in (0, 1).
$$

Now we can establish the following result:

**Lemma 1** Consider CRRA preferences $u(c) = \frac{c^\alpha}{\alpha}$, a match value function $h(\omega, \sigma)$ and distributions $F, G$ ordered by FOSD. Then there is positive (negative) assortative matching provided

$$
V_{xy} + \frac{1 - \alpha}{\alpha} \frac{V_x V_y}{V} > 0,
$$

where $V(x, y) = \int_\omega \int_\sigma f(\omega|x)g(\sigma|y)\frac{h(\omega, \sigma)^\alpha}{\alpha}d\omega d\sigma$. A sufficient condition for Positive Assortative Matching is that $h(\omega, \sigma)$ is $\alpha$-rootsupermodular, i.e., $\frac{h(\omega, \sigma)^\alpha}{\alpha}$ is supermodular.

**Proof.** In Appendix. ■

To get some insight into the mechanism behind sorting, consider the case where $h(\omega, \sigma) = \omega + \sigma$. Then there are no complementarities in the match value ($h_{\omega\sigma} = 0$). Now $V_{xy} < 0$, since $\frac{\partial^2}{\partial \omega \partial \sigma} \frac{(\omega + \sigma)^\alpha}{\alpha} = (\alpha - 1)(\omega + \sigma)^{\alpha-2} < 0$. At the same time,

Now we can immediately analyze the logarithmic utility function as the limit case of CRRA where $\alpha \to 0$. 

8
**Corollary 1** Consider logarithmic preferences \( u(c) = \log c \), a match value function \( h(\omega, \sigma) \) and distributions \( F, G \) ordered by FOSD. Then there is positive (negative) assortative matching provided

\[
V_{xy} + V_x V_y > 0,
\]

where \( V(x, y) = \int_{\omega} f(\omega|x) g(\sigma|y) \log(h(\omega, \sigma)) d\omega d\sigma \). A sufficient condition for Positive Assortative Matching is that \( h(\omega, \sigma) \) is logsupermodular, i.e., \( \log h(\omega, \sigma) \) is supermodular.

**Proof.** In Appendix. ■

**Location Family.** The objective is to keep the riskiness constant while exclusively changing the first order stochastic dominance.

Consider a class of distributions \( F(\omega|x) = F(\omega - x_s) \) and \( G(\sigma|y) = G(\sigma - y_t) \) where \( s \) and \( t \) are positive constants that scale the distribution, and \( x \) and \( y \) determine the location of the distribution.

**Proposition 3** Consider CRRA preferences, a match value function \( h(\omega, \sigma) = \omega + \sigma \) and distributions ordered in the location family, \( F(\omega|x) = F(\omega - x) \) and \( G(\sigma|y) = G(\sigma - y) \). Then there is always Negative Assortative Matching for any distributions \( F \) and \( G \).

**Proof.** In Appendix. ■

**4.2 Constant Absolute Risk Aversion (CARA)**

The preferences under Constant Absolute Risk Aversion are represented by \( u_x(c) = -e^{-\rho_x c}, u_y(v) = -e^{-\rho_y v} \).

**Proposition 4** Consider CARA preferences \( u_x(c) = -e^{-\rho_x c}, u_y(v) = -e^{-\rho_y v} \), a match value function \( h(\omega, \sigma) \) and distributions \( F, G \) ordered by FOSD. Then there is Positive (Negative) Assortative Matching provided \( h(\omega, \sigma) \) is supermodular \( h_{\omega\sigma} > 0 \) (submodular \( h_{\omega\sigma} < 0 \)).

**Proof.** In Appendix. ■

Under CARA preferences, the pattern of sorting is completely determined by the properties of the match value function \( h(\omega, \sigma) \) (viz., whether it is supermodular or submodular). Risk sharing between the matched pair does not affect the pattern of sorting. Of course, that is due to the fact that there is no income effect. Across different types, the value of risk is invariant, and as a result, no matter which type of partner an agent matches with, she the value of insurance to her will be constant. Therefore, this is priced uniformly across types and it does not affect the allocation pattern.
4.3 One-sided Risk Neutrality

Proposition 5 Consider one side of the market is risk neutral. Then for any order on $F$ and $G$, and for any preferences of the other side of the market, the sorting pattern is determined by the properties of $h(\omega, \sigma)$: there is PAM (NAM) $\iff h_{\omega\sigma} > (<)0$.

Proof. To Be Completed. ■

When one side of the market is risk neutral (say $x$), all agents of type $x$ potentially provide full insurance to all risk averse agents $y$. As a result, the risk that a given type $y$ faces is priced equally at all types $x$. As a result, there is no motive for sorting based on the risk sharing motive. As a result, the sorting pattern is completely determined by the complementarities (substitutes) in the match value $h(\omega, \sigma)$.

The next corollary establishes that the properties of the matching pattern under CARA and with one-sided risk neutrality are the same as those under transferable utility.

Corollary 2 Let the match value function be $h(\omega, \sigma)$. Then under CARA preferences or when one side of the market is risk neutral, the sorting pattern is effectively as under transferable utility.

5 Identification

Can we identify the degree of complementarity between ex post types, i.e., the properties of $h(\omega, \sigma)$?

Following Athey and Haile (2002), we posit that a model is identified if, given the implications of equilibrium behavior in a matching market, the joint distribution of bidders consumptions (and incomes?) is uniquely determined by the joint distribution of observables. Our model $(\mathbb{W}, \Theta)$ is defined as a set of joint distributions $\mathbb{W}$ over the underlying random variables $(x, y, \omega, \sigma)$, and a collection of mappings $\Theta : \mathbb{W} \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of joint distributions over the observable random variables.

Definition 2 A model $(\mathbb{W}, \Phi)$ is identified off for every $(W, \hat{W}) \in \mathbb{W}^2$ and $(\theta, \hat{\theta}) \in \Theta^2$, $\theta(W) = \hat{\theta}(\hat{W})$ implies $(W, \theta) = (\hat{W}, \hat{\theta})$.

5.1 Non-Transferable Utility – CRRA

Denote the ex ante payoff to the men by $v(x)$ and the ex ante payoff to the women by $w(y)$. Observe that $w(y) = \Phi(x, y, v(x))$ evaluated at the equilibrium allocation.
To this end, we write equilibrium consumption of the male $c^m$ and the female $c^f$ as:

$$c^m(\omega, \sigma | x, y) = \left( \frac{v(x)}{V(x, y)} \right)^{\frac{1}{\alpha}} h(\omega, \sigma)$$

$$c^f(\omega, \sigma | x, y) = \left[ 1 - \left( \frac{v(x)}{V(x, y)} \right)^{\frac{1}{\alpha}} \right] h(\omega, \sigma),$$

where $y$ is evaluated at the equilibrium allocation $y = \mu(x)$ and $v(x)$ is the equilibrium payoff to type $x$. 

$$V(x, y) = \int_{\omega}^{\sigma} \int_{\sigma}^{\sigma} f(\omega|x)g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha} d\omega d\sigma.$$

The value of $v$ of each matched pair depends on the equilibrium allocation of ex ante types $x, y$. Under PAM, the allocation $y = \mu(x)$ satisfies $\int d\Gamma(x) = \int d\Psi(y)$. For example, if $\Gamma(\cdot) = \Psi(\cdot)$ then $y = \mu(x) = x$.

The first question is whether based on observed consumption we can identify the properties of $h(\omega, \sigma)$. This is trivial if $V(x, y)$ and $v(x)$ (depending on $x$) are known. Provided the system of two equations is invertible we can solve for $(\omega, \sigma)$ given the consumption bundle of the pair $c_x, c_y$.

Both partners choose the other partner optimally. For example, a woman of type $y$ chooses type $x$ to maximize $\Phi$. Observe then that we can calculate $v(x)$ from integrating the first order condition evaluated along the equilibrium allocation $\mu(x)$:

$$v^*(x) = \int_{0}^{x} \frac{-\Phi_x(x, \mu(x), v)}{\Phi_v(x, \mu(x), v)} dx$$

where $\Phi = \left( V(x, y)^{\frac{1}{\alpha}} - v^{\frac{1}{\alpha}} \right)^\alpha$. Therefore

$$v^*(x) = \int_{0}^{x} \left( \frac{V(x, \mu(x))}{v(x)} \right)^{\frac{1}{\alpha} - 1} V_x(x, \mu(x)) dx$$

The challenge is how to solve for $V(x, y)$ and $v(y)$. While this does not depend on $(\omega, \sigma)$ because it integrates over the entire distributions $f, g$, it does depend on the function $h(\cdot, \cdot)$, precisely what we want to estimate. We can also write the expression for $w^*(y)$

$$w^*(y) = \int_{0}^{y} \frac{-\Phi_y^{-1}(\mu^{-1}(y), y, w)}{\Phi_w^{-1}(\mu^{-1}(y), y, w)} dy$$
where $\Phi^{-1} = \left( V(x, y)^{\frac{1}{\alpha}} - w^{\frac{1}{\alpha}} \right)^{\alpha}$ and hence

$$w^*(y) = \int_0^y \left( \frac{V(\mu^{-1}(y), y)}{w} \right)^{\frac{1}{\alpha}-1} V_y(\mu^{-1}(y), y) dy$$

### 5.2 Symmetry

Consider a symmetric setup between men and women. Alternatively, one can think of two-sided matching from a single population, e.g., same-sex marriage. Then $\Gamma(x) = \Psi(y), f(\omega|x) = g(\sigma|y)$ and $h(\omega, \sigma) = h(\sigma, \omega)$. Symmetry of $\Gamma$ and $\Psi$ implies that $\mu(x) = x$. Therefore, under PAM, we can write

$$c^m(\omega, \sigma|x, y) = \left( \frac{v(x)}{V(x, x)} \right)^{\frac{1}{\alpha}} h(\omega, \sigma)$$

and

$$c^f(\omega, \sigma|x, y) = \left( \frac{w(x)}{V(x, x)} \right)^{\frac{1}{\alpha}} h(\omega, \sigma) = \left[ 1 - \left( \frac{v(x)}{V(x, x)} \right)^{\frac{1}{\alpha}} \right] h(\omega, \sigma).$$

From the FOC, we know that the optimal allocation satisfies $\Phi_x - \Phi_v v' = 0$ or, given $\Phi(x, y, v) = \left( V(x, y)^{\frac{1}{\alpha}} - v(x)^{\frac{1}{\alpha}} \right)^{\alpha}$, we get:

$$\alpha \left( V(x, y)^{\frac{1}{\alpha}} - v(x)^{\frac{1}{\alpha}} \right)^{\alpha-1} \left( \frac{1}{\alpha} V(x, y)^{\frac{1}{\alpha}} - v(x)^{\frac{1}{\alpha}} - V_x - \frac{1}{\alpha} v(x)^{\frac{1}{\alpha}} v'(x) \right) = 0$$

and likewise $w'(y) = \left( \frac{v(x,y)}{w(y)} \right)^{\frac{1}{\alpha}-1} V_y(x, y)$. Now observe the following:

1. Along the equilibrium allocation, $x = y$ so that $w'(x) = \left( \frac{V(x,x)}{w(x)} \right)^{\frac{1}{\alpha}-1} V_y(x, x)$

2. Symmetry implies $V_y(x, x) = V_x(x, x)$, so that $w'(x) = V_x(x, s)$

$\Rightarrow w'(x) = v'(x)$ and hence $w(x) = v(x)$ since the boundary conditions are the same.

Now this implies that

$$\left( \frac{v(x)}{V(x, x)} \right)^{\frac{1}{\alpha}} = \left( \frac{w(x)}{V(x, x)} \right)^{\frac{1}{\alpha}} = \left[ 1 - \left( \frac{v(x)}{V(x, x)} \right)^{\frac{1}{\alpha}} \right] h(\omega, \sigma)$$

and therefore

$$\left( \frac{v}{V} \right)^{\frac{1}{\alpha}} = \frac{1}{2}.$$
Now identification follows immediately since the share of the surplus is always one half and \( h(\cdot, \cdot) \) is monotonic. As a result, any change in \( h(\omega, \sigma) \) induces a change in consumption \( c^m(\omega, \sigma|x, y) \) (and therefore \( c^f \)). Of course, the symmetry premise will be rejected whenever we observe \( c^m \neq c^f \) within a married couple. This is an extremely strong identification restriction that is testable.

5.3 Asymmetry – An Example under CRRA

Consider the following example. Types \( x \) and \( y \) are drawn from general distributions \( \Gamma \) and \( \Psi \) on support \([0, 1]\). The ex post distributions of types \( f(\omega|x) \) and \( g(\sigma|y) \) have support \( \{\omega, \overline{\omega}\} \) and \( \{\sigma, \overline{\sigma}\} \), and such that the probability \( F(\omega) = 1 - x \) and \( F(\overline{\omega}) = x \); likewise, the probability \( G(\sigma) = 1 - y \) and \( G(\overline{\sigma}) = y \). To reduce notation, denote \( h(\omega, \sigma) = h \), \( h(\omega, \overline{\sigma}) = h' \), \( h(\overline{\omega}, \sigma) = h'' \), and set \( h(\omega, \overline{\sigma}) = 0 \) without loss of generality. We also make the assumption that \( \Psi(y) \) is uniform, which is also without loss since what matters is the allocation \( \mu \) under PAM that \( y = \mu(x) = \Psi^{-1}(\Gamma(x)) \), which can immediately be normalized by a transformation of the function \( \Gamma \).

Under PAM, \( \mu(x) = \Gamma(x) \). Hence, given CRRA, \( V(x, \mu(x)) \) satisfies:

\[
V(x, \Gamma(x)) = \int_0^\omega \int_0^\sigma f(\omega|x)g(\sigma|\Gamma(x)) \frac{h(\omega, \sigma)^\alpha}{\alpha} d\omega d\sigma \\
= \frac{1}{\alpha} (x \Gamma(x) \gamma^\alpha + x(1 - \Gamma(x)) \beta^\alpha + \Gamma(x)(1 - x) \delta^\alpha)
\]

In order to analyze consumption, we need to derive the ratio \( v/V \). Then, from the FOC

\[
v(x) = \int_0^x \frac{-\Phi_x(x, \mu(x), v)}{\Phi_v(x, \mu(x), v)} dx
\]

where \( \Phi = \left(V(x, y)^{\frac{1}{\alpha}} - v^{\frac{1}{\alpha}}\right)^\alpha \) so that

\[
v(x) = \left(V(x, y)^{\frac{1}{\alpha}} - w^{\frac{1}{\alpha}}\right)^\alpha = \int_0^x \left(\frac{V(x, \Gamma(x))}{v}\right)^{\frac{1}{\alpha} - 1} V_s(x, \Gamma(x)) dx.
\]

This gives an implicit expression for \( v \). The problem is that we cannot obtain an explicit expression for \( v \), except in the case where \( \alpha = 1 \). Since we will use a method that needs to calculate the consumption share explicitly, we therefore restrict attention to the case where \( \alpha = 1 \), risk neutrality. While we can generally identify the model with risk neutrality and therefore TU, we like to interpret this model as one with CRRA preferences in the limit as \( \alpha \to 1 \).

Then for this example with \( \alpha = 1 \), we have \( V(x, y) = xy\gamma + x(1 - y)\beta + y(1 - x)\delta \) and \( v(x) = \ldots \)
where \( A \) change is given by:

\[
\int_0^x V_x(s, \Gamma(s), s) ds.
\]

Partially differentiating \( V \) with respect to \( x \) and evaluating at \( y = \Gamma(x) \), we obtain \( V_x(x, \Gamma(x)) = \beta + (\gamma - \beta - \delta)\Gamma(x) \). Therefore,

\[
v(x) = \beta x + (\gamma - \beta - \delta) \int_0^x \Gamma(r) dr
\]

Then we can write consumption \( c^m(\omega, \sigma) \) along the equilibrium allocation as:

\[
e^m(\gamma, \beta, \delta; i) = \frac{v(x)}{V(x, \Gamma(x))} = \frac{\beta x + (\gamma - \beta - \delta) \int_0^x \Gamma(s) ds}{x \Gamma(x) y + x(1 - \Gamma(x)) \beta + \Gamma(x)(1 - x) \delta} \bigg|_i
\]

We want to evaluate the change in consumption \( dc \) as the function \( h(\omega, \sigma) \) changes. For this example, a change in the function \( h \) is completely captured by a change in \((\gamma, \beta, \delta)\). In particular, we can evaluate explicitly the change in consumption evaluated at a given point of \( h \), denoted by \( i \in \{\gamma, \beta, \delta\} \). To simplify notation, in what follows we will denote \( \int_0^x \Gamma(s) ds = A \) and \( c^m = \frac{N}{\Gamma} \).

The change in consumption evaluated at any point \( i \), given the change in the entire function \( h \) his change is given by:

\[
dc|_i = \frac{\partial c}{\partial \gamma} |_i d\gamma + \frac{\partial c}{\partial \beta} |_i d\beta + \frac{\partial c}{\partial \delta} |_i d\delta
\]

The change in consumption will be uniquely determined for a given change in the technology \((\gamma, \beta, \delta)\) provided the matrix \( C \) is non-singular, i.e., its determinant is non-zero.

\[
|C| = \frac{1}{D^2} \left| \begin{array}{ccc}
(AD - N x \Gamma) \gamma + ND & (AD - N x \Gamma) \gamma & (AD - N x \Gamma) \gamma \\
((x - A)D - N x(1 - \Gamma)) \beta & ((x - A)D - N x(1 - \Gamma)) \beta + ND & ((x - A)D - N x(1 - \Gamma)) \beta \\
(-AD - N \Gamma(1 - x)) \delta & (-AD - N \Gamma(1 - x)) \delta + ND & (-AD - N \Gamma(1 - x)) \delta + ND
\end{array} \right| \neq 0
\]

\[
= \frac{1}{D^2} \left| \begin{array}{ccc}
A_\gamma + ND & A_\gamma & A_\gamma \\
A_\beta & A_\beta + ND & A_\beta \\
A_\delta & A_\delta & A_\delta + ND
\end{array} \right| \neq 0
\]

\[
= \frac{A_\gamma A_\beta A_\delta}{D^2} \left[ \left(1 + \frac{ND}{A_\gamma} \right) \left( \frac{ND}{A_\beta} + \frac{ND}{A_\delta} + \frac{N^2 D^2}{A_\beta A_\delta} \right) - \frac{ND}{A_\beta} - \frac{ND}{A_\delta} \right]
\]

\[
= N^2 [A_\gamma + A_\beta + A_\delta + ND]
\]

where \( A_\gamma = (AD - N x \Gamma) \gamma, A_\beta = ((x - A)D - N x(1 - \Gamma)) \beta, A_\delta = (-AD - N \Gamma(1 - x)) \delta \). Therefore

\[
|C| = N^2 [(AD - N x \Gamma)(\gamma - \beta - \delta) - N \Gamma \delta + x D \beta - N x \beta + ND]
\]

\[
= N^2 [(AD - N x \Gamma)(\gamma - \beta - \delta - N \Gamma \delta + x D \beta - N x \beta + ND)]]
\]

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It is sufficient that $|C|$ is non-zero at any point in the domain of $x$. We therefore check its value at $x = 1$:

$$|C| = N^2 [(AD - N)(\gamma - \beta - \delta) - N\delta + D\beta - N\beta + ND]$$

$$= N^2 [AD(\gamma - \beta - \delta) - N\gamma + D\beta + ND]$$

At $x = 0$:

$$|C| = N^2 [AD(\gamma - \beta - \delta) + ND] > 0$$

6 Concluding Remarks

In order to capture mismatch due to evolving types, we have proposed a simple model that generalizes the frictionless matching model with deterministic types (Becker (1973)) to a matching model with stochastic types. The sorting pattern – whether there is monotone matching and if so, whether there is positive or negative assortative matching – depends on the characteristics of the match value and the nature of the order on the distributions. For the baseline case for example, we find that first order stochastic dominance together with super modularity of the match value ensures positive sorting.

The stochastic nature of the model offers a natural setting to analyze risk sharing in partnerships. Now, the equilibrium allocation in addition also depends on the properties of the utility function. We can characterize the sorting pattern for broad classes of preferences such as constant relative risk aversion. This then allows us to investigate whether the underlying match value can be identified from observed realized matches.
Appendix

Proof of Proposition 1

Proof. Integrating by parts twice, we obtain:

\[
V(x, y) = \int_{\sigma}^{\sigma} \left[ \int_{\omega}^{\omega} f(\omega|x)h(\omega, \sigma)d\omega \right] g(\sigma|y)d\sigma = \int_{\sigma}^{\sigma} \left[ F(\omega|x)h(\omega, \sigma)|^{\omega}_{\omega} - \int_{\omega}^{\omega} F(\omega|x)h_{\omega}(\omega, \sigma)d\omega \right] g(\sigma|y)d\sigma
\]

\[
= \int_{\sigma}^{\sigma} \left[ h(\omega, \sigma) - \int_{\omega}^{\omega} F(\omega|x)h_{\omega}(\omega, \sigma)d\omega \right] g(\sigma|y)d\sigma = \int_{\sigma}^{\sigma} h(\omega, \sigma)g(\sigma|y)d\sigma - \int_{\sigma}^{\sigma} z(x, \omega)g(\sigma|y)d\sigma
\]

\[
= \int_{\sigma}^{\sigma} v(\omega, \sigma)g(\sigma|y)d\sigma - \int_{\omega}^{\omega} F(\omega|x)h_{\omega}(\omega, \sigma)d\omega + \int_{\sigma}^{\sigma} \int_{\omega}^{\omega} F(\omega|x)G(\sigma|y)h_{\omega\sigma}(\omega, \sigma)d\sigma d\omega
\]

(2)

where \( z(x, \omega) = \int_{\omega}^{\omega} F(\omega|x)h_{\omega}(\omega, \sigma)d\omega \).

\( V(x, y) \) is supermodular in \((x, y)\) if and only if \( \int_{\sigma}^{\sigma} \int_{\omega}^{\omega} F(\omega|x)G(\sigma|y)h_{\omega\sigma}(\omega, \sigma)d\sigma d\omega \) is supermodular in \((x, y)\) given the first two terms in \( V(x, y) \) only depend exclusively on either \( x \) or \( y \). That is,

\[
V_{xy} = \int_{\sigma}^{\sigma} \int_{\omega}^{\omega} F_{x}(\omega|x)G_{y}(\sigma|y)h_{\omega\sigma}(\omega, \sigma)d\sigma d\omega > 0
\]

and given FOSD implies \( F_{x}(\omega|x) < 0, G_{y}(\sigma|y) < 0 \), supermodularity of \( h(\omega, \sigma) \), i.e., \( h_{\omega\sigma}(\omega, \sigma) > 0 \) implies supermodularity of \( V \). The same logic applies for \( h_{\omega\sigma}(\omega, \sigma) < 0 \) which induces NAM.

To investigate the role of MPS, we start from the expression (2) and further integrate by parts twice
more to obtain the expression for \(V(x, y)\):

\[
\begin{align*}
\int_{\sigma} \int_{\omega} h(\omega, \sigma) g(\sigma|y) d\sigma & - \int_{\sigma} \left[ \left( \int_{\omega} F(\omega|x) \right) h(\omega, \sigma) \right] d\sigma - \int_{\omega} F(\omega|x) h(\omega, \sigma) g(\sigma|y) d\sigma \\
& = \int_{\omega} \left( \int_{\omega} F(\omega|x) \right) h(\omega, \sigma) g(\sigma|y) d\sigma + \int_{\omega} F(\omega|x) h(\omega, \sigma) g(\sigma|y) d\sigma \\
& = \int_{\omega} \left( \int_{\omega} F(\omega|x) \right) h(\omega, \sigma) g(\sigma|y) d\sigma + \int_{\omega} F(\omega|x) h(\omega, \sigma) g(\sigma|y) d\sigma \\
& = \int_{\omega} \left( \int_{\omega} G(\sigma|y) d\sigma \right) \int_{\omega} F(\omega|x) d\omega h(\omega, \sigma|\omega) d\sigma \\
& = \int_{\omega} \left( \int_{\omega} F(\omega|x) d\omega \right) h(\omega, \sigma|\omega) d\sigma
\end{align*}
\]

By assumption of mean preserving spread, \(\int_{\omega} F(\omega|x) d\omega\) is independent of \(x\) and \(\int_{\omega} G(\sigma|y) d\sigma\) is inde-

dependent of \(y\). Then \(V(x, y)\) is supermodular in \((x, y)\) if and only if

\[
\int_{\omega} \int_{\omega} G(\sigma|y) d\sigma \int_{\omega} F(\omega|x) d\omega h(\omega, \sigma) d\sigma
\]

is supermodular in \((x, y)\). Given \(\int_{\omega} G(\sigma|y) d\sigma \int_{\omega} F(\omega|x) d\omega\) is supermodular by definition of mean preserving spread, \(V(x, y)\) is supermodular whenever \(h(\omega, \sigma|\omega) d\sigma\) is positive. The same logic applies for \(h(\omega, \sigma|\sigma) < 0\) which induces NAM.

**Proof of Lemma 1**

**Proof.** The maximization problem in the constraint then reduces to:

\[
(h(\omega, \sigma) - c)^{\alpha - 1} = \lambda c^{\alpha - 1} \Rightarrow c = \frac{\lambda^{\frac{1}{\alpha}}}{1 + \lambda^{\frac{1}{\alpha}}} h(\omega, \sigma)
\]
Then the constraint is:

\[
\int_{\omega} \int_{\sigma} f(\omega|x) g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha} \left( \frac{\lambda^{\frac{1}{1-\alpha}}}{1 + \lambda^{\frac{1}{1-\alpha}}} \right) d\omega d\sigma = v,
\]

or

\[
\frac{\lambda^{\frac{1}{1-\alpha}}}{1 + \lambda^{\frac{1}{1-\alpha}}} = \left( \frac{v}{\int_{\omega} \int_{\sigma} f(\omega|x) g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha}} \right)^{\frac{1}{\alpha}}.
\]

Denoting \( V(x, y) = \int_{\omega} \int_{\sigma} f(\omega|x) g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha} d\omega d\sigma \) we can write

\[
\Phi(x, y, v) = \int_{\omega} \int_{\sigma} f(\omega|x) g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha} \left( \frac{1}{\alpha} - \left( \frac{v}{\int_{\omega} \int_{\sigma} f(\omega|x) g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha}} \right)^{\frac{1}{\alpha}} \right)^{\alpha} d\omega d\sigma
\]

\[
= \left( V(x, y) \frac{1}{\alpha} - v \frac{1}{\alpha} \right)^{\alpha}.
\]

Observe the CES functional form in \( V(x, y) \) and \( v \).

We calculate the cross-partial total derivative of \( \Phi \):

\[
\Phi_x = \Phi^{\frac{n-1}{\alpha}} V(x, y)^{\frac{1}{\alpha}-1} V_x
\]

\[
\Phi_y = \Phi^{\frac{n-1}{\alpha}} V(x, y)^{\frac{1}{\alpha}-1} V_y
\]

\[
\Phi_v = -\Phi^{\frac{n-1}{\alpha}} v^{\frac{1}{\alpha}-1}
\]

\[
\Phi_{xy} = -\Phi^{\frac{n-1}{\alpha}} v^{\frac{1}{\alpha}-1} (\alpha - 1) \frac{1}{\alpha} V(x, y)^{\frac{1}{\alpha}-1} V_y
\]

\[
\Phi_{xy} = \Phi^{\frac{n-1}{\alpha}} V(x, y)^{\frac{1}{\alpha}-1} V_{xy} + A^{\alpha-1} \left( \frac{1}{\alpha} - 1 \right) V(x, y)^{\frac{1}{\alpha}-1} V_{xy} + A^{\alpha-2}(\alpha - 1) \frac{1}{\alpha} V(x, y)^{\frac{2}{\alpha}-2} V_x V_y
\]

There is PAM provided \( \frac{d^2\Phi}{dxdy} = \Phi_{xy} - \frac{\Phi_x}{\Phi_v} \Phi_{vy} > 0 \) or

\[
\frac{d^2\Phi}{dxdy} = V_{xy} + \frac{1 - \alpha}{\alpha} \frac{V_x V_y}{V} > 0.
\]

The result follows nearly immediately if one considers that the problem with \( h \) has the same condition for PAM as before \( V_{xy} + \frac{1 - \alpha}{\alpha} \frac{V_x V_y}{V} > 0 \), where

\[
V(x, y) = \int_{\omega} \int_{\sigma} f(\omega|x) g(\sigma|y) \frac{h(\omega, \sigma)^\alpha}{\alpha} d\omega d\sigma.
\]
Now we now from the first result on FOSD under TU that after twice integrating by parts and taking the cross partial, we can write

$$V_{xy} = \int_{\omega}^{\sigma} \int_{\sigma}^{\sigma} F_x(\omega|x)G_y(\sigma|y) \frac{\partial^2}{\partial \omega \partial \sigma} \frac{h(\omega, \sigma)^\alpha}{\alpha} d\omega d\sigma.$$ 

Note also that $V_x V_y > 0$ since $V_x < 0$ and $V_y < 0$. To see this, observe that from equation (2)

$$V(x, y) = \int_{\sigma}^{\sigma} v(\omega, \sigma) g(\sigma|y)d\sigma - \int_{\omega}^{\omega} \int_{\sigma}^{\sigma} F(\omega|x) g(\sigma|y) v_\omega(\omega, \sigma)d\omega d\sigma$$

we obtain that $V_x = -\int_{\omega}^{\omega} \int_{\sigma}^{\sigma} F_x(\omega|x) g(\sigma|y) v_\omega(\omega, \sigma) d\omega d\sigma > 0$ since $F_x < 0$ from FOSD and $v_\omega > 0$. Now immediate that it is sufficient that $V_{xy} > 0$, which is satisfied when $\frac{h(\omega, \sigma)^\alpha}{\alpha}$ is supermodular, i.e., $h$ is $\alpha$-rootsupermodular. ■

**Proof of Corollary 1**

**Proof.** With logarithmic utility, the maximization problem in the constraint can be written as:

$$\frac{1}{\omega + \sigma - c} = \lambda \frac{1}{c}$$

and therefore

$$c = \frac{\lambda}{1 + \lambda} (\omega + \sigma)$$

and the constraint is therefore:

$$\int_{\omega}^{\omega} \int_{\sigma}^{\sigma} f(\omega|x) g(\sigma|y) \log(\omega + \sigma) d\omega d\sigma = v(x)$$

$$\log \left( \frac{\lambda}{1 + \lambda} \right) + \int_{\omega}^{\omega} \int_{\sigma}^{\sigma} f(\omega|x) g(\sigma|y) \log(\omega + \sigma) d\omega d\sigma = v(x)$$

Denoting $V(x, y) = \int_{\omega}^{\omega} \int_{\sigma}^{\sigma} f(\omega|x) g(\sigma|y) \log(\omega + \sigma) d\omega d\sigma$ and $a = \frac{\lambda}{1 + \lambda}$ we can write

$$\log (a) = v(x) - V(x, y)$$

$$a = e^{v(x)-V(x,y)}$$

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Since \( \omega + \sigma - c = (1 - a)(\omega + \sigma) \), we can write the objective of the problem as

\[
\Phi(x, y, v(x)) = \int_{\omega}^{\overline{\omega}} \int_{\sigma}^{\overline{\sigma}} f(\omega|x)g(\sigma|y)[\log(1 - a) + \log(\omega + \sigma)] \, d\omega d\sigma
\]

\[
= \log(1 - a) + V(x, y)
\]

\[
= \log \left( 1 - e^{v(x)}e^{-V(x, y)} \right) + \log e^{V(x, y)}
\]

\[
= \log \left( e^{V(x, y)} - e^{v(x)} \right)
\]

Now we calculate the cross-partial total derivative of \( \Phi \):

\[
\Phi_x = \frac{V_x e^V}{e^V - e^{v(x)}}
\]

\[
\Phi_y = \frac{V_y e^V}{e^V - e^{v(x)}}
\]

\[
\Phi_v = -\frac{e^{v(x)}}{e^V - e^{v(x)}}
\]

\[
\Phi_{vy} = \frac{V_y e^{v(x)} e^V}{e^V - e^{v(x)}}
\]

\[
\Phi_{xy} = \frac{e^V}{e^V - e^{v(x)}} \left[ V_{xy} - \frac{V_x V_y e^{v(x)}}{e^V - e^{v(x)}} \right]
\]

and therefore there is PAM provided

\[
\frac{d^2 \Phi}{dxdy} = V_{xy} + V_x V_y > 0,
\]

where \( V(x, y) = \int_{\omega}^{\overline{\omega}} \int_{\sigma}^{\overline{\sigma}} f(\omega|x)g(\sigma|y) \log(\omega + \sigma) d\omega d\sigma \).

Observe that under FOSD, \( V_{xy} = F_x G_y \frac{\partial^2 \log(\omega + \sigma)}{\partial \omega \partial \sigma} = -F_x G_y (\omega + \sigma)^{-2} < 0 \). This indicates that if \( V_x, V_y \) are not too large (i.e. \( F_x, G_y \) are not too large), there will be NAM.

Proof of Proposition 3

Proof. We can write the function \( V(x, y) \) as:

\[
V(x, y) = \frac{1}{\alpha} \int \int \left( \frac{\omega - x}{s} \right) g \left( \frac{\sigma - y}{t} \right) (\omega + \sigma)^{\alpha} d\omega d\sigma
\]

\[
= \frac{1}{\alpha} \int g \left( \frac{\sigma - y}{t} \right) \left[ (\overline{\omega} + \sigma)^{\alpha} - \int F \left( \frac{\omega - x}{s} \right) \alpha (\omega + \sigma)^{\alpha-1} d\omega \right] d\sigma
\]
Then:

\[ V_x = \frac{1}{s} \int \int f\left( \frac{\omega - x}{s} \right) g\left( \frac{\sigma - y}{t} \right) (\omega + \sigma)^{\alpha - 1} d\omega d\sigma \]

and integrating by parts:

\[ V_x = \frac{1}{s} \int f\left( \frac{\omega - x}{s} \right) (\omega + \sigma)^{\alpha - 1} - \int G\left( \frac{\sigma - y}{t} \right) (\alpha - 1)(\omega + \sigma)^{\alpha - 2} d\sigma \] \hspace{1em} d\omega.

Differentiating with respect to \( y \) then yields

\[ V_{xy} = -\frac{1 - \alpha}{st} \int \int f\left( \frac{\omega - x}{s} \right) g\left( \frac{\sigma - y}{t} \right) (\omega + \sigma)^{\alpha - 2} d\omega d\sigma. \]

The condition for negative sorting is (where we drop the argument of the distribution functions for notational convenience):

\[ V_{xy} + \frac{1 - \alpha}{\alpha} V_x V_y < 0 \]

\[ (1 - \alpha) \left[ -\int \int fg(\omega + \sigma)^{\alpha - 2} d\omega d\sigma + \left( \int \int fg(\omega + \sigma)^{\alpha - 1} d\omega d\sigma \right)^2 \right] < 0, \]

or

\[ \left( \int \int fg(\omega + \sigma)^{\alpha - 1} d\omega d\sigma \right)^2 - \int \int fg(\omega + \sigma)^{\alpha - 2} d\omega d\sigma \int \int fg(\omega + \sigma)^{\alpha} d\omega d\sigma < 0. \]

Observe that

\[ \text{cov}\{X, Y\} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \]

and let

\[ \mathbb{E}[X] = \int \int fg(\omega + \sigma)^{\alpha - 2} d\omega d\sigma \]

\[ \mathbb{E}[Y] = \int \int fg(\omega + \sigma)^{\alpha} d\omega d\sigma \]

\[ \mathbb{E}[XY] = \int \int fg(\omega + \sigma)^{2\alpha - 2} = \int \int f g ( (\omega + \sigma)^{\alpha - 1})^2 d\omega d\sigma. \]
The covariance $\text{cov}\{(\omega + \sigma)^{\alpha - 2}, (\omega + \sigma)^{\alpha}\} < 0$ since they are inversely related for $\alpha \in (0, 1]$. But by Jensen’s inequality

$$\int \int fg((\omega + \sigma)^{\alpha - 1})^2 \, d\omega d\sigma > \left(\int \int fg(\omega + \sigma)^{\alpha - 1}\right)^2 \, d\omega d\sigma.$$ 

Therefore, the condition for sorting

$$\left(\int \int fg(\omega + \sigma)^{\alpha - 1} \, d\omega d\sigma\right)^2 - \int \int fg(\omega + \sigma)^{\alpha - 2} \, d\omega d\sigma \int \int fg(\omega + \sigma)^{\alpha} \, d\omega d\sigma < 0.$$

Proof of Proposition 4

**Proof.** The first-order conditions in the constraint are:

$$\rho_y e^{-\rho_y (h(\omega, \sigma) - c)} = \lambda \rho_x e^{-\rho_x c} \Rightarrow c = \frac{\rho_y h(\omega, \sigma) + \log \frac{\rho_x}{\rho_y}}{\rho_x + \rho_y},$$

and the constraint is therefore:

$$\int_{\omega}^{\bar{\omega}} \int_{\sigma}^{\bar{\sigma}} f(\omega|x) g(\sigma|y) e^{-\rho_x \frac{-\rho_x}{\rho_x + \rho_y} \left(\rho_y h(\omega, \sigma) + \log \frac{\rho_x}{\rho_y}\right)} \, d\omega d\sigma = e^{-\rho_x c(x)}$$

or

$$\lambda \frac{\rho_x}{\rho_x + \rho_y} = \frac{\left(\frac{\rho_x}{\rho_y}\right)^{-\rho_x} \int_{\omega}^{\bar{\omega}} \int_{\sigma}^{\bar{\sigma}} f(\omega|x) g(\sigma|y) e^{-\rho_x \frac{-\rho_x}{\rho_x + \rho_y} h(\omega, \sigma)} \, d\omega d\sigma}{e^{-\rho_x c(x)}}.$$
We can write the objective of the problem as

\[
\Phi(x, y, v) = -\int_{\omega}^{\bar{\omega}} \int_{\sigma}^{\bar{\sigma}} f(\omega|x)g(\sigma|y)e^{-\rho_y(h(\omega, \sigma)c)} d\omega d\sigma
\]

\[
= -\int_{\omega}^{\bar{\omega}} \int_{\sigma}^{\bar{\sigma}} f(\omega|x)g(\sigma|y)e^{-\rho_y h(\omega, \sigma)} \lambda^{\rho_y h(\omega, \sigma)} \left( \frac{\rho_x}{\rho_y} \right)^{\rho_y h(\omega, \sigma)} d\omega d\sigma
\]

\[
= -\left( \frac{\rho_x}{\rho_y} \right)^{\rho_y h(\omega, \sigma)} \left( \int_{\omega}^{\bar{\omega}} \int_{\sigma}^{\bar{\sigma}} f(\omega|x)g(\sigma|y)e^{-\rho_y h(\omega, \sigma)} d\omega d\sigma \right) \int_{\omega}^{\bar{\omega}} \int_{\sigma}^{\bar{\sigma}} \left( f(\omega|x)g(\sigma|y)e^{-\rho_y h(\omega, \sigma)} d\omega d\sigma \right) e^{-\rho_y c(x)}
\]

Let \( h(\omega, \sigma) = \omega + \sigma \), then

\[
\Phi(x, y, v) = -e^{\rho_y c(x)} \left( \int_{\omega}^{\bar{\omega}} f(\omega|x)e^{-\rho_y h(\omega, \sigma)} d\omega \right)^{1+\frac{\rho_y}{\rho_x} m} \left( \int_{\sigma}^{\bar{\sigma}} g(\sigma|y)e^{-\rho_y h(\omega, \sigma)} d\sigma \right)^{1+\frac{\rho_y}{\rho_x} n}
\]

and denote \( \Phi(x, y, v) \) by

\[
\Phi(x, y, v) = m(x)^{1+\frac{\rho_y}{\rho_x} m} n(y)^{1+\frac{\rho_y}{\rho_x} n}.
\]

Now we calculate the cross-partial total derivative of \( \Phi \) using the partial derivatives:

\[
\Phi_x = v^{\rho_y \rho_x} \left( 1 + \frac{\rho_y}{\rho_x} \right) m^{\rho_y \rho_x} n^{1+\frac{\rho_y}{\rho_x} m'}
\]

\[
\Phi_y = -\frac{\rho_y}{\rho_x} v^{\rho_y \rho_x} m^{1+\frac{\rho_y}{\rho_x} n} n^{1+\frac{\rho_y}{\rho_x} m'}
\]

\[
\Phi_{xy} = -\frac{\rho_y}{\rho_x} v^{\rho_y \rho_x} m^{1+\frac{\rho_y}{\rho_x} n} \left( 1 + \frac{\rho_y}{\rho_x} \right) n^{m_x n'}
\]

\[
\frac{\Phi_x}{\Phi_y} = v^{\rho_y \rho_x} \left( 1 + \frac{\rho_y}{\rho_x} \right)^2 m^{m'v^{\rho_y \rho_x} m^{\rho_y \rho_x} n^{m_x n'}}
\]

\[
\frac{\Phi_y}{\Phi_x} = v^{\rho_y \rho_x} \left( 1 + \frac{\rho_y}{\rho_x} \right)^2 m^{m'v^{\rho_y \rho_x} m^{\rho_y \rho_x} n^{m_x n'}}.
\]

Inspection immediately reveals that \( \Phi_{xy} = \frac{\Phi_x}{\Phi_y} \Phi_{xy} \) for all \((x, y, v)\) and therefore \( \Phi \) is modular and the allocation is indeterminate.

\[ \blacksquare \]
References


