A Model of Monetary Exchange in Over-the-Counter Markets*

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Abstract

We develop a model of monetary exchange in over-the-counter (OTC) markets and use it to study the effects of inflation on asset prices, as well as on standard measures of financial liquidity, such as the size of bid-ask spreads, trade volume, and the incentives of dealers to supply immediacy, both by choosing to participate in the market-making activity, and by holding asset inventories on their own account.

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1 Introduction

We develop a model of monetary exchange in financial over-the-counter markets and use it to study some elementary questions in financial and monetary economics. The model builds on two strands of literature: the Search Theory of Money (e.g., Lagos and Wright, 2005) and search-based models of financial trade in over-the-counter markets (e.g., Duffie et al., 2007 and Lagos and Rocheteau, 2009). Specifically, we consider a setting in which periodically, there are gains from trade among investors who hold an asset that represents a flow of consumption goods (e.g., an equity or a real bond). In order to realize these gains from trade, investors participate in a bilateral market with random search that is intermediated by specialized dealers who have access to a well-functioning interdealer market. In this bilateral market, which has all the stylized features of an typical over-the-counter (OTC) market structure, investors and dealers negotiate trades in the financial asset, using fiat money as a means of payment. Periodically, both dealers and investors are able to rebalance their portfolios in a frictionless (Walrasian) market. Thus the background environment is similar to Lagos and Wright (2005), and trade in the OTC market is similar to Lagos and Rocheteau (2009).

We use the model to investigate some questions that have received much attention in the recent work, such as the effects of anticipated inflation on real asset prices. In contrast to most of the related literature (e.g., Geromichalos et al., 2007, Lagos and Rocheteau, 2008, Lagos 2010, 2011, Lester et al., 2012, Nosal and Rocheteau, 2013), we find that real asset prices are decreasing in the rate of anticipated inflation.¹ We also study the effects of inflation on standard measures of financial liquidity of OTC markets, such as the size of bid-ask spreads, the volume of trade, and the incentives of dealers to provide liquidity, both by choosing to participate in the market-making activity, as well as by holding asset inventories on their own account.

2 The model

Time is represented by a sequence of periods indexed by $t = 0, 1, \dots$. Each time-period is divided into two subperiods where different activities take place. There is a continuum of infinitely lived agents called *investors*, each identified with a point in the set $\mathcal{I} = [0, 1]$. There

¹There are some models that build on Lagos and Wright (2005) where agents can use a real asset as collateral to borrow money that they subsequently use to purchase consumption goods. In this context, anticipated inflation can reduce the real price of the collateral asset (see, e.g., He et al., 2012, and Li and Li, 2012). As we explain below, the economic mechanism that we identify is novel.

is also a continuum of infinitely lived agents called *dealers*, each identified with a point in the set $\mathcal{D} = [0, v]$, where $v \in \mathbb{R}_+$. There is a fixed measure $A \in \mathbb{R}_{++}$ of productive units, each of which yields an exogenous quantity $y \in \mathbb{R}_{++}$ of a perishable consumption good at the end of the first subperiod of every period. We will refer to these productive units as *trees*, and to the consumption good that they yield, as *fruit* (or *dividend*). In the second subperiod of every period, every agent has access to a linear production technology that transforms a unit of the agent's effort into a unit of another kind of perishable homogeneous consumption good.

Each tree has outstanding one durable and perfectly divisible equity share that represents the bearer's ownership of the tree and confers him the right to collect the dividends. There is a second financial instrument, money, which is intrinsically useless (it is not an argument of any utility or production function, and unlike equity, ownership of money does not constitute a right to collect any resources). The stock of money at time t is denoted M_t . The initial stock of money, $M_0 \in \mathbb{R}_{++}$, is given, and $M_{t+1} = \gamma M_t$, with $\gamma \in \mathbb{R}_{++}$. For simplicity, we assume that the monetary authority injects or withdraws money via lump-sum transfers or taxes to investors in the second subperiod of every period.² At the beginning of period $t = 0$, each dealer and each investor is endowed with a given portfolio of money and equity shares. All financial instruments are perfectly recognizable, cannot be forged, and can be traded among agents in every subperiod.

In the second subperiod of every period, all agents can trade the consumption good produced in that subperiod, equity shares, and money, in a spot Walrasian market. In the first subperiod of every period, trading is organized as follows: Investors and dealers can trade equity shares and money in a random bilateral *OTC market*, while dealers can trade with each other continuously equity shares and money in a spot Walrasian *interdealer market*. We use $\delta \in (0, \min(v, 1)]$ to denote the probability that an individual investor is able to make contact with a dealer in the OTC market. The probability that a dealer contacts some investor is $\delta/v \equiv \eta \in (0, 1]$. Once a dealer and an investor have contacted each other, the pair negotiates the quantity of equity shares that the dealer will buy from, or sell to the investor in exchange for money. We assume that the terms of the trade between an investor and a dealer in the OTC market are chosen by the investor with probability $\theta \in [0, 1]$, and by the dealer with probability $1 - \theta$. After the transaction has been completed, the dealer and the investor part ways.³ The timing assumption

²Our substantive results do not depend on who are the agents that stand on the other side of these lump-sum interventions.

³See Zhang (2012) for an OTC model with long-term relationships between investors and dealers.

is that the round of OTC trade between investors and dealers takes place in the first subperiod of a typical period t , and ends before trees yield fruit. Hence equity is traded *cum dividend* in the OTC market (and in the interdealer market) of the first subperiod, but *ex dividend* in the Walrasian market of the second subperiod. We assume that agents cannot make binding commitments, that there is no enforcement, and that histories of actions are private in a way that precludes any borrowing and lending, so any trade must be *quid pro quo*. This assumption and the structure of preferences described below create the need for a medium of exchange.⁴

All agents discount one-period ahead payoffs with the same factor $\beta \in (0, 1)$. An individual dealer's preferences are given by

$$\mathbb{E}_0^d \sum_{t=0}^{\infty} \beta^t (c_{td} - h_{td})$$

where c_{td} is his consumption of the homogeneous good that is produced, traded and consumed in the second subperiod of period t , and h_{td} is the utility cost from exerting h_{td} units of effort to produce this good. The expectation operator $\mathbb{E}_0^d$ is with respect to the probability measure induced by the random trading process in the OTC market. Dealers get no utility from fruit.⁵ An individual investor i 's preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\varepsilon_{ti} y_{ti} + c_{ti} - h_{ti})$$

where y_{ti} is the quantity of fruit that investor i consumes at the end of the first subperiod of period t , c_{ti} is his consumption of the homogeneous good that is produced, traded and consumed in the second subperiod of period t , and h_{ti} is the utility cost from exerting h_{ti} units of effort to produce this good. The variable ε_{ti} denotes the realization of a preference shock that is distributed independently over time and across agents, with a differentiable cumulative distribution function G on the support $[\varepsilon_L, \varepsilon_H] \subseteq [0, \infty]$, and $\bar{\varepsilon} = \int \varepsilon dG(\varepsilon) < \infty$. Investor i learns his realization ε_{ti} at the beginning of period t , before the OTC trading round. The expectation operator $\mathbb{E}_0$ is with respect to the probability measure induced by the investor's preference shock and the random trading process in the OTC market.

⁴Notice that under these conditions there cannot exist a futures market for fruit, so an agent who wishes to consume the fruit dividend must be holding the equity share at the time the dividend is paid. A similar assumption is typically made in search models of financial OTC trade, e.g., see Duffie et al. (2005) and Lagos and Rocheteau (2009).

⁵This assumption implies that dealers have no direct consumption motive for holding the equity share. It is easy to relax, but it is the standard benchmark in the literature, e.g., see Duffie et al. (2005) and Lagos and Rocheteau (2009), Lagos, Rocheteau and Weill (2011), and Weill (2007).

3 Equilibrium

We begin with the formulation of the individual dealer's problems during a typical period. Let $V_t^d(\mathbf{a}_t)$ denote the expected discounted payoff of a dealer who is holding portfolio $\mathbf{a}_t \equiv (a_t^m, a_t^s)$ at the beginning of the first subperiod of period t . Let $W_t^d(\mathbf{a}_t)$ denote the expected discounted payoff of a dealer who is holding portfolio $\mathbf{a}_t$ at the beginning of the second subperiod of period t . Let ϕ_t^m be the real price of money, and ϕ_t^s be the real *ex dividend* price of equity in the second subperiod of period t (in terms of the second-subperiod consumption good). Then,

$$\begin{aligned} W_t^d(\mathbf{a}_t) &= \max_{c_t, h_t, \mathbf{a}_{t+1}} \left[c_t - h_t + \beta V_{t+1}^d(\mathbf{a}_{t+1}) \right] \\ \text{s.t. } c_t + \phi_t \mathbf{a}_{t+1} &\leq h_t + \phi_t \mathbf{a}_t, \quad c_t, h_t \in \mathbb{R}_+, \quad \mathbf{a}_{t+1} \in \mathbb{R}_+^2, \end{aligned} \quad (1)$$

where $\phi_t \equiv (\phi_t^m, \phi_t^s)$, $\mathbf{a}_{t+1} \equiv (a_{t+1}^m, a_{t+1}^s)$, and $\phi_t \mathbf{a}_{t+1}$ denotes the dot product of both vectors.

Let $\hat{W}_t^d(\mathbf{a}_t)$ denote the expected discounted payoff of a dealer with portfolio $\mathbf{a}_t$ in the first subperiod of period t , conditional on not having contacted an investor in the OTC market. Since the dealer can still access the interdealer market,

$$\begin{aligned} \hat{W}_t^d(\mathbf{a}_t) &= \max_{\hat{a}_t^m, \hat{a}_t^s} W_t^d(\hat{a}_t^m, \hat{a}_t^s) \\ \text{s.t. } \hat{a}_t^m + p_t \hat{a}_t^s &\leq a_t^m + p_t a_t^s, \quad \hat{a}_t^m, \hat{a}_t^s \in \mathbb{R}_+, \end{aligned} \quad (2)$$

where p_t is the dollar price of equity in the interdealer market of period t .

Next consider the situation of a dealer who enters the first subperiod of period t with portfolio $\mathbf{a}_{td}$, and who contacts an investor with portfolio $\mathbf{a}_{ti}$ and preference type ε in the OTC market. With probability θ the terms of trade are determined by a take-it-or-leave-it offer by the investor, and the resulting post-trade portfolios of the investor and the dealer are denoted

$$(\bar{a}_{i*}^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_{i*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)) \text{ and } (\bar{a}_d^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_d^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)),$$

respectively, where $\boldsymbol{\psi}_t \equiv (p_t, \phi_t)$. With probability $1 - \theta$ the terms of trade are determined by a take-it-or-leave-it offer by the dealer, and the resulting post-trade portfolios of the investor and the dealer are

$$(\bar{a}_i^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_i^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)) \text{ and } (\bar{a}_{d*}^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_{d*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)),$$

respectively.⁶ We can now write the value function of a dealer who enters the first subperiod

⁶In what follows, we will sometimes use $\bar{a}_{ti*}^m$ to denote $\bar{a}_{i*}^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)$, $\bar{a}_{td}^s$ to denote $\bar{a}_d^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)$, etc.

of period t with portfolio $\mathbf{a}_{td}$,

$$\begin{aligned} V_t^d(\mathbf{a}_{td}) &= \eta\theta \int \hat{W}_t^d[\bar{a}_d^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_d^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)] dH_t(\mathbf{a}_{ti}, \varepsilon) \\ &\quad + \eta(1-\theta) \int \hat{W}_t^d[\bar{a}_{d*}^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_{d*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)] dH_t(\mathbf{a}_{ti}, \varepsilon) \\ &\quad + (1-\eta) \hat{W}_t^d(\mathbf{a}_{td}), \end{aligned} \quad (3)$$

where H_t is the joint cumulative distribution function over the preference types and portfolios held by the random investor whom the dealer may contact in the OTC market of period t .

Next consider the investor's problems in a typical period. Let $V_t^i(\mathbf{a}_{ti}, \varepsilon)$ denote the expected discounted payoff of an investor who has preference type ε and is holding portfolio $\mathbf{a}_{ti} \equiv (a_{ti}^m, a_{ti}^s)$ at the beginning of the first subperiod of period t . Let $W_t^i(\mathbf{a}_t)$ denote the expected discounted payoff of an investor who is holding portfolio $\mathbf{a}_t$ at the beginning of the second subperiod of period t (after the trees have borne their fruit dividends). Then,

$$\begin{aligned} W_t^i(\mathbf{a}_t) &= \max_{c_t, h_t, \mathbf{a}_{t+1}} \left[c_t - h_t + \beta \int V_{t+1}^i(\mathbf{a}_{t+1}, \varepsilon) dG(\varepsilon) \right] \\ \text{s.t. } c_t + \phi_t \mathbf{a}_{t+1} &\leq h_t + \phi_t \mathbf{a}_t + T_t, \quad c_t, h_t \in \mathbb{R}_+, \quad \mathbf{a}_{t+1} \in \mathbb{R}_+^2, \end{aligned} \quad (4)$$

where T_t is the real value of the time- t lump-sum monetary transfer (tax, if negative). Since ε is iid over time, $W_t^i(\mathbf{a}_t)$ is independent of ε , so the portfolio that each investor chooses to carry into period $t+1$ is independent of ε . Consequently, in what follows we can write $dH_t(\mathbf{a}_{ti}, \varepsilon) = dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon)$, where F_t^I is the period- t joint cumulative distribution function of investors' portfolios.

The value function of an investor who enters the first subperiod of period t with portfolio $\mathbf{a}_{ti}$ and preference type ε , is

$$\begin{aligned} V_t^i(\mathbf{a}_{ti}, \varepsilon) &= \delta\theta \int \{ \varepsilon y \bar{a}_{i*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) + W_t^i[\bar{a}_{i*}^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_{i*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)] \} dF_t^D(\mathbf{a}_{td}) \\ &\quad + \delta(1-\theta) \int \{ \varepsilon y \bar{a}_i^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) + W_t^i[\bar{a}_i^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t), \bar{a}_i^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)] \} dF_t^D(\mathbf{a}_{td}) \\ &\quad + (1-\delta) [\varepsilon y a_{ti}^s + W_t^i(\mathbf{a}_{ti})], \end{aligned} \quad (5)$$

where F_t^D is the cumulative distribution function over portfolios held by the random dealer whom the dealer may contact in the OTC market of period t . Next, we characterize the outcomes of trades in the OTC and the interdealer markets.

The maximization problem (2) represents the portfolio problem of a dealer who did not contact an investor in the OTC market of period t . The solution is summarized as follows:

Lemma 1 *A dealer d who enters period t with portfolio $\mathbf{a}_t$ and does not contact an investor, enters the second subperiod of period t with portfolio $(\hat{a}_{td}^m, \hat{a}_{td}^s) \equiv (\hat{a}_d^m(\mathbf{a}_t; \boldsymbol{\psi}_t), \hat{a}_d^s(\mathbf{a}_t; \boldsymbol{\psi}_t))$, given by*

$$\hat{a}_{td}^m = \begin{cases} = 0 & \text{if } p_t \phi_t^m < \phi_t^s \\ \in [0, a_t^m + p_t a_t^s] & \text{if } p_t \phi_t^m = \phi_t^s \\ = a_t^m + p_t a_t^s & \text{if } \phi_t^s < p_t \phi_t^m \end{cases} \quad \text{and} \quad \hat{a}_{td}^s = \begin{cases} a_t^s + \frac{1}{p_t} a_t^m & \text{if } p_t \phi_t^m < \phi_t^s \\ a_t^s + \frac{1}{p_t} (a_t^m - \hat{a}_{td}^m) & \text{if } p_t \phi_t^m = \phi_t^s \\ 0 & \text{if } \phi_t^s < p_t \phi_t^m, \end{cases}$$

and his expected discounted payoff is

$$\hat{W}_t^d(\mathbf{a}_t) = \bar{\phi}_t (a_t^m + p_t a_t^s) + W_t^d(\mathbf{0}) \quad (6)$$

where $\bar{\phi}_t \equiv \max(\phi_t^m, \phi_t^s/p_t)$, and

$$W_t^d(\mathbf{0}) = \max_{\mathbf{a}_{t+1} \in \mathbb{R}_+^2} [\beta V_{t+1}^d(\mathbf{a}_{t+1}) - \phi_t \mathbf{a}_{t+1}]. \quad (7)$$

If $p_t \phi_t^m < \phi_t^s$, then a dealer who holds any cash in the interdealer market can use a dollar to buy $1/p_t$ equity shares, and the net return from this trade equals ϕ_t^s/p_t (the real value of the equities in the Walrasian market of the second subperiod of period t) minus ϕ_t^m (the real cost of the trading strategy also in terms of the consumption that is traded in the second subperiod of period t), which is strictly positive. Hence, it is optimal for the dealer to sell off all his cash for equity if $p_t \phi_t^m < \phi_t^s$. Conversely, if $\phi_t^s < p_t \phi_t^m$, it is optimal for the dealer to sell off any equity holdings he may have and carry only cash into the second subperiod of period t .

Consider the bargaining problem between an investor with preference type ε and portfolio (a_{ti}^m, a_{ti}^s) who contacts a dealer with portfolio (a_{td}^m, a_{td}^s) in the OTC market of period t . With probability θ the investor has the power to make a take-it-or-leave-it offer to the dealer. The investor chooses his offer of post-trade portfolios for himself, $(\bar{a}_{ti}^m, \bar{a}_{ti}^s)$, and for the dealer, $(\bar{a}_{td}^m, \bar{a}_{td}^s)$, by solving

$$\begin{aligned} & \max_{\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s} [\varepsilon y \bar{a}_{ti}^s + W_t^i(\bar{a}_{ti}^m, \bar{a}_{ti}^s)] \\ \text{s.t. } & \bar{a}_{ti}^m + \bar{a}_{td}^m + p_t(\bar{a}_{ti}^s + \bar{a}_{td}^s) \leq a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) \\ & \hat{W}_t^d(\bar{a}_{td}^m, \bar{a}_{td}^s) \geq \hat{W}_t^d(a_{td}^m, a_{td}^s) \\ & \bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s \in \mathbb{R}_+. \end{aligned}$$

The first constraint says that the combined value of the post-trade portfolios held by the investor and the dealer cannot exceed the combined value of their pre-trade portfolios, and the second is the dealer's participation constraint.

With complementary probability $1 - \theta$, the dealer has the power to make a take-it-or-leave-it offer to the investor. The dealer chooses his offer of post-trade portfolios for himself, $(\bar{a}_{td}^m, \bar{a}_{td}^s)$, and for the dealer, $(\bar{a}_{ti}^m, \bar{a}_{ti}^s)$, by solving

$$\begin{aligned} & \max_{\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s} \hat{W}_t^d(\bar{a}_{td}^m, \bar{a}_{td}^s) \\ \text{s.t. } & \bar{a}_{ti}^m + \bar{a}_{td}^m + p_t(\bar{a}_{ti}^s + \bar{a}_{td}^s) \leq a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) \\ & \varepsilon y \bar{a}_{ti}^s + W_t^i(\bar{a}_{ti}^m, \bar{a}_{ti}^s) \geq \varepsilon y a_{ti}^s + W_t^i(a_{ti}^m, a_{ti}^s) \\ & \bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s \in \mathbb{R}_+. \end{aligned}$$

The first constraint again requires that the combined value of the pre-trade portfolios is enough to finance the post-trade portfolios, and the second is the investor's participation constraint. The following result summarizes the outcome of the bargaining game between an investor and a dealer.

Lemma 2 *Consider the bargaining problem between an investor i with portfolio (a_{ti}^m, a_{ti}^s) and preference type ε , and a dealer d with portfolio (a_{td}^m, a_{td}^s) in the OTC market of period t .*

(i) *With probability θ the investor chooses the terms of trade, and in this case the investor exits the meeting with post-trade portfolio $(\bar{a}_{ti}^m, \bar{a}_{ti}^s)$ given by*

$$\bar{a}_{ti}^m \begin{cases} = 0 & \text{if } \varepsilon_t^* < \varepsilon \\ \in [0, a_{ti}^m + p_t a_{ti}^s] & \text{if } \varepsilon = \varepsilon_t^* \\ = a_{ti}^m + p_t a_{ti}^s & \text{if } \varepsilon < \varepsilon_t^* \end{cases} \quad \text{and} \quad \bar{a}_{ti}^s = \begin{cases} a_{ti}^s + \frac{1}{p_t} a_{ti}^m & \text{if } \varepsilon_t^* < \varepsilon \\ a_{ti}^s + \frac{1}{p_t} (a_{ti}^m - \bar{a}_{ti}^m) & \text{if } \varepsilon = \varepsilon_t^* \\ 0 & \text{if } \varepsilon < \varepsilon_t^* \end{cases}$$

where

$$\varepsilon_t^* \equiv \frac{p_t \phi_t^m - \phi_t^s}{y}. \quad (8)$$

The dealer's portfolio $(\bar{a}_{td}^m, \bar{a}_{td}^s)$ that results from trading with the investor is given by

$$\bar{a}_{td}^m \in [0, a_{td}^m + p_t a_{td}^s] \quad \text{and} \quad \bar{a}_{td}^s = a_{td}^s + \frac{1}{p_t} (a_{td}^m - \bar{a}_{td}^m).$$

(ii) *With probability $1 - \theta$ the dealer chooses the terms of trade, and in this case the investor exits the meeting with post-trade portfolio $(\bar{a}_{ti}^m, \bar{a}_{ti}^s)$ given by*

$$\bar{a}_{ti}^m \begin{cases} = 0 & \text{if } \varepsilon_t^* < \varepsilon \\ \in [0, a_{ti}^m + p_t a_{ti}^s] & \text{if } \varepsilon = \varepsilon_t^* \\ = a_{ti}^m + \frac{\varepsilon y + \phi_t^s}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s & \text{if } \varepsilon < \varepsilon_t^* \end{cases} \quad \text{and} \quad \bar{a}_{ti}^s = \begin{cases} a_{ti}^s + \left(\frac{\varepsilon_t^* y + \phi_t^s}{\varepsilon y + \phi_t^s} \right) \frac{1}{p_t} a_{ti}^m & \text{if } \varepsilon_t^* < \varepsilon \\ a_{ti}^s + \frac{1}{p_t} (a_{ti}^m - \bar{a}_{ti}^m) & \text{if } \varepsilon = \varepsilon_t^* \\ 0 & \text{if } \varepsilon < \varepsilon_t^*. \end{cases}$$

The dealer's portfolio $(\bar{a}_{td*}^m, \bar{a}_{td*}^s)$ that results from trading with the investor is given by

$$\bar{a}_{td*}^m \begin{cases} \in \left[0, a_{td}^m + p_t a_{td}^s + \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} a_{ti}^m\right] & \text{if } \varepsilon_t^* < \varepsilon \\ \in [0, a_{td}^m + p_t a_{td}^s] & \text{if } \varepsilon = \varepsilon_t^* \\ \in \left[0, a_{td}^m + p_t a_{td}^s + \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s\right] & \text{if } \varepsilon < \varepsilon_t^* \end{cases}$$

and

$$\bar{a}_{td*}^s = \begin{cases} a_{td}^s + \frac{1}{p_t} \left[a_{td}^m - \bar{a}_{td*}^m + \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} a_{ti}^m \right] & \text{if } \varepsilon_t^* < \varepsilon \\ a_{td}^s + \frac{1}{p_t} (a_{td}^m - \bar{a}_{td*}^m) & \text{if } \varepsilon = \varepsilon_t^* \\ a_{td}^s + \frac{1}{p_t} \left[a_{td}^m - \bar{a}_{td*}^m + \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s \right] & \text{if } \varepsilon < \varepsilon_t^*. \end{cases}$$

To interpret the results in Lemma 2, observe that (8) defines the preference type of the “marginal investor.” That is, investors with preference type $\varepsilon < \varepsilon_t^*$ want to sell equity for cash, investors with preference type $\varepsilon > \varepsilon_t^*$ want to buy equity with cash, and the marginal investors with preference type $\varepsilon = \varepsilon_t^*$ are indifferent between buying or selling equity, as they have no gain from trading in the OTC market.⁷ Consider an investor who has drawn preference type ε and meets a dealer in period t . If $p_t \phi_t^m < \varepsilon y + \phi_t^s$, or equivalently, if $\varepsilon_t^* < \varepsilon$, then the real value of a dollar to the investor is ϕ_t^m (the amount of general goods he can buy in the following centralized market), which is smaller than $(\varepsilon y + \phi_t^s) / p_t$, namely the value to the investor of the (cum-dividend) equity position that can be purchased with a dollar in the interdealer market. Naturally, in this case the bargaining outcome is that the investor sells all his cash and uses it to purchase equity, regardless of whether the dealer or the investor has the bargaining power. Formally, in Lemma 2, $\bar{a}_{ti*}^m = \bar{a}_{ti}^m = 0$ if $\varepsilon_t^* < \varepsilon$. Analogously, if $\varepsilon < \varepsilon_t^*$, then the investor sells all his equity for cash, both when he makes the offer, and when the dealer makes the offer (i.e., $\bar{a}_{ti*}^s = \bar{a}_{ti}^s = 0$ if $\varepsilon < \varepsilon_t^*$ in the lemma). If $\varepsilon = \varepsilon_t^*$, the investor is indifferent between holding equity or cash; there are no gains from trade between him and the dealer.

The quantity of equity shares the investor gets for his cash holdings when $\varepsilon_t^* < \varepsilon$, and the amount of cash he gets for his equity shares when $\varepsilon < \varepsilon_t^*$, however, do depend on whether the investor or the dealer has the bargaining power. If the investor has the bargaining power, he can effectively trade money for equity at the interdealer market price, p_t , i.e., he pays p_t dollars per share when he buys equity, and gets p_t dollars per share when he sells equity. Formally, in Lemma 2, $\bar{a}_{ti*}^s = a_{ti}^s + \frac{1}{p_t} a_{ti}^m$ if $\varepsilon_t^* < \varepsilon$, and $\bar{a}_{ti*}^m = a_{ti}^m + p_t a_{ti}^s$ if $\varepsilon < \varepsilon_t^*$. If the dealer has

⁷Another way to interpret (8) is that given ε_t^* , $p_t \phi_t^m = \varepsilon_t^* y + \phi_t^s$ is the *cum dividend* real value of equity to the marginal investor in period t .

the bargaining power, he offers less favorable terms of trade to the investor. Effectively, the bargaining outcome is that the dealer lets any investor trade at

$$p_t^d(\varepsilon) \equiv \left(\frac{\varepsilon y + \phi_t^s}{\varepsilon_t^* y + \phi_t^s} \right) p_t \quad (9)$$

dollars per share, rather than p_t dollars per share. Notice that $\partial p_t^d(\varepsilon) / \partial \varepsilon > 0$, so investors with higher preference type are offered a higher price per share. Also, note that $p_t^d(\varepsilon) > p_t$ if and only if $\varepsilon_t^* < \varepsilon$. Thus the dealer charges $p_t^d(\varepsilon) > p_t$ dollars per share to an investor who wishes to buy equity (i.e., an investor with $\varepsilon_t^* < \varepsilon$), and pays $p_t^d(\varepsilon) < p_t$ dollars per share to an investor who wishes to sell equity (an investor with $\varepsilon < \varepsilon_t^*$). In other words, in a meeting where the dealer has the bargaining power, $p_t^d(\varepsilon)$ is the nominal *bid price* for investors who want to sell equity, or the nominal *ask price* for investors who want to buy equity. In terms of the lemma, this is why $\bar{a}_{ti}^s = a_{ti}^s + \frac{1}{p_t^d(\varepsilon)} a_{ti}^m$ if $\varepsilon_t^* < \varepsilon$, and $\bar{a}_{ti}^m = a_{ti}^m + p_t^d(\varepsilon) a_{ti}^s$ if $\varepsilon < \varepsilon_t^*$. The indeterminacy in the dealer's portfolio follows from the fact that he can immediately retrade in the interdealer market after having traded with the investor, so all he cares about is the value of his *combined* post-trade portfolio. In fact, as the following corollary shows, the post-trade value of the dealer and the investor portfolios are uniquely pinned down.

Corollary 1 *Consider the bargaining problem between an investor i with portfolio (a_{ti}^m, a_{ti}^s) and preference type ε , and a dealer d with portfolio (a_{td}^m, a_{td}^s) in the OTC market of period t .*

(i) *With probability θ the investor chooses the terms of trade, and in this case the dollar value of the investor's and the dealer's post-trade portfolios are, respectively,*

$$\begin{aligned} \bar{a}_{ti}^m + p_t \bar{a}_{ti}^s &= a_{ti}^m + p_t a_{ti}^s \\ \bar{a}_{td}^m + p_t \bar{a}_{td}^s &= a_{td}^m + p_t a_{td}^s. \end{aligned}$$

(ii) *With probability $1 - \theta$ the dealer chooses the terms of trade, and in this case the dollar value of the investor's and the dealer's post-trade portfolios are, respectively,*

$$\begin{aligned} \bar{a}_{ti}^m + p_t \bar{a}_{ti}^s &= \begin{cases} a_{ti}^m + p_t a_{ti}^s - \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} a_{ti}^m & \text{if } \varepsilon_t^* \leq \varepsilon \\ a_{ti}^m + p_t a_{ti}^s - \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s & \text{if } \varepsilon < \varepsilon_t^* \end{cases} \\ \bar{a}_{td}^m + p_t \bar{a}_{td}^s &= \begin{cases} a_{td}^m + p_t a_{td}^s + \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} a_{ti}^m & \text{if } \varepsilon_t^* \leq \varepsilon \\ a_{td}^m + p_t a_{td}^s + \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s & \text{if } \varepsilon < \varepsilon_t^*. \end{cases} \end{aligned}$$

Corollary 1 shows that the dealer extracts a transaction fee from the investor only when he has the bargaining power. For example, when the dealer encounters an investor with $\varepsilon > \varepsilon_t^*$ who wishes to purchase x shares, the dealer extracts $p_t^d(\varepsilon) - p_t = \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon_t^*y + \phi_t^s} p_t$ dollars per share purchased by the investor, for a total fee of $\frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon_t^*y + \phi_t^s} p_t x$ dollars. In Lemma 2 and Corollary 1, $x = \bar{a}_{ti}^s - a_{ti}^s = \frac{1}{p_t^d(\varepsilon)} a_{ti}^m = \left(\frac{\varepsilon_t^*y + \phi_t^s}{\varepsilon y + \phi_t^s} \right) \frac{1}{p_t} a_{ti}^m$, so the total fee equals $\frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} a_{ti}^m$ dollars. Similarly, when the dealer encounters an investor with $\varepsilon < \varepsilon_t^*$ who wishes to sell $\bar{a}_{ti}^s - a_{ti}^s$ shares, the dealer extracts $p_t - p_t^d(\varepsilon) = \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^*y + \phi_t^s} p_t$ dollars per share sold by the investor.

The bargaining outcomes can be substituted in the value functions (3) and (5) to obtain the following result.

Lemma 3 *Let A_t^I and M_t^I denote the quantity of shares and money held by all investors at the beginning of period t , respectively, i.e., $A_t^I = \int a_{ti}^s dF_t^I(\mathbf{a}_{ti})$ and $M_t^I = \int a_{ti}^m dF_t^I(\mathbf{a}_{ti})$.*

(i) *The value function of a dealer who enters the first subperiod of period t with portfolio $\mathbf{a}_{td} = (a_{td}^m, a_{td}^s)$ is given by*

$$V_t^d(a_{td}^m, a_{td}^s) = \bar{\phi}_t(a_{td}^m + p_t a_{td}^s) + V_t^d(\mathbf{0}) \quad (10)$$

where

$$V_t^d(\mathbf{0}) \equiv \eta(1 - \theta) \bar{\phi}_t \left[p_t A_t^I \int_{\varepsilon_L}^{\varepsilon_t^*} \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^*y + \phi_t^s} dG(\varepsilon) + M_t^I \int_{\varepsilon_t^*}^{\varepsilon_H} \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} dG(\varepsilon) \right] + W_t^d(\mathbf{0}).$$

(ii) *The value function of an investor who enters the first subperiod of period t with portfolio $\mathbf{a}_{ti} = (a_{ti}^m, a_{ti}^s)$ and preference type ε is given by*

$$\begin{aligned} V_t^i(a_{ti}^m, a_{ti}^s, \varepsilon) &= \left[\mathbb{I}_{\{\varepsilon_t^* \leq \varepsilon\}} \delta \theta \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon_t^*y + \phi_t^s} + 1 \right] \phi_t^m a_{ti}^m \\ &\quad + \left[\mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}} \delta \theta (\varepsilon_t^* - \varepsilon)y + (\varepsilon y + \phi_t^s) \right] a_{ti}^s + W_t^i(\mathbf{0}) \end{aligned} \quad (11)$$

where $\mathbb{I}_{\{\varepsilon_t^* \leq \varepsilon\}}$ is an indicator function that takes the value 1 if $\varepsilon_t^* \leq \varepsilon$, and 0 otherwise.

Notice that the first term on the right side of $V_t^d(\mathbf{0})$ is the expected fee earned by a dealer in the OTC market of period t (the term inside the square bracket on the right side of $V_t^d(\mathbf{0})$ is the expected fee earned by a dealer when he makes an offer to an investor whose preference type, ε , is a random draw from G). To interpret (11), notice that the factor that multiplies the indicator function $\mathbb{I}_{\{\varepsilon_t^* \leq \varepsilon\}}$, i.e., $\delta \theta \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon_t^*y + \phi_t^s} \phi_t^m a_{ti}^m$, is the expected gain to the investor from exchanging money

for shares in the OTC market.⁸ Similarly, the factor that multiplies the indicator function $\mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}}$, i.e., $\delta\theta(\varepsilon_t^* - \varepsilon)ya_{ti}^s$, is the expected gain to the investor from exchanging shares for money in the OTC market.⁹

The following result uses Lemma 3, to characterize the solutions to the portfolio problems that a typical dealer and a typical investor solve in the Walrasian market of the second subperiod of period t .

Lemma 4 *Let (a_{t+1d}^m, a_{t+1d}^s) and (a_{t+1i}^m, a_{t+1i}^s) denote the portfolios chosen by a dealer d and an investor i , respectively, in the Walrasian market of the second subperiod of period t . The first-order necessary and sufficient conditions for optimization that these portfolios must satisfy are*

$$\phi_t^m \geq \beta \max(\phi_{t+1}^m, \phi_{t+1}^s/p_{t+1}), \text{ “} = \text{” if } a_{t+1d}^m > 0 \quad (12)$$

$$\phi_t^s \geq \beta \max(p_{t+1}\phi_{t+1}^m, \phi_{t+1}^s), \text{ “} = \text{” if } a_{t+1d}^s > 0 \quad (13)$$

for the dealer, and

$$\phi_t^m \geq \beta\phi_{t+1}^m + \beta\delta\theta y \frac{\phi_{t+1}^m}{\varepsilon_{t+1}^* y + \phi_{t+1}^s} \int_{\varepsilon_{t+1}^*}^{\varepsilon_H} (\varepsilon - \varepsilon_{t+1}^*) dG(\varepsilon), \text{ “} = \text{” if } a_{t+1i}^m > 0 \quad (14)$$

$$\phi_t^s \geq \beta(\bar{\varepsilon}y + \phi_{t+1}^s) + \beta\delta\theta y \int_{\varepsilon_L}^{\varepsilon_{t+1}^*} (\varepsilon_{t+1}^* - \varepsilon) dG(\varepsilon), \text{ “} = \text{” if } a_{t+1i}^s > 0 \quad (15)$$

for the investor.

Condition (12) is a dealer's Euler equation for money. The left side is the real cost of purchasing a dollar (in terms of the homogeneous good) in the second subperiod of period t . The right side is the discounted gain from this marginal dollar in the following period, i.e., the dealer can choose to hold on to the dollar until the second subperiod of $t + 1$ to obtain ϕ_{t+1}^m homogeneous consumption goods, or he can sell the dollar in the interdealer market of the

⁸Conditional on having drawn $\varepsilon \geq \varepsilon_t^*$, the investor contacts a dealer with probability δ and has the bargaining power with probability θ . In this case the investor sells a_{ti}^m dollars for $\frac{1}{p_t}a_{ti}^m$ shares, and his net payoff from this transaction is

$$(\varepsilon y + \phi_t^s) \frac{1}{p_t} a_{ti}^m - \phi_t^m a_{ti}^m = (\varepsilon y + \phi_t^s) \frac{\phi_t^m}{\varepsilon_t^* y + \phi_t^s} a_{ti}^m - \phi_t^m a_{ti}^m = \frac{(\varepsilon - \varepsilon_t^*) y}{\varepsilon_t^* y + \phi_t^s} \phi_t^m a_{ti}^m.$$

⁹Conditional on having drawn $\varepsilon < \varepsilon_t^*$, the investor contacts a dealer with probability δ and has the bargaining power with probability θ . In this case the investor sells a_{ti}^s shares for p_t dollars each. The investor's net payoff from this transaction is $\phi_t^m p_t a_{ti}^s - (\varepsilon y + \phi_t^s) a_{ti}^s = (\varepsilon_t^* - \varepsilon) y a_{ti}^s$.

following OTC round for $1/p_{t+1}$ equity shares, each of which will be worth ϕ_{t+1}^s homogeneous goods in the second subperiod of $t + 1$. Naturally, the dealer will choose the best of these two trading strategies. He holds no money overnight if the left side of (12) exceeds the right side. Condition (13) is a dealer's Euler equation for equity shares. The left side is the real cost of purchasing a share (in terms of the homogeneous good) in the second subperiod of t . The right side is the discounted gain from this marginal share in the following period, i.e., the dealer can sell the share in the interdealer market of the following OTC round for p_{t+1} dollars, each of which will be worth ϕ_{t+1}^m homogeneous goods in the second subperiod of $t + 1$, or he can choose to hold on to the share until the second subperiod of period $t + 1$ to obtain ϕ_{t+1}^s homogeneous consumption goods. The dealer will choose the best of these two trading strategies. He holds no equity overnight if the left side of (13) exceeds the right side.

Condition (14) is the investor's Euler equation for money. To interpret it, it is useful to rewrite it as

$$\phi_t^m \geq \beta \left\{ \delta \theta [1 - G(\varepsilon_{t+1}^*)] \frac{\bar{\varepsilon}_{t+1}^{*h} y + \phi_{t+1}^s}{\varepsilon_{t+1}^* y + \phi_{t+1}^s} \phi_{t+1}^m + \{1 - \delta \theta [1 - G(\varepsilon_{t+1}^*)]\} \phi_{t+1}^m \right\},$$

with “=” if $a_{t+1i}^m > 0$, where $\bar{\varepsilon}_{t+1}^{*h} \equiv \int_{\varepsilon_{t+1}^*}^{\varepsilon^H} \varepsilon \frac{dG(\varepsilon)}{1 - G(\varepsilon_{t+1}^*)}$. The left side is the real cost of purchasing a dollar (in terms of the homogeneous good) in the second subperiod of period t . The right side is the discounted expected gain from this marginal dollar in the following period. With probability $\delta \theta [1 - G(\varepsilon_{t+1}^*)]$ the investor contacts a dealer in the OTC market, has bargaining power, and wishes to purchase equity. In this event, he uses the marginal dollar which is worth ϕ_{t+1}^m general goods to purchase $\frac{1}{\varepsilon_{t+1}^* y + \phi_{t+1}^s}$ equity shares each of which yields expected utility from the dividend $\bar{\varepsilon}_{t+1}^{*h} y$, and a resale value of ϕ_{t+1}^s homogeneous goods in the second subperiod of $t + 1$.¹⁰ With probability $1 - \delta \theta [1 - G(\varepsilon_{t+1}^*)] = 1 - \delta + \delta(1 - \theta) + \delta \theta G(\varepsilon_{t+1}^*)$ the investor gets no additional return from the marginal dollar aside from the resale value, ϕ_{t+1}^m , since money confers him no gain in the OTC market if he either does not contact a dealer (with probability $1 - \delta$), or if he contacts a dealer but has no bargaining power (with probability $\delta(1 - \theta)$), or if he contacts a dealer and has bargaining power but wishes to sell equity (with probability $\delta \theta G(\varepsilon_{t+1}^*)$).

Condition (15) is the investor's Euler equation for equity. To interpret it, it is useful to

¹⁰ As it will become clear below (Proposition 1), $\varepsilon_{t+1}^* y + \phi_{t+1}^s$ is the (implicit) equilibrium relative price of equity in terms of the homogeneous consumption good in the interdealer market of the first subperiod of $t + 1$.

rewrite it as

$$\phi_t^s \geq \beta \left\{ G(\varepsilon_{t+1}^*) \left[\delta\theta(\varepsilon_{t+1}^* y + \phi_{t+1}^s) + (1 - \delta\theta)(\bar{\varepsilon}_{t+1}^{*l} y + \phi_{t+1}^s) \right] + [1 - G(\varepsilon_{t+1}^*)] (\bar{\varepsilon}_{t+1}^{*h} y + \phi_{t+1}^s) \right\}$$

with “=” if $a_{t+1i}^s > 0$, where $\bar{\varepsilon}_{t+1}^{*l} \equiv \int_{\varepsilon_L}^{\varepsilon_{t+1}^*} \varepsilon \frac{dG(\varepsilon)}{G(\varepsilon_{t+1}^*)}$. The left side is the real cost of purchasing an additional equity share (in terms of the homogeneous good) in the second subperiod of t . The right side is the discounted expected gain to the investor from this marginal share in the following period. With probability $G(\varepsilon_{t+1}^*)$ the investor will want to sell the share in the OTC market, and conditional on wanting to sell the equity, he may either meet a dealer and have the bargaining power, or not. In the former case (with conditional probability $\delta\theta$), the investor sells the equity share to the dealer in exchange for $\varepsilon_{t+1}^* y + \phi_{t+1}^s$ real money balances (i.e., he obtains $\varepsilon_{t+1}^* y + \phi_{t+1}^s / \phi_{t+1}^m$ dollars per share from the dealer). In this case the investor's payoff, $\varepsilon_{t+1}^* y + \phi_{t+1}^s$, is higher than the expected payoff he would get if he did not trade the equity, i.e., $\bar{\varepsilon}_{t+1}^{*l} y + \phi_{t+1}^s$. With complementary probability, $1 - \delta\theta$, the investor gets no gain from trade either because he does not meet a dealer, or because he meets a dealer who makes him a take-it-or-leave-it offer that makes him indifferent between trading or not; either way, his expected payoff is $\bar{\varepsilon}_{t+1}^{*l} y + \phi_{t+1}^s$. With probability $1 - G(\varepsilon_{t+1}^*)$, the investor does not sell assets, so he keeps any shares he brought into the period and enjoys expected real return $\bar{\varepsilon}_{t+1}^{*h} y + \phi_{t+1}^s$ per share.

Let A_t^D and M_t^D denote the quantity of shares and money held by all dealers at the beginning of period t , respectively, i.e., $A_t^D = v \int a_{td}^s dF_t^D(\mathbf{a}_{td})$, and $M_t^D = v \int a_{td}^m dF_t^D(\mathbf{a}_{td})$. Let $\bar{A}_t^D$ and $\bar{A}_t^I$ denote the quantity of shares and money held after the OTC round of trade of period t by dealers and investors who are able to trade in the first subperiod, i.e.,

$$\begin{aligned} \bar{A}_t^D &= \eta v \theta \int \hat{a}_d^s [\bar{a}_d(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t); \psi_t] dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon) \\ &\quad + \eta v (1 - \theta) \int \hat{a}_d^s [\bar{a}_d^*(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t); \psi_t] dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon) \\ &\quad + (1 - \eta) v \int \hat{a}_d^s(\mathbf{a}_{td}; \psi_t) dF_t^D(\mathbf{a}_{td}) \end{aligned}$$

and

$$\bar{A}_t^I = \delta \int [\theta \bar{a}_{i*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t) + (1 - \theta) \bar{a}_i^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t)] dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon)$$

where $\bar{a}_d(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t) \equiv (\bar{a}_d^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t), \bar{a}_d^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t))$. Similarly, let $\bar{M}_t^D$ and $\bar{M}_t^I$ denote the quantity of money and shares held by dealers and investors, respectively, after the

OTC round of trade of period t , i.e.,

$$\begin{aligned}\bar{M}_t^D &= \eta v \theta \int \hat{a}_d^m [\bar{a}_d(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t); \boldsymbol{\psi}_t] dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon) \\ &\quad + \eta v (1 - \theta) \int \hat{a}_d^m [\bar{a}_{d*}(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t); \boldsymbol{\psi}_t] dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon) \\ &\quad + (1 - \eta) v \int \hat{a}_d^m(\mathbf{a}_{td}; \boldsymbol{\psi}_t) dF_t^D(\mathbf{a}_{td})\end{aligned}$$

and

$$\bar{M}_t^I = \delta \int [\theta \bar{a}_{i*}^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) + (1 - \theta) \bar{a}_i^m(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)] dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon).$$

We are now ready to define equilibrium.

Definition 1 *An equilibrium is a sequence of terms of trade in the OTC market*

$$\{\bar{a}_{ti*}^m, \bar{a}_{ti*}^s, \bar{a}_{td}^m, \bar{a}_{td}^s, \bar{a}_{td*}^m, \bar{a}_{td*}^s, \bar{a}_{ti}^m, \bar{a}_{ti}^s, \hat{a}_{td}^m, \hat{a}_{td}^s\}_{t=0}^\infty,$$

as given in Lemma 1 and Lemma 2, together with a sequence of asset holdings

$$\{\langle a_{t+1d}^m, a_{t+1d}^s \rangle_{d \in \mathcal{D}}, \langle a_{t+1i}^m, a_{t+1i}^s \rangle_{i \in \mathcal{I}}\}_{t=0}^\infty$$

and prices $\{\boldsymbol{\psi}_t\}_{t=0}^\infty \equiv \{p_t, \phi_t^m, \phi_t^s\}_{t=0}^\infty$, such that for all t , (i) the asset allocation solves the individual optimization problems (1) and (4) taking prices as given, and (ii) prices are such that all Walrasian markets clear, i.e., $A_t^D + A_t^I = A$ (the end-of-period Walrasian market for equity), $M_t^D + M_t^I = M_t$ (the end-of-period Walrasian market for money), as well as the OTC interdealer market for equity and money, i.e., $\bar{A}_t^D + \bar{A}_t^I = A_t^D + \delta A_t^I$ and $\bar{M}_t^D + \bar{M}_t^I = M_t^D + \delta M_t^I$. An equilibrium is “monetary” if $\phi_t^m > 0$ for all t , and “nonmonetary” otherwise.

In what follows, we specialize the analysis to stationary equilibria where all real variables are constant, and nominal variables grow at the same rate as the money supply. Specifically, for all t , $A_t^D = A^D$, $A_t^I = A^I$, $\phi_t^s = \phi^s$, $p_t \phi_t^m \equiv \bar{\phi}^s$, $\varepsilon_t^* = \frac{\bar{\phi}^s - \phi^s}{y} \equiv \varepsilon^*$, $\phi_t^m M_t \equiv Z$, and $\phi_t^m / \phi_{t+1}^m = p_{t+1} / p_t = M_{t+1} / M_t = \gamma$. For the analysis we maintain the assumption $\gamma > \beta$, but later we also consider the limiting case $\gamma \rightarrow \beta$.

For the analysis that follows, it is convenient to define

$$\hat{\gamma} \equiv \beta \left[1 + \frac{(1 - \delta\theta)(1 - \beta)(\hat{\varepsilon} - \bar{\varepsilon})}{\hat{\varepsilon}} \right] \quad \text{and} \quad \bar{\gamma} \equiv \beta \left[1 + \frac{\delta\theta(1 - \beta)(\bar{\varepsilon} - \varepsilon_L)}{\beta\bar{\varepsilon} + (1 - \beta)\varepsilon_L} \right] \quad (16)$$

where $\hat{\varepsilon} \in [\bar{\varepsilon}, \varepsilon_H]$ is the unique solution to

$$\bar{\varepsilon} - \hat{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\hat{\varepsilon}} G(\varepsilon) d\varepsilon = 0. \quad (17)$$

Lemma 5 (in the appendix) establishes that $\hat{\gamma} < \bar{\gamma}$. The following proposition summarizes the equilibrium set.

Proposition 1 (i) *There exists a nonmonetary equilibrium for any parametrization.* (ii) *There is no stationary monetary equilibrium if $\gamma \geq \bar{\gamma}$.* (iii) *In the nonmonetary equilibrium, $A^I = A - A^D = A$ (only investors hold equity), there is no trade in the OTC market, and the equity price in the Walrasian market is*

$$\phi^s = \frac{\beta}{1-\beta} \bar{\varepsilon} y.$$

(iv) *If $\gamma \in (\beta, \bar{\gamma})$, then there is one stationary monetary equilibrium; asset holdings of dealers and investors at the beginning of period t are $M_t^D = M_t - M_t^I = 0$ and*

$$A^D = A - A^I \begin{cases} = A & \text{if } \beta < \gamma < \hat{\gamma} \\ \in [0, A] & \text{if } \gamma = \hat{\gamma} \\ = 0 & \text{if } \hat{\gamma} < \gamma < \bar{\gamma}, \end{cases}$$

and asset prices are

$$\phi^s = \begin{cases} \frac{\beta}{1-\beta} \varepsilon^* y & \text{if } \beta < \gamma \leq \hat{\gamma} \\ \frac{\beta}{1-\beta} \left[\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon \right] y & \text{if } \hat{\gamma} < \gamma < \bar{\gamma} \end{cases} \quad (18)$$

$$p_t = \left\{ \theta [1 - G(\varepsilon^*)] + (1 - \theta) \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} dG(\varepsilon) \right\} \frac{\delta M_t}{A^D + \delta G(\varepsilon^*) A^I} \quad (19)$$

$$\bar{\phi}^s = \varepsilon^* y + \phi^s \quad (20)$$

$$\phi_t^m = \frac{\bar{\phi}^s}{p_t}, \quad (21)$$

where, for any $\gamma \in (\beta, \bar{\gamma})$, $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$ is the unique solution to

$$\frac{(1 - \beta) \int_{\varepsilon^*}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\varepsilon^* + \beta \left[\bar{\varepsilon} - \varepsilon^* + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon \right] \mathbb{I}_{\{\hat{\gamma} < \gamma\}}} - \frac{\gamma - \beta}{\beta \delta \theta} = 0. \quad (22)$$

(v) (a) *As $\gamma \rightarrow \bar{\gamma}$, $\varepsilon^* \rightarrow \varepsilon_L$ and $\phi^s \rightarrow \frac{\beta}{1-\beta} \bar{\varepsilon} y$.* (b) *As $\gamma \rightarrow \beta$, $\varepsilon^* \rightarrow \varepsilon_H$ and $\phi^s \rightarrow \frac{\beta}{1-\beta} \varepsilon_H y$.*

In the nonmonetary equilibrium, dealers are inactive and equity shares are held only by investors. With no valued money, investors and dealers cannot exploit the gains from trade

that arise from the heterogeneity in preference types in the first subperiod of every period, and the equilibrium real asset price, $\phi^s = \frac{\beta}{1-\beta}\bar{\varepsilon}y$, is equal to the expected discounted value to an investor of holding the equity share forever. (Shares can be traded in the Walrasian market of the second subperiod, but gains from trade at that stage are null.) The stationary monetary equilibrium exists only if the inflation rate is not too high, i.e., if $\gamma < \bar{\gamma}$. In the monetary equilibrium, the marginal preference type, ε^* , which according to Lemma 2 partitions the set of investors into those who buy and those who sell the asset in the OTC market, is characterized in part (iv) of Proposition 1. In the OTC market, investors with $\varepsilon < \varepsilon^*$ sell all their equity holdings for money, and investors with $\varepsilon > \varepsilon^*$ spend all their money buying equity. Thus unlike what happens in the nonmonetary equilibrium, the OTC market is active in the monetary equilibrium, and it is easy to show that the marginal type, ε^* , is strictly decreasing in the rate of inflation, i.e., $\frac{\partial \varepsilon^*}{\partial \gamma} < 0$ both for $\gamma \in (\beta, \hat{\gamma})$, and for $\gamma \in (\hat{\gamma}, \bar{\gamma})$ (see Corollary 3 in the appendix). Intuitively, as γ increases the real value of money falls, and the marginal investor type, ε^* , decreases, reflecting the fact that under the higher inflation rate, the old marginal investor is no longer indifferent between carrying cash and equity out of the OTC market (he prefers equity).

According to Proposition 1, $\phi^s < \phi^s + \varepsilon^*y = p_t\phi_t^m$ in the monetary equilibrium, so Lemma 1 implies that dealers hold no equity shares at the end of the OTC round: all equity is held by investors, in particular, by those investors who carried equity into the period but were unable to contact a dealer, and by those investors who purchased equity shares from dealers. After the round of OTC trade, all the money supply is held by the investors who carried cash into the period but were unable to contact a dealer, by the investors who sold equity shares to dealers, and by those dealers who had bargaining power in the OTC negotiations or carried equity into the OTC market. Another feature of the monetary equilibrium is that dealers never hold money overnight: at the beginning of every period t , the money supply is all in the hands of investors, i.e., $M_t^D = 0$ and $M_t^I = M_t$. The reason is that access to the interdealer market allows dealers to intermediate assets without having to carry cash.¹¹ Whether it is investors or dealers who hold the equity overnight, depends on the inflation rate: if the inflation rate is low, i.e., if $\gamma \in (\beta, \hat{\gamma})$, then only dealers hold equity shares overnight, that is, $A_t^D = A$ and $A_t^I = 0$ for all t . Conversely, if the inflation rate is high, i.e., if $\gamma \in (\hat{\gamma}, \bar{\gamma})$, then at the beginning of every period t , all equity shares are in the hands of investors, i.e., $A_t^D = 0$ and $A_t^I = A$. To

¹¹To see this formally, notice that in a stationary equilibrium, (12) becomes $\gamma\bar{\phi}^s > \beta \max(\bar{\phi}^s, \phi^s) = \beta\bar{\phi}^s$.

understand this result, it is useful to inspect the Euler equations for equity. In a stationary equilibrium, (13) reduces to

$$1 \geq \beta R_d^s(\varepsilon^*), \text{ “} = \text{” if } A_t^D > 0 \quad (23)$$

where

$$R_d^s(\varepsilon^*) \equiv \frac{\varepsilon^* y + \phi^s}{\phi^s} = \frac{\bar{\phi}^s}{\phi^s}.$$

Dealers do not wish to hold equity overnight if (23) holds with strict inequality. The equilibrium return to a dealer from holding equity overnight, $R_d^s(\varepsilon^*)$, consists of the expected capital gain from purchasing equity in the second subperiod and reselling it in the OTC market of the following period. Similarly, in a stationary equilibrium (15) reduces to

$$1 \geq \beta R_i^s(\varepsilon^*), \text{ “} = \text{” if } A^I > 0 \quad (24)$$

where

$$\begin{aligned} R_i^s(\varepsilon^*) &\equiv G(\varepsilon^*) \left\{ \delta \theta \frac{\bar{\phi}^s}{\phi^s} + [(1 - \delta) + \delta(1 - \theta)] \frac{\bar{\varepsilon}^{*l} y + \phi^s}{\phi^s} \right\} + [1 - G(\varepsilon^*)] \frac{\bar{\varepsilon}^{*h} y + \phi^s}{\phi^s} \\ &= \frac{\bar{\varepsilon} y + \delta \theta G(\varepsilon^*) (\varepsilon^* - \bar{\varepsilon}^{*l}) y + \phi^s}{\phi^s}, \end{aligned}$$

$\bar{\varepsilon}^{*l} \equiv \int_{\varepsilon_L}^{\varepsilon^*} \varepsilon \frac{dG(\varepsilon)}{G(\varepsilon^*)}$ and $\bar{\varepsilon}^{*h} \equiv \int_{\varepsilon^*}^{\varepsilon_H} \varepsilon \frac{dG(\varepsilon)}{1 - G(\varepsilon^*)}$. Investors do not wish to hold equity overnight if (24) holds with strict inequality. The equilibrium expected return to an investor from holding equity overnight, $R_i^s(\varepsilon^*)$, can be thought of as a weighted average of four gross returns. The first, $\frac{\bar{\phi}^s}{\phi^s}$, is the capital gain of an investor who sells equity in the OTC market at the interdealer market price ($\bar{\phi}^s$), which is what occurs with probability $G(\varepsilon^*) \delta \theta$, i.e., when the investor draws a preference type lower than ε^* (so he wishes to sell in the OTC market), contacts a dealer, and has the bargaining power. The second, $\frac{\bar{\varepsilon}^{*l} y + \phi^s}{\phi^s}$, is the expected equity return to an investor who wishes to sell in the OTC market but fails to contact a dealer, which occurs with probability $G(\varepsilon^*) (1 - \delta)$. In this case the expected equity payoff consists of the expected value of the period dividend conditional on wanting to sell, $\bar{\varepsilon}^{*l} y$, and the resale value of the equity in the following Walrasian round of trade, ϕ^s . The third, also $\frac{\bar{\varepsilon}^{*l} y + \phi^s}{\phi^s}$, is the expected capital gain of an investor who sells equity in the OTC market at the dealer's expected bid price, $\bar{\varepsilon}^{*l} y + \phi^s$, that (in expected value) reaps all the gains from trade from an investor who wishes to sell, an event that occurs with probability $G(\varepsilon^*) \delta \theta$, i.e., when the investor draws a preference type lower than ε^* (so he wishes to sell in the OTC market), contacts a dealer, and the dealer has

the bargaining power. The fourth, $\frac{\bar{\varepsilon}^* y + \phi^s}{\phi^s}$, is the expected equity return of an investor who does not wish to sell in the OTC market and therefore keeps the equity share for a full period, which occurs with probability $1 - G(\varepsilon^*)$. In this case the expected equity payoff consists of the expected value of the period dividend conditional on not wanting to sell, $\bar{\varepsilon}^* y$, and the resale value of the equity in the following Walrasian round of trade, ϕ^s .

From (23) and (24), dealers hold all equity shares overnight (i.e., $A_t^D = A$ and $A_t^I = 0$) if and only if $R_i^s(\varepsilon^*) < R_d^s(\varepsilon^*)$, i.e., if and only if $\bar{\varepsilon} + \delta\theta G(\varepsilon^*)(\varepsilon^* - \bar{\varepsilon}^{*l}) < \varepsilon^*$. This condition is equivalent to

$$\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon < \varepsilon^*,$$

which is in turn equivalent to $\hat{\varepsilon} < \varepsilon^*$, where $\hat{\varepsilon}$ is defined by (17). Intuitively then, all equity is held by dealers overnight if the marginal preference type that partitions investors between buyers and sellers is large enough (larger than $\hat{\varepsilon}$), else all equity is held by investors overnight, and strictly speaking, dealers only provide brokerage services in the OTC market.¹²

Given the marginal preference type, ε^* , part (iv) of Proposition 1 gives all asset prices in closed form. The real price of equity (in terms of the homogeneous consumption good) in the Walrasian round of trade, ϕ^s , is given by (22). The dollar price of equity in the OTC market, p_t , is given by (19). The real price of money (in terms of the homogeneous consumption good) in the Walrasian round of trade, ϕ_t^m , is given by (21). The real price of equity (in terms of the homogeneous consumption good) in the OTC market, $p_t \phi_t^m = \bar{\phi}^s$ is given by (20).

Finally, part (v)(a) states that as the rate of money creation rises toward $\bar{\gamma}$, ε^* approaches the lowest bound of the type distribution, ε_L , so no investor wishes to sell equity in the OTC market, and as a result the allocations and prices of the monetary equilibrium approach those of the nonmonetary equilibrium. Part (v)(b) states that as the rate of money creation falls toward β , ε^* increases toward the upper bound of the type distribution, ε_H , so only investors

¹²To get some more intuition, notice that

$$\frac{\partial R_i^s(\varepsilon^*)}{\partial \varepsilon^*} = \delta\theta G(\varepsilon^*) \frac{y}{\phi^s} < \frac{y}{\phi^s} = \frac{\partial R_d^s(\varepsilon^*)}{\partial \varepsilon^*}$$

and

$$\frac{\varepsilon_L y + \phi^s}{\phi^s} = R_d^s(\varepsilon_L) < R_i^s(\varepsilon_L) = \frac{\bar{\varepsilon} y + \phi^s}{\phi^s} < \frac{[\delta\theta \varepsilon_H + (1 - \delta\theta) \bar{\varepsilon}] y + \phi^s}{\phi^s} = R_i^s(\varepsilon_H) < R_d^s(\varepsilon_H) = \frac{\varepsilon_H y + \phi^s}{\phi^s}.$$

This reasoning is in terms of ε^* while Proposition 1 is stated in terms of the parameter γ . However, there is a monotonic relationship between γ and ε^* , and as it is shown in the proof of the proposition, $\hat{\varepsilon} < \varepsilon^*$ if and only if $\gamma < \hat{\gamma}$.

with the highest preference type purchase equity in the OTC market (all other investors wish to sell it).

4 Efficiency

Consider a social planner who wishes to maximize the sum of all agents' expected discounted utilities, subject to the same meeting frictions that agents face in the decentralized formulation. Specifically, in the first subperiod of every period, the planner can only reallocate assets among all dealers and a measure δ of investors chosen at random from the population. Restrict attention to symmetric allocations (where identical agents receive equal treatment), and let c_{td} and h_{td} denote a dealer's consumption and production of the homogeneous consumption good in the second subperiod of period t , let $c_{ti}(\varepsilon)$ and $h_{ti}(\varepsilon)$ denote consumption and production of the homogeneous consumption good in the second subperiod of period t by an investor with idiosyncratic time- t preference type equal to ε . Let a_{td} denote the beginning-of-period- t asset holding of a dealer, and let a'_{td} denote asset holding of a dealer at the end of the first subperiod of period t . Similarly, let a_{ti} denote the beginning-of-period- t asset holding of an investor with preference type ε , and let a'_{ti} denote a measure on $\mathcal{F}([\varepsilon_L, \varepsilon_H])$, the Borel σ -field defined on $[\varepsilon_L, \varepsilon_H]$. This measure is interpreted as the distribution of post-trade asset holdings among investors with different preference types who contacted a dealer in the first subperiod of period t . With this notation, the planner's problem consists of choosing a nonnegative allocation,

$$\left\{ a_{td}, a'_{td}, c_{td}, h_{td}, a_{ti}, a'_{ti}, [c_{ti}(\varepsilon), h_{ti}(\varepsilon)]_{\varepsilon \in [\varepsilon_L, \varepsilon_H]} \right\}_{t=0}^{\infty}, \quad (25)$$

in order to maximize

$$\sum_{t=0}^{\infty} \beta^t \left[\delta \int_{[\varepsilon_L, \varepsilon_H]} \varepsilon y a'_{ti}(d\varepsilon) + \int_{\varepsilon_L}^{\varepsilon_H} [(1 - \delta) \varepsilon y a_{ti} + c_{ti}(\varepsilon) - h_{ti}(\varepsilon)] dG(\varepsilon) + v(c_{td} - h_{td}) \right]$$

subject to the feasibility constraints

$$v a_{td} + a_{ti} \leq A \quad (26)$$

$$v a'_{td} + \delta \int_{[\varepsilon_L, \varepsilon_H]} a'_{ti}(d\varepsilon) \leq v a_{td} + \delta a_{ti} \quad (27)$$

$$\int_{\varepsilon_L}^{\varepsilon_H} c_{ti}(\varepsilon) dG(\varepsilon) + v c_{td} \leq \int_{\varepsilon_L}^{\varepsilon_H} h_{ti}(\varepsilon) dG(\varepsilon) + v h_{td} \quad (28)$$

for each t .

Proposition 2 *A feasible allocation is efficient if and only if (a) $a_{td} = (A - a_{ti})/v = A/v$ for all t , and (b) it has the property that among those investors who have a trading opportunity in the first subperiod of any period, only those with the highest preference type carry equity shares into the second subperiod, i.e., $a'_{ti}(E) = \mathbb{I}_{\{\varepsilon_H \in E\}}A/\delta$, where $\mathbb{I}_{\{\varepsilon_H \in E\}}$ is an indicator function that takes the value 1 if $\varepsilon_H \in E$, and 0 otherwise, for any $E \in \mathcal{F}([\varepsilon_L, \varepsilon_H])$.*

The following corollary is immediate from Proposition 2 and part (v)(b) of Proposition 1.

Corollary 2 *The allocation implemented by the stationary monetary equilibrium converges to the symmetric efficient allocation as $\gamma \rightarrow \beta$, i.e., the Friedman rule is efficient.*

A Proofs

Proof of Lemma 1. Notice that (1) can be written as

$$W_t^d(\mathbf{a}_t) = \phi_t \mathbf{a}_t + W_t^d(\mathbf{0}) \quad (29)$$

with $W_t^d(\mathbf{0})$ given by (7). With (29), (2) is equivalent to

$$\hat{W}_t^d(\mathbf{a}_t) = \max_{\hat{a}_t^m, \hat{a}_t^s} [\phi_t^m \hat{a}_t^m + \phi_t^s \hat{a}_t^s + \lambda(a_t^m + p_t a_t^s - \hat{a}_t^m - p_t \hat{a}_t^s) + \mu_m \hat{a}_t^m + \mu_s \hat{a}_t^s] + W_t^d(\mathbf{0})$$

where λ is a Lagrange multiplier on the budget constraint $\hat{a}_t^m + p_t \hat{a}_t^s \leq a_t^m + p_t a_t^s$, and μ_m and μ_s are the multipliers on the nonnegativity constraints $\hat{a}_t^m \geq 0$ and $\hat{a}_t^s \geq 0$. The corresponding first-order necessary and sufficient conditions for $\hat{a}_t^m$ and $\hat{a}_t^s$ are

$$-\lambda + \phi_t^m + \mu_m = 0 \quad (30)$$

$$-\lambda p_t + \phi_t^s + \mu_s = 0 \quad (31)$$

$$\lambda(a_t^m + p_t a_t^s - \hat{a}_t^m - p_t \hat{a}_t^s) = 0 \quad (32)$$

Clearly $\hat{a}_t^m = \hat{a}_t^s = 0$ is the solution if and only if $a_t^m = a_t^s = 0$, but more generally the solution could take one of three forms: (i) $\mu_s = 0 < \mu_m$, (ii) $\mu_s = \mu_m = 0$, or (iii) $\mu_m = 0 < \mu_s$. In case (i), (30)-(32) imply $\hat{a}_t^m = 0$, $\hat{a}_t^s = a_t^s + \frac{1}{p_t} a_t^m$, and $p_t \phi_t^m < \phi_t^s$. In case (ii), (30)-(32) imply $\hat{a}_t^m \in [0, a_t^m + p_t a_t^s]$, $\hat{a}_t^s = a_t^s + \frac{1}{p_t} (a_t^m - \hat{a}_t^m)$, and $\phi_t^s = p_t \phi_t^m$. In case (iii), (30)-(32) imply $\hat{a}_t^s = 0$, $\hat{a}_t^m = a_t^m + p_t a_t^s$, and $\phi_t^s < p_t \phi_t^m$. The expressions for $\hat{a}_{td}^m$ and $\hat{a}_{td}^s$ in Lemma 1 follow from three cases. The value function (6) is obtained by substituting the optimal portfolio $(\hat{a}_{td}^m, \hat{a}_{td}^s)$ into (2). ■

Proof of Lemma 2. (i) Notice that (4) can be written as

$$W_t^i(\mathbf{a}_t) = \phi_t \mathbf{a}_t + W_t^i(\mathbf{0}) \quad (33)$$

where

$$W_t^i(\mathbf{0}) = T_t + \max_{\mathbf{a}_{t+1} \in \mathbb{R}_+^2} \left[\beta \int V_{t+1}^i(\mathbf{a}_{t+1}, \varepsilon) dG(\varepsilon) - \phi_t \mathbf{a}_{t+1} \right].$$

With (6) and (33) the problem of the investor when he makes the ultimatum offer becomes

$$\max_{\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s} [\varepsilon y \bar{a}_{ti}^s + \phi_t^m \bar{a}_{ti}^m + \phi_t^s \bar{a}_{ti}^s]$$

$$\begin{aligned}
\text{s.t. } \bar{a}_{ti}^m + \bar{a}_{td}^m + p_t(\bar{a}_{ti}^s + \bar{a}_{td}^s) &\leq a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) \\
\bar{a}_{td}^m + p_t\bar{a}_{td}^s &\geq a_{td}^m + p_ta_{td}^s \\
\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s &\in \mathbb{R}_+.
\end{aligned}$$

The corresponding Lagrangian is

$$\begin{aligned}
\mathcal{L} = & (\phi_t^m + \mu_i^m - \lambda) \bar{a}_{ti}^m + (\varepsilon y + \phi_t^s + \mu_i^s - \lambda p_t) \bar{a}_{ti}^s \\
& + (\rho + \mu_d^m - \lambda) \bar{a}_{td}^m + (\rho p_t + \mu_d^s - \lambda p_t) \bar{a}_{td}^s + K,
\end{aligned}$$

where $K \equiv \lambda[a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s)] - \rho(a_{td}^m + p_ta_{td}^s)$, $\lambda \in \mathbb{R}_+$ is the Lagrange multiplier associated with the budget constraint, $\rho \in \mathbb{R}_+$ is the multiplier on the dealer's participation constraint, and $\mu_i^m, \mu_i^s, \mu_d^m, \mu_d^s \in \mathbb{R}_+$ are the multipliers for the nonnegativity constraints on $\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s$, respectively. The first-order necessary and sufficient conditions are

$$\phi_t^m + \mu_i^m - \lambda = 0 \quad (34)$$

$$\varepsilon y + \phi_t^s + \mu_i^s - \lambda p_t = 0 \quad (35)$$

$$\rho + \mu_d^m - \lambda = 0 \quad (36)$$

$$\rho p_t + \mu_d^s - \lambda p_t = 0 \quad (37)$$

and the complementary slackness conditions

$$\lambda \{a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) - [\bar{a}_{ti}^m + \bar{a}_{td}^m + p_t(\bar{a}_{ti}^s + \bar{a}_{td}^s)]\} = 0 \quad (38)$$

$$\rho [\bar{a}_{td}^m + p_t\bar{a}_{td}^s - (a_{td}^m + p_ta_{td}^s)] = 0 \quad (39)$$

$$\mu_i^m \bar{a}_{ti}^m = 0 \quad (40)$$

$$\mu_i^s \bar{a}_{ti}^s = 0 \quad (41)$$

$$\mu_d^m \bar{a}_{td}^m = 0 \quad (42)$$

$$\mu_d^s \bar{a}_{td}^s = 0. \quad (43)$$

First, notice that $\lambda > 0$ at an optimum. To see this, assume the contrary, i.e., $\lambda = 0$. Then (35) implies $\varepsilon y + \phi_t^s = -\mu_i^s \leq 0$ which is a contradiction since $\varepsilon y + \phi_t^s > 0$. If $\rho > 0$, then (39) implies

$$\bar{a}_{td}^m + p_t\bar{a}_{td}^s = a_{td}^m + p_ta_{td}^s. \quad (44)$$

If instead $\rho = 0$, then (36) and (37) imply $\mu_d^m = \lambda > 0$ and $\mu_d^s = \lambda p_t > 0$, which (using (42) and (43)) in turn imply $\bar{a}_{td}^m = \bar{a}_{td}^s = 0$. This can only be a solution if $a_{td}^m + p_ta_{td}^s = 0$ (since

$\bar{a}_{td}^m + p_t \bar{a}_{td}^s \geq a_{td}^m + p_t a_{td}^s$ must hold at an optimum) in which case (44) also holds. Thus, we conclude that (44) must always hold at an optimum (and with $\rho > 0$ unless $a_{td}^m + p_t a_{td}^s = 0$). Since $\lambda > 0$, (38) and (44) imply

$$\bar{a}_{ti}^m + p_t \bar{a}_{ti}^s = a_{ti}^m + p_t a_{ti}^s. \quad (45)$$

From (44) it is immediate that if $a_{td}^m + p_t a_{td}^s = 0$, then $\bar{a}_{td}^m = \bar{a}_{td}^s = 0$. So suppose $a_{td}^m + p_t a_{td}^s > 0$. In this case μ_d^m and μ_d^s cannot both be strictly positive. (To see this, assume the contrary, i.e., that $\mu_d^m > 0$ and $\mu_d^s > 0$. Then (42) and (43) imply $\bar{a}_{td}^m = \bar{a}_{td}^s = 0$, and (44) implies $a_{td}^m + p_t a_{td}^s = 0$, a contradiction.) Moreover, conditions (36) and (37) imply $\mu_d^s = \mu_d^m p_t$, so $\mu_d^s = \mu_d^m = 0$ must hold at an optimum. Hence when he makes the ultimatum offer, the investor is indifferent between offering the dealer any nonnegative pair $(\bar{a}_{td}^m, \bar{a}_{td}^s)$ that satisfies (44).

From (45) it is immediate that $\bar{a}_{ti}^m = \bar{a}_{ti}^s = 0$ if $a_{ti}^m + p_t a_{ti}^s = 0$. So suppose $a_{ti}^m + p_t a_{ti}^s > 0$. In this case μ_i^m and μ_i^s cannot both be strictly positive (if they were, then (40) and (41) would imply $\bar{a}_{ti}^m = \bar{a}_{ti}^s = 0$, and in turn (45) would imply $a_{ti}^m + p_t a_{ti}^s = 0$, a contradiction). There are three possible cases: (a) $\mu_i^s = 0 < \mu_i^m$, (b) $\mu_i^s = \mu_i^m = 0$, or (c) $\mu_i^m = 0 < \mu_i^s$. In every case, from (34) and (35), we obtain

$$\varepsilon y + \phi_t^s + \mu_i^s = p_t \phi_t^m + p_t \mu_i^m. \quad (46)$$

In case (a), (40) implies $\bar{a}_{ti}^m = 0$, (45) implies $\bar{a}_{ti}^s = a_{ti}^m/p_t + a_{ti}^s$, and (46) implies that ε must satisfy $\varepsilon > \varepsilon_t^*$, where ε_t^* is as defined in (8). In case (b), (46) implies that ε must satisfy $\varepsilon = \varepsilon_t^*$ and the investor is indifferent between making any offer that leaves him with a nonnegative post-trade portfolio $(\bar{a}_{ti}^m, \bar{a}_{ti}^s)$ that satisfies (45). In case (c), (41) implies $\bar{a}_{ti}^s = 0$, (45) implies $\bar{a}_{ti}^m = a_{ti}^m + p_t a_{ti}^s$, and (46) implies that ε must satisfy $\varepsilon < \varepsilon_t^*$. The first, second, and third lines on the right side of the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td}^m$, and $\bar{a}_{td}^s$ in part (i) of the statement of the lemma correspond cases (a), (b), and (c), respectively.

(ii) With (6) and (33) the problem of the dealer when it is his turn to make the ultimatum offer is equivalent to

$$\begin{aligned} & \max_{\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s} \bar{\phi}_t [\bar{a}_{td}^m + p_t \bar{a}_{td}^s] \\ \text{s.t. } & \bar{a}_{ti}^m + \bar{a}_{td}^m + p_t (\bar{a}_{ti}^s + \bar{a}_{td}^s) \leq a_{ti}^m + a_{td}^m + p_t (a_{ti}^s + a_{td}^s) \end{aligned} \quad (47)$$

$$\phi_t^m \bar{a}_{ti}^m + (\varepsilon y + \phi_t^s) \bar{a}_{ti}^s \geq \phi_t^m a_{ti}^m + (\varepsilon y + \phi_t^s) a_{ti}^s \quad (48)$$

$$\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s \in \mathbb{R}_+.$$

The corresponding Lagrangian is

$$\begin{aligned}\mathcal{L}' = & (\bar{\phi}_t + \mu_d^m - \lambda)\bar{a}_{td}^m + (\bar{\phi}_t p_t + \mu_d^s - \lambda p_t)\bar{a}_{td}^s \\ & + (\rho\phi_t^m + \mu_i^m - \lambda)\bar{a}_{ti}^m + [\rho(\varepsilon y + \phi_t^s) + \mu_i^s - \lambda p_t]\bar{a}_{ti}^s + K',\end{aligned}$$

where $K' \equiv \lambda[a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s)] - \rho[\phi_t^m a_{ti}^m + (\varepsilon y + \phi_t^s)a_{ti}^s]$, $\lambda \in \mathbb{R}_+$ is the Lagrange multiplier associated with the budget constraint, $\rho \in \mathbb{R}_+$ is the multiplier on the dealer's participation constraint, and $\mu_i^m, \mu_i^s, \mu_d^m, \mu_d^s \in \mathbb{R}_+$ are the multipliers for the nonnegativity constraints on $\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td}^m, \bar{a}_{td}^s$, respectively. The first-order necessary and sufficient conditions are

$$\bar{\phi}_t + \mu_d^m - \lambda = 0 \quad (49)$$

$$\bar{\phi}_t p_t + \mu_d^s - \lambda p_t = 0 \quad (50)$$

$$\rho\phi_t^m + \mu_i^m - \lambda = 0 \quad (51)$$

$$\rho(\varepsilon y + \phi_t^s) + \mu_i^s - \lambda p_t = 0 \quad (52)$$

and the complementary slackness conditions

$$\lambda \{a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) - [\bar{a}_{ti}^m + \bar{a}_{td}^m + p_t(\bar{a}_{ti}^s + \bar{a}_{td}^s)]\} = 0 \quad (53)$$

$$\rho \{\phi_t^m \bar{a}_{ti}^m + (\varepsilon y + \phi_t^s) \bar{a}_{ti}^s - [\phi_t^m a_{ti}^m + (\varepsilon y + \phi_t^s) a_{ti}^s]\} = 0 \quad (54)$$

$$\mu_i^m \bar{a}_{ti}^m = 0 \quad (55)$$

$$\mu_i^s \bar{a}_{ti}^s = 0 \quad (56)$$

$$\mu_d^m \bar{a}_{td}^m = 0 \quad (57)$$

$$\mu_d^s \bar{a}_{td}^s = 0. \quad (58)$$

First, notice that $\lambda > 0$ at an optimum. To see this, note that if $\lambda = 0$ then (49) implies $\bar{\phi}_t + \mu_d^m = 0$ which is a contradiction since the left side is strictly positive ($\bar{\phi}_t > 0$ and $\mu_d^m \geq 0$ in a monetary equilibrium, and $\bar{\phi}_t = 0 < \mu_d^m$ in a nonmonetary equilibrium). Hence, at an optimum,

$$\bar{a}_{ti}^m + \bar{a}_{td}^m + p_t(\bar{a}_{ti}^s + \bar{a}_{td}^s) = a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s). \quad (59)$$

Second, observe that conditions (49) and (50), imply $p_t \mu_d^m = \mu_d^s$, so μ_d^m and μ_d^s have the same sign, i.e., either both are positive or both are zero.

If $\rho = 0$, then (51) and (52) imply $\mu_i^m = \lambda > 0$ and $\mu_i^s = \lambda p_t > 0$, which (using (55) and (56)) in turn imply $\bar{a}_{ti}^m = \bar{a}_{ti}^s = 0$. From the buyer's participation constraint (48) it follows that

this can be a solution only if $\phi_t^m a_{ti}^m + (\varepsilon y + \phi_t^s) a_{ti}^s = 0$, or equivalently only if $a_{ti}^m = a_{ti}^s = 0$. To obtain $(\bar{a}_{td*}^m, \bar{a}_{td*}^s)$, consider two cases: (a) $\mu_d^m = \mu_d^s = 0$, in which case $(\bar{a}_{td*}^m, \bar{a}_{td*}^s)$ need only satisfy $\bar{a}_{td*}^m + p_t \bar{a}_{td*}^s = a_{td}^m + p_t a_{td}^s$, or (b) $\mu_d^m > 0$ and $\mu_d^s > 0$, in which case $\bar{a}_{td*}^m = \bar{a}_{td*}^s = 0$, which according to (47), is only possible if $a_{td}^m = a_{td}^s = 0$. It is easy to see that the solution for case (a) can be obtained from the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td*}^m$, and $\bar{a}_{td*}^s$ in part (ii) of the statement of the lemma simply by setting $a_{ti}^m = a_{ti}^s = 0$, and the solution for case (b) can be obtained similarly, by setting $a_{ti}^m = a_{ti}^s = a_{td}^m = a_{td}^s = 0$.

If $\rho > 0$, then (54) implies

$$\phi_t^m \bar{a}_{ti}^m + (\varepsilon y + \phi_t^s) \bar{a}_{ti}^s = \phi_t^m a_{ti}^m + (\varepsilon y + \phi_t^s) a_{ti}^s. \quad (60)$$

There are eight possible configurations of to be considered: [Configuration 1] $\mu_i^s = \mu_d^m = \mu_d^s = 0 < \mu_i^m$. In this case (55) implies $\bar{a}_{ti}^m = 0$. Conditions (49)-(52) imply $\mu_i^m = (\varepsilon - \varepsilon_t^*) \bar{\phi}_t y / (\varepsilon y + \phi_t^s)$, and therefore $\varepsilon_t^* < \varepsilon$. Then from (59) and (60) it follows that

$$\bar{a}_{ti}^s = a_{ti}^s + \left(\frac{\varepsilon_t^* y + \phi_t^s}{\varepsilon y + \phi_t^s} \right) \frac{1}{p_t} a_{ti}^m$$

and $(\bar{a}_{td*}^m, \bar{a}_{td*}^s)$ is any nonnegative pair that satisfies

$$\bar{a}_{td*}^m + p_t \bar{a}_{td*}^s = a_{td}^m + p_t a_{td}^s + \frac{(\varepsilon - \varepsilon_t^*) y}{\varepsilon y + \phi_t^s} a_{ti}^m.$$

[Configuration 2] $\mu_i^m = \mu_i^s = \mu_d^m = \mu_d^s = 0$. In this case conditions (49)-(52) imply $\varepsilon = \varepsilon_t^*$, and (59) and (60) yield

$$\bar{a}_{ti}^m + p_t \bar{a}_{ti}^s = a_{ti}^m + p_t a_{ti}^s \quad (61)$$

$$\bar{a}_{td*}^m + p_t \bar{a}_{td*}^s = a_{td}^m + p_t a_{td}^s. \quad (62)$$

Hence the dealer is indifferent between making any offer $(\bar{a}_{ti}^m, \bar{a}_{ti}^s, \bar{a}_{td*}^m, \bar{a}_{td*}^s)$ such that $(\bar{a}_{ti}^m, \bar{a}_{ti}^s) \in \mathbb{R}_+$ satisfies (61), and $(\bar{a}_{td*}^m, \bar{a}_{td*}^s) \in \mathbb{R}_+$ satisfies (62). [Configuration 3] $\mu_i^m = \mu_d^m = \mu_d^s = 0 < \mu_i^s$. In this case condition (56) implies $\bar{a}_{ti}^s = 0$. Conditions (51) and (52) imply $\mu_i^s = (\varepsilon_t^* - \varepsilon) y \rho$, and therefore $\varepsilon < \varepsilon_t^*$. Then from (59) and (60) it follows that

$$\bar{a}_{ti}^m = a_{ti}^m + \frac{\varepsilon y + \phi_t^s}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s$$

and $(\bar{a}_{td*}^m, \bar{a}_{td*}^s)$ is any nonnegative pair that satisfies

$$\bar{a}_{td*}^m + p_t \bar{a}_{td*}^s = a_{td}^m + p_t a_{td}^s + \frac{(\varepsilon_t^* - \varepsilon) y}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s.$$

[Configuration 4] $\mu_d^m = \mu_d^s = 0$, $0 < \mu_i^m$ and $0 < \mu_i^s$. In this case conditions (55) and (56) imply $\bar{a}_{ti}^m = \bar{a}_{ti}^s = 0$, which according to (60), is only possible if $a_{ti}^m = a_{ti}^s = 0$. Then $(\bar{a}_{td^*}^m, \bar{a}_{td^*}^s)$ is any nonnegative pair that satisfies (62). [Configuration 5] $\mu_i^s = 0 < \mu_i^m$, $0 < \mu_d^m$ and $0 < \mu_d^s$. In this case conditions (55), (57) and (58) imply $\bar{a}_{ti}^m = \bar{a}_{td^*}^m = \bar{a}_{td^*}^s = 0$. Conditions (51) and (52) imply $\varepsilon_t^* < \varepsilon$. Then from (59) and (60) it follows that the following condition must hold:

$$a_{td}^s + \frac{1}{p_t} a_{td}^m = - \left[\frac{(\varepsilon - \varepsilon_t^*) y}{(\varepsilon - \varepsilon_t^*) y + p_t \phi_t^m} \right] \frac{1}{p_t} a_{ti}^m.$$

The term on the left side of the equality is nonnegative and the term on the right side of the equality is nonpositive (since $\varepsilon_t^* < \varepsilon$), so this condition can hold only if $a_{ti}^m = a_{td}^m = a_{td}^s = 0$. Therefore (59) implies $\bar{a}_{ti}^s = a_{ti}^s$. [Configuration 6] $\mu_i^m = \mu_i^s = 0$, $0 < \mu_d^m$ and $0 < \mu_d^s$. In this case conditions (57) and (58) imply $\bar{a}_{td^*}^m = \bar{a}_{td^*}^s = 0$. Conditions (51) and (52) imply $\varepsilon = \varepsilon_t^*$, and in turn conditions (59) and (60) imply $a_{td}^m + p_t a_{td}^s = 0$, or equivalently, $a_{td}^m = a_{td}^s = 0$ must hold, and $(\bar{a}_{ti}^m, \bar{a}_{ti}^s)$ is any nonnegative pair that satisfies (61). [Configuration 7] $\mu_i^m = 0 < \mu_i^s$, $0 < \mu_d^s$ and $0 < \mu_d^m$. In this case conditions (56)-(58) imply $\bar{a}_{ti}^s = \bar{a}_{td^*}^m = \bar{a}_{td^*}^s = 0$. Conditions (51) and (52) imply $\varepsilon < \varepsilon_t^*$. Then from (59) and (60) it follows that the following condition must hold:

$$\phi_t^m (a_{td}^m + p_t a_{td}^s) = - (\varepsilon_t^* - \varepsilon) y a_{ti}^s.$$

The term on the left side of the equality is nonnegative and the term on the right side of the equality is nonpositive (since $\varepsilon < \varepsilon_t^*$), so this condition can hold only if $\phi_t^m (a_{td}^m + p_t a_{td}^s) = a_{ti}^s = 0$. Therefore (60) implies $\bar{a}_{ti}^m = a_{ti}^m$. [Configuration 8] $0 < \mu_i^m$, $0 < \mu_i^s$, $0 < \mu_d^s$ and $0 < \mu_d^m$. In this case conditions (55)-(58) imply $\bar{a}_{ti}^m = \bar{a}_{ti}^s = \bar{a}_{td^*}^m = \bar{a}_{td^*}^s = 0$, which according to (59) is only possible, and the only possible solution if $a_{ti}^m = a_{ti}^s = a_{td}^m = a_{td}^s = 0$. To conclude, notice that the solutions for Configurations 1, 2, and 3, correspond to the first, second, and third lines of the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td^*}^m$, and $\bar{a}_{td^*}^s$ in part (ii) of the statement of the lemma. Similarly, the solution for Configuration 5 corresponds to the first line of the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td^*}^m$, and $\bar{a}_{td^*}^s$ in part (ii) of the statement of the lemma, with $a_{ti}^m = a_{td}^m = a_{td}^s = 0$. The solution for Configuration 6 corresponds to the second line of the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td^*}^m$, and $\bar{a}_{td^*}^s$ in part (ii) of the statement of the lemma, with $a_{td}^m = a_{td}^s = 0$. The solution for Configuration 7 corresponds to the third line of the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td^*}^m$, and $\bar{a}_{td^*}^s$ in part (ii) of the statement of the lemma, with $\phi_t^m (a_{td}^m + p_t a_{td}^s) = a_{ti}^s = 0$. Finally, it is easy to see that the solution for Configuration 4 can be obtained from the expressions for $\bar{a}_{ti}^m$, $\bar{a}_{ti}^s$, $\bar{a}_{td^*}^m$, and $\bar{a}_{td^*}^s$ in

part (ii) of the statement of the lemma simply by setting $a_{ti}^m = a_{ti}^s = 0$, and the solution for case Configuration 8 can be obtained similarly, by setting $a_{ti}^m = a_{ti}^s = a_{td}^m = a_{td}^s = 0$. ■

Proof of Lemma 3. (i) With Lemma 1, (3) becomes

$$\begin{aligned} V_t^d(a_{td}^m, a_{td}^s) &= \eta\theta \int \bar{\phi}_t [\bar{a}_{td}^m + p_t \bar{a}_{td}^s - (a_{td}^m + p_t a_{td}^s)] dH_t(\mathbf{a}_{ti}, \varepsilon) \\ &\quad + \eta(1-\theta) \int \bar{\phi}_t [\bar{a}_{td}^{m*} + p_t \bar{a}_{td}^{s*} - (a_{td}^m + p_t a_{td}^s)] dH_t(\mathbf{a}_{ti}, \varepsilon) \\ &\quad + \bar{\phi}_t(a_{td}^m + p_t a_{td}^s) + W_t^d(\mathbf{0}) \end{aligned}$$

where we have used the more compact notation introduced in Lemma 2, i.e., $\bar{a}_{ti*}^k \equiv \bar{a}_{i*}^k(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t)$, $\bar{a}_{td}^k \equiv \bar{a}_d^k(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t)$, $\bar{a}_{ti}^k \equiv \bar{a}_i^k(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t)$, and $\bar{a}_{td*}^k \equiv \bar{a}_{d*}^k(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \psi_t)$, for $k = m, s$. Use Corollary 1 to arrive at

$$\begin{aligned} V_t^d(a_{td}^m, a_{td}^s) &= \eta(1-\theta) \int \bar{\phi}_t \left[\mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}} \frac{(\varepsilon_t^* - \varepsilon)y}{\varepsilon_t^* y + \phi_t^s} p_t a_{ti}^s + \mathbb{I}_{\{\varepsilon_t^* \leq \varepsilon\}} \frac{(\varepsilon - \varepsilon_t^*)y}{\varepsilon y + \phi_t^s} a_{ti}^m \right] dH_t(\mathbf{a}_{ti}, \varepsilon) \\ &\quad + \bar{\phi}_t(a_{td}^m + p_t a_{td}^s) + W_t^d(\mathbf{0}) \end{aligned}$$

where $\mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}}$ is an indicator function that takes the value 1 if $\varepsilon < \varepsilon_t^*$, and 0 otherwise. To obtain (10) simply use the fact that $dH_t(\mathbf{a}_{ti}, \varepsilon) = dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon)$.

(ii) With (33) and the notation introduced in Lemma 2, (5) becomes

$$\begin{aligned} V_t^i(a_{ti}^m, a_{ti}^s, \varepsilon) &= \delta\theta \int [\phi_t^m (\bar{a}_{ti*}^m - a_{ti}^m) + (\varepsilon y + \phi_t^s) (\bar{a}_{ti*}^s - a_{ti}^s)] dF_t^D(\mathbf{a}_{td}) \\ &\quad + \delta(1-\theta) \int \{\phi_t^m (\bar{a}_{ti}^m - a_{ti}^m) + (\varepsilon y + \phi_t^s) (\bar{a}_{ti}^s - a_{ti}^s)\} dF_t^D(\mathbf{a}_{td}) \\ &\quad + \phi_t^m a_{ti}^m + (\varepsilon y + \phi_t^s) a_{ti}^s + W_t^i(\mathbf{0}). \end{aligned}$$

Using Lemma 2 to substitute for $\bar{a}_{ti*}^m - a_{ti}^m$, $\bar{a}_{ti*}^s - a_{ti}^s$, $\bar{a}_{ti}^m - a_{ti}^m$ and $\bar{a}_{ti}^s - a_{ti}^s$, delivers (11). ■

Proof of Lemma 4. With Lemma 3, the dealer's problem in the second subperiod of period t , (7), becomes

$$W_t^d(\mathbf{0}) = \max_{\mathbf{a}_{t+1} \in \mathbb{R}_+^2} [(\beta \bar{\phi}_{t+1} - \phi_t^m) a_{t+1}^m + (\beta \bar{\phi}_{t+1} p_{t+1} - \phi_t^s) a_{t+1}^s] + \beta V_{t+1}^d(\mathbf{0}). \quad (63)$$

The investor's problem in the second subperiod of period t , (4), can be written as in (33), where

$$\begin{aligned} W_t^i(\mathbf{0}) = & \max_{a_{t+1}^m \in \mathbb{R}_+} \left\{ \beta \int \left[\mathbb{I}_{\{\varepsilon_{t+1}^* \leq \varepsilon\}} \delta \theta \frac{(\varepsilon - \varepsilon_{t+1}^*) y}{\varepsilon_{t+1}^* y + \phi_{t+1}^s} + 1 \right] \phi_{t+1}^m dG(\varepsilon) - \phi_t^m \right\} a_{t+1}^m + \\ & \max_{a_{t+1}^s \in \mathbb{R}_+} \left\{ \beta \int \left[\mathbb{I}_{\{\varepsilon < \varepsilon_{t+1}^*\}} \delta \theta (\varepsilon_{t+1}^* - \varepsilon) y + (\varepsilon y + \phi_{t+1}^s) \right] dG(\varepsilon) - \phi_t^s \right\} a_{t+1}^s \\ & + T_t + \beta W_{t+1}^i(\mathbf{0}). \end{aligned} \quad (64)$$

The first-order necessary and sufficient conditions for optimization of (63) are (12) and (13).

The first-order necessary and sufficient conditions for optimization of (64) are (14) and (15). ■

Lemma 5 Consider $\hat{\gamma}$ and $\bar{\gamma}$ as defined in (16). Then $\hat{\gamma} < \bar{\gamma}$.

Proof of Lemma 5. Define $\Gamma(\zeta) : \mathbb{R} \rightarrow \mathbb{R}$ by $\Gamma(\zeta) \equiv \beta[1 + \delta\theta(1 - \beta)\zeta]$. Let $\hat{\zeta} \equiv \frac{(1 - \delta\theta)(\bar{\varepsilon} - \bar{\varepsilon})}{\delta\theta\bar{\varepsilon}}$ and $\bar{\zeta} \equiv \frac{\bar{\varepsilon} - \varepsilon_L}{\beta\bar{\varepsilon} + (1 - \beta)\varepsilon_L}$, so that $\hat{\gamma} = \Gamma(\hat{\zeta})$ and $\bar{\gamma} = \Gamma(\bar{\zeta})$. Since Γ is strictly increasing, $\hat{\gamma} < \bar{\gamma}$ if and only if $\hat{\zeta} < \bar{\zeta}$. With (17) and the fact that $\bar{\varepsilon} \equiv \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon dG(\varepsilon) = \varepsilon_H - \int_{\varepsilon_L}^{\varepsilon_H} G(\varepsilon) d\varepsilon$,

$$\hat{\zeta} = \frac{\int_{\hat{\varepsilon}}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\hat{\varepsilon}} G(\varepsilon) d\varepsilon},$$

so clearly,

$$\hat{\zeta} < \frac{\int_{\varepsilon_L}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\bar{\varepsilon}} = \frac{\bar{\varepsilon} - \varepsilon_L}{\bar{\varepsilon}} < \bar{\zeta}.$$

Hence $\hat{\gamma} < \bar{\gamma}$. ■

Proof of Proposition 1. In an equilibrium with no money (or no valued money), there is no trade in the OTC market. The first-order conditions for a dealer d in the Walrasian market are

$$\begin{aligned} \phi_t^s &\geq \beta \phi_{t+1}^s, \text{ “=” if } a_{t+1d}^s > 0 \\ \phi_t^m &\geq \beta \phi_{t+1}^m, \text{ “=” if } a_{t+1d}^m > 0. \end{aligned}$$

The first-order conditions for an investor i in the Walrasian market are

$$\begin{aligned} \phi_t^s &\geq \beta (\bar{\varepsilon} y + \phi_{t+1}^s), \text{ “=” if } a_{t+1i}^s > 0 \\ \phi_t^m &\geq \beta \phi_{t+1}^m, \text{ “=” if } a_{t+1i}^m > 0. \end{aligned}$$

In a stationary equilibrium, $\phi_t^s / \phi_{t+1}^s = 1 > \beta$ and $\phi_t^m / \phi_{t+1}^m = \gamma > \beta$, so no dealer holds equity, and no dealer or investor holds money at the beginning of a period. The Walrasian market for

equity can only clear if $\phi^s = \frac{\beta}{1-\beta} \bar{\varepsilon} y$. This establishes parts (i) and (iii) in the statement of the proposition. Next, we turn to monetary equilibria. In a stationary equilibrium, (12)-(15) can be written as

$$\gamma \geq \beta, \text{ “=” if } a_{t+1d}^m > 0 \quad (65)$$

$$\phi^s \geq \beta \bar{\phi}^s, \text{ “=” if } a_{t+1d}^s > 0 \quad (66)$$

$$1 \geq \frac{\beta}{\gamma} \left[1 + \delta \theta y \frac{\int_{\varepsilon^*}^{\varepsilon^H} [1 - G(\varepsilon)] d\varepsilon}{\varepsilon^* y + \phi^s} \right], \text{ “=” if } a_{t+1i}^m > 0 \quad (67)$$

$$\phi^s \geq \frac{\beta}{1-\beta} \left[\bar{\varepsilon} + \delta \theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon \right] y, \text{ “=” if } a_{t+1i}^s > 0. \quad (68)$$

(In (65) we have used the fact that $\bar{\phi}^s = \varepsilon^* y + \phi^s > \phi^s$.) Under our maintained assumption $\beta < \gamma$, (65) implies $a_{t+1d}^m = 0$, so in a monetary equilibrium (67) must hold with equality for some investor i . Thus in order to find a monetary equilibrium, there are three possible equilibrium configurations to consider, depending on the various binding patterns of the complementary slackness conditions (65)-(68). The following three conditions will apply to all three configurations:

$$\varepsilon^* = \frac{\bar{\phi}^s - \phi^s}{y} \quad (69)$$

$$\bar{\phi}^s = p_t \phi_t^m \quad (70)$$

$$\begin{aligned} & \theta \int \{ \hat{a}_d^s [\bar{\mathbf{a}}_d(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t); \boldsymbol{\psi}_t] + \bar{a}_{i*}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) \} dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon) \\ & + (1 - \theta) \int \{ \hat{a}_d^s [\bar{\mathbf{a}}_d^*(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t); \boldsymbol{\psi}_t] + \bar{a}_i^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) \} dF_t^D(\mathbf{a}_{td}) dF_t^I(\mathbf{a}_{ti}) dG(\varepsilon) \\ & = \int a_{td}^s dF_t^D(\mathbf{a}_{td}) + \int a_{ti}^s dF_t^I(\mathbf{a}_{ti}) + \frac{(1 - \eta)v}{\delta} \int [a_{td}^s - \hat{a}_d^s(\mathbf{a}_{td}; \boldsymbol{\psi}_t)] dF_t^D(\mathbf{a}_{td}). \end{aligned} \quad (71)$$

The last condition is $\bar{A}_t^D + \bar{A}_t^I = A_t^D + \delta A_t^I$ after using the fact that $\delta = \eta v$. Hereafter the proof proceeds by construction, in three steps.

Step 1: Try to construct a stationary monetary equilibrium with $a_{t+1d}^s = 0$ for all $d \in \mathcal{D}$, and $a_{t+1i}^s > 0$ for some $i \in \mathcal{I}$. The equilibrium conditions for this case are (69), (70) and (71)

together with

$$\phi^s > \beta \bar{\phi}^s \quad (72)$$

$$1 = \frac{\beta}{\gamma} \left[1 + \delta\theta y \frac{\int_{\varepsilon^*}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\varepsilon^* y + \phi^s} \right] \quad (73)$$

$$\phi^s = \frac{\beta}{1 - \beta} \left[\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon \right] y \quad (74)$$

$$a_{t+1d}^m = 0 \text{ for all } d \in \mathcal{D} \quad (75)$$

$$a_{t+1i}^m \geq 0, \text{ with “} > \text{” for some } i \in \mathcal{I} \quad (76)$$

$$a_{t+1d}^s = 0 \text{ for all } d \in \mathcal{D} \quad (77)$$

$$a_{t+1i}^s \geq 0, \text{ with “} > \text{” for some } i \in \mathcal{I}. \quad (78)$$

The five conditions (69), (70), (71), (73), and (74), are to be solved for the five unknowns: ε^* , ϕ^s , $\bar{\phi}^s$, p_t , ϕ_t^m . Substitute (74) into (73) to obtain

$$1 = \frac{\beta}{\gamma} \left[1 + \delta\theta \frac{\int_{\varepsilon^*}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\varepsilon^* + \frac{\beta}{1-\beta} \left[\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon \right]} \right] \quad (79)$$

which is a single equation in ε^* . Define

$$T(x) \equiv \frac{\int_x^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\frac{1}{1-\beta}x + \frac{\beta}{1-\beta}\hat{T}(x)} - \frac{\gamma - \beta}{\beta\delta\theta} \quad (80)$$

with

$$\hat{T}(x) \equiv \bar{\varepsilon} - x + \delta\theta \int_{\varepsilon_L}^x G(\varepsilon) d\varepsilon, \quad (81)$$

and notice that ε^* solves (79) if and only if it satisfies $T(\varepsilon^*) = 0$. Notice that T is a continuous real-valued function on $[\varepsilon_L, \varepsilon_H]$, with

$$T(\varepsilon_L) = \frac{\bar{\varepsilon} - \varepsilon_L}{\varepsilon_L + \frac{\beta}{1-\beta}\bar{\varepsilon}} - \frac{\gamma - \beta}{\beta\delta\theta},$$

$$T(\varepsilon_H) = -\frac{\gamma - \beta}{\beta\delta\theta} < 0,$$

and

$$T'(x) = -\frac{[1 - G(x)] \left\{ x + \frac{\beta}{1-\beta} \left[\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^x G(\varepsilon) d\varepsilon \right] \right\} + \left[\int_x^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon \right] \left\{ 1 + \frac{\beta}{1-\beta} \delta\theta G(x) \right\}}{\left\{ x + \frac{\beta}{1-\beta} \left[\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^x G(\varepsilon) d\varepsilon \right] \right\}^2} < 0.$$

Hence if $T(\varepsilon_L) > 0$, or equivalently, if $\gamma < \bar{\gamma}$ (with $\bar{\gamma}$ is as defined in (16)) then there exists a unique $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$ that satisfies $T(\varepsilon^*) = 0$ (and $\varepsilon^* \downarrow \varepsilon_L$ as $\gamma \uparrow \bar{\gamma}$). Given ε^* , ϕ^s is given by (74). Given ε^* and ϕ^s , $\bar{\phi}^s$ is given by (69). Given $\bar{\phi}^s$, we can then use (70) and (71) to obtain p_t and ϕ_t^m as follows. From Lemma 1 and Lemma 2 we know that

$$\bar{a}_{ti}^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) = \left(a_{ti}^s + \frac{1}{p_t} a_{ti}^m \right) \mathbb{I}_{\{\varepsilon^* < \varepsilon\}} \quad (82)$$

$$\bar{a}_i^s(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t) = \left[a_{ti}^s + \left(\frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} \right) \frac{1}{p_t} a_{ti}^m \right] \mathbb{I}_{\{\varepsilon^* < \varepsilon\}} \quad (83)$$

$$\hat{a}_d^s[\bar{\mathbf{a}}_d(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t); \boldsymbol{\psi}_t] = 0 \quad (84)$$

$$\hat{a}_d^s[\bar{\mathbf{a}}_{d^*}(\mathbf{a}_{ti}, \mathbf{a}_{td}, \varepsilon; \boldsymbol{\psi}_t); \boldsymbol{\psi}_t] = 0 \quad (85)$$

$$\hat{a}_d^s(\mathbf{a}_{td}; \boldsymbol{\psi}_t) = 0. \quad (86)$$

Conditions (84)-(86) follow from Lemma 1, given that (69) and (70) imply $\phi^s < \varepsilon^* y + \phi^s = p_t \phi_t^m$.

With (82)-(86) we can rewrite (71) as

$$\begin{aligned} & \delta \left\{ \theta [1 - G(\varepsilon^*)] + (1 - \theta) \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} dG(\varepsilon) \right\} \frac{1}{p_t} \int a_{ti}^m dF_t^I(\mathbf{a}_{ti}) \\ &= \delta G(\varepsilon^*) \int a_{ti}^s dF_t^I(\mathbf{a}_{ti}) + \int v a_{td}^s dF_t^D(\mathbf{a}_{td}). \end{aligned} \quad (87)$$

Since $\beta < \gamma$ implies $a_{td}^m = 0$ and (72) implies $a_{td}^s = 0$ for all d , (87) simplifies to

$$\left\{ \theta [1 - G(\varepsilon^*)] + (1 - \theta) \int_{\varepsilon^*}^{\varepsilon_H} \left(\frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} \right) dG(\varepsilon) \right\} \frac{1}{p_t} M_t = G(\varepsilon^*) A$$

where we have used the equilibrium conditions $A_t^I + A_t^D = A_t^I = A$ and $M_t^I + M_t^D = M_t^I = M_t$.

This condition in turn implies

$$p_t = \left\{ \theta [1 - G(\varepsilon^*)] + (1 - \theta) \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} dG(\varepsilon) \right\} \frac{M_t}{G(\varepsilon^*) A}. \quad (88)$$

Given p_t and $\bar{\phi}^s$, (70) can then be used to obtain ϕ_t^m . To conclude this step, notice that for this case to be an equilibrium (72) must hold, or equivalently, using (69) and (74), it must be that $\hat{T}(\varepsilon^*) > 0$, where $\hat{T}$ is the continuous function on $[\varepsilon_L, \varepsilon_H]$ defined in (81). Notice that $\hat{T}'(x) = -[1 - \delta\theta G(x)] < 0$, and $\hat{T}(\varepsilon_H) = -(1 - \delta\theta)(\varepsilon_H - \bar{\varepsilon}) < 0 < \bar{\varepsilon} - \varepsilon_L = \hat{T}(\varepsilon_L)$, so there exists a unique $\hat{\varepsilon} \in (\varepsilon_L, \varepsilon_H)$ such that $\hat{T}(\hat{\varepsilon}) = 0$. (Since $\hat{T}(\bar{\varepsilon}) > 0$, and $\hat{T}' < 0$, it follows that $\bar{\varepsilon} < \hat{\varepsilon}$.) Then $\hat{T}'(x) < 0$ implies $\hat{T}(\varepsilon^*) \geq 0$ if and only if $\varepsilon^* \leq \hat{\varepsilon}$, with “=” for $\varepsilon^* = \hat{\varepsilon}$. With (80), we know that $\varepsilon^* < \hat{\varepsilon}$ if and only if $T(\hat{\varepsilon}) < 0 = T(\varepsilon^*)$, i.e., if and only if

$$\beta \left[1 + \frac{\delta\theta(1 - \beta) \int_{\hat{\varepsilon}}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\beta\bar{\varepsilon} + (1 - \beta)\hat{\varepsilon} + \beta\delta\theta \int_{\varepsilon_L}^{\hat{\varepsilon}} G(\varepsilon) d\varepsilon} \right] < \gamma.$$

Since $\hat{T}(\hat{\varepsilon}) = (1 - \delta\theta)(\bar{\varepsilon} - \hat{\varepsilon}) + \delta\theta \int_{\hat{\varepsilon}}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon = 0$, this last condition is equivalent to $\hat{\gamma} < \gamma$, where $\hat{\gamma}$ is as defined in (16). The allocations and asset prices described in this step correspond to those in the statement of the proposition for the case with $\gamma \in (\hat{\gamma}, \bar{\gamma})$.

Step 2: Try to construct a stationary monetary equilibrium with $a_{t+1d}^s > 0$ for some $d \in \mathcal{D}$, and $a_{t+1i}^s = 0$ for all $i \in \mathcal{I}$. The equilibrium conditions are (69), (70) and (71) together with (73), (75), (76),

$$\phi^s = \beta \bar{\phi}^s \quad (89)$$

$$\phi^s > \frac{\beta}{1 - \beta} \left[\bar{\varepsilon} + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon \right] y, \text{ “=” if } a_{t+1i}^s > 0. \quad (90)$$

$$a_{t+1d}^s \geq 0, \text{ with “>” for some } d \in \mathcal{D} \quad (91)$$

$$a_{t+1i}^s = 0, \text{ for all } i \in \mathcal{I}. \quad (92)$$

The five conditions (69), (70), (71), (73), and (89), are to be solved for the five unknowns: ε^* , ϕ^s , $\bar{\phi}^s$, p_t , ϕ_t^m . First combine (69) and (89) to obtain

$$\phi^s = \frac{\beta}{1 - \beta} \varepsilon^* y. \quad (93)$$

Substitute (93) in (73) to obtain

$$1 = \frac{\beta}{\gamma} \left[1 + \frac{\delta\theta(1 - \beta) \int_{\varepsilon^*}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\varepsilon^*} \right] \quad (94)$$

which is a single equation in ε^* . Define

$$R(x) \equiv \frac{\int_x^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\frac{1}{1 - \beta} x} - \frac{\gamma - \beta}{\beta\delta\theta} \quad (95)$$

and notice that ε^* solves (94) if and only if it satisfies $R(\varepsilon^*) = 0$. Notice that R is a continuous real-valued function on $[\varepsilon_L, \varepsilon_H]$, with

$$R(\varepsilon_L) = \frac{\bar{\varepsilon} - \varepsilon_L}{\frac{1}{1 - \beta} \varepsilon_L} - \frac{\gamma - \beta}{\beta\delta\theta}$$

$$R(\varepsilon_H) = -\frac{\gamma - \beta}{\beta\delta\theta}$$

and

$$R'(x) = -\frac{[1 - G(x)] \frac{1}{1 - \beta} x + \frac{1}{1 - \beta} \int_x^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\left(\frac{1}{1 - \beta} x\right)^2} < 0.$$

Hence if $R(\varepsilon_L) > 0$, or equivalently, if

$$\gamma < \beta \left[1 + \frac{\delta\theta(1-\beta)(\bar{\varepsilon} - \varepsilon_L)}{\varepsilon_L} \right] \equiv \tilde{\gamma}$$

then there exists a unique $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$ that satisfies $R(\varepsilon^*) = 0$ (and $\varepsilon^* \downarrow \varepsilon_L$ as $\gamma \uparrow \tilde{\gamma}$). Having solved for ε^* , ϕ^s is obtained from (93), and given ϕ^s , $\bar{\phi}^s$ is obtained from (89). Next we use the market-clearing condition (71) (or equivalently, (87)) to obtain p_t . We start from (87) and impose: (a) $a_{ti}^s = 0$ (condition (92)) which implies $A_t^I + A_t^D = A_t^D = A$, and (b) $a_{td}^m = 0$ (implied by the maintained assumption $\beta < \gamma$, which in turn implies $M_t^I + M_t^D = M_t^I = M_t$), to obtain

$$p_t = \left\{ \theta [1 - G(\varepsilon^*)] + (1 - \theta) \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} dG(\varepsilon) \right\} \frac{\delta M_t}{A}. \quad (96)$$

Having obtained p_t (and $\bar{\phi}^s$), ϕ_t^m is obtained from (70). Notice that for this case to be an equilibrium (90) must hold, or equivalently, using (93), it must be that $\hat{T}(\varepsilon^*) < 0$, which is in turn equivalent to $\hat{\varepsilon} < \varepsilon^*$. With (95), we know that $\hat{\varepsilon} < \varepsilon^*$ if and only if $R(\varepsilon^*) = 0 < R(\hat{\varepsilon})$, i.e., if and only if

$$\gamma < \beta \left[1 + \frac{\delta\theta(1-\beta) \int_{\hat{\varepsilon}}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\hat{\varepsilon}} \right],$$

which using the fact that $\hat{T}(\hat{\varepsilon}) = 0$, can be written as $\gamma < \hat{\gamma}$. To summarize, the prices and allocations constructed in this step constitute a stationary monetary equilibrium provided $\gamma \in (\beta, \min(\hat{\gamma}, \tilde{\gamma}))$. To conclude this step, we show that $\hat{\gamma} < \bar{\gamma} < \tilde{\gamma}$, which together with the previous step will mean that there is no stationary monetary equilibrium for $\gamma \geq \bar{\gamma}$ (thus establishing part (ii) in the statement of the proposition). It is clear that $\bar{\gamma} < \tilde{\gamma}$, and we know that $\hat{\gamma} < \bar{\gamma}$ from Lemma 5. Therefore the allocations and asset prices described in this step correspond to those in the statement of the proposition for the case with $\gamma \in (\beta, \min(\hat{\gamma}, \tilde{\gamma})) = (\beta, \hat{\gamma})$.

Step 3: Try to construct a stationary monetary equilibrium with $a_{t+1d}^s > 0$ for some $d \in \mathcal{D}$, and $a_{t+1i}^s > 0$ for some $i \in \mathcal{I}$. The equilibrium conditions are (69), (70) and (71) together with (73), (74), (75), (76), and (89)

$$a_{t+1i}^s \geq 0 \text{ and } a_{t+1d}^s \geq 0, \text{ with “} > \text{” for some } i \in \mathcal{I} \text{ or some } d \in \mathcal{D}.$$

The five unknowns: ε^* , ϕ^s , $\bar{\phi}^s$, p_t , ϕ_t^m , must satisfy the six conditions (69), (70), (71), (73), (74) and (89).

Notice that ε^* , ϕ^s and $\bar{\phi}^s$ are obtained as in Step 2. Now, however, (74) must also hold, which together with (93) implies that

$$0 = \bar{\varepsilon} - \varepsilon^* + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon) d\varepsilon$$

or equivalently, (since the right side is just $\hat{T}(\varepsilon^*)$), that $\varepsilon^* = \hat{\varepsilon}$. In other words, this condition requires $R(\hat{\varepsilon}) = \hat{T}(\hat{\varepsilon})$, or equivalently, we must have $\gamma = \hat{\gamma}$. Next we use the market-clearing condition (71) (or equivalently, (87)) to obtain p_t . We start from (87) and impose $a_{td}^m = 0$ (implied by the maintained assumption $\beta < \gamma$, which in turn implies $M_t^I + M_t^D = M_t^I = M_t$) to obtain

$$p_t = \left\{ \theta [1 - G(\varepsilon^*)] + (1 - \theta) \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon^* y + \phi^s}{\varepsilon y + \phi^s} dG(\varepsilon) \right\} \frac{\delta M_t}{A_t^D + \delta G(\varepsilon^*) A_t^I}. \quad (97)$$

Having obtained p_t (and $\bar{\phi}^s$), ϕ_t^m is obtained from (70). The allocations and asset prices described in this step correspond to those in the statement of the proposition for the case with $\gamma = \hat{\gamma}$.

Combined, Steps 1-3 prove part (iv) in the statement of the proposition. Part (v)(a) is immediate from (74) and (80), and part (v)(b) from (93) and (95). ■

Corollary 3 *The marginal type, ε^* , characterized in Proposition 1 is strictly decreasing in the rate of inflation, i.e., $\frac{\partial \varepsilon^*}{\partial \gamma} < 0$ both for $\gamma \in (\beta, \hat{\gamma})$, and for $\gamma \in (\hat{\gamma}, \bar{\gamma})$.*

Proof of Corollary 3. For $\gamma \in (\beta, \hat{\gamma})$, implicitly differentiate $R(\varepsilon^*) = 0$ (with R given by (95)), and for $\gamma \in (\hat{\gamma}, \bar{\gamma})$, implicitly differentiate $T(\varepsilon^*) = 0$ (with T given by (80)) to obtain

$$\frac{\partial \varepsilon^*}{\partial \gamma} = \begin{cases} -\frac{\varepsilon^*}{\beta \delta \theta (1 - \beta) [1 - G(\varepsilon^*)] + \gamma - \beta} & \text{if } \beta < \gamma < \hat{\gamma} \\ -\frac{\beta \delta \theta \int_{\varepsilon^*}^{\varepsilon_H} [1 - G(\varepsilon)] d\varepsilon}{\left\{ 1 + \beta \delta \theta \left[\frac{G(\varepsilon^*)}{1 - \beta} + \frac{1 - G(\varepsilon^*)}{\gamma - \beta} \right] \right\} (\gamma - \beta)^2} & \text{if } \hat{\gamma} < \gamma < \bar{\gamma}. \end{cases}$$

Clearly, $\frac{\partial \varepsilon^*}{\partial \gamma} < 0$ for $\gamma \in (\beta, \hat{\gamma})$, and for $\gamma \in (\hat{\gamma}, \bar{\gamma})$. ■

Proof of Proposition 2. The choice variable a'_{td} does not appear in the Planner's objective function, so $a'_{td} = 0$ at an optimum. Also, (28) must bind for every t at an optimum, so the planner's problem is equivalent to

$$\begin{aligned} \max_{\{a_{td}, a_{ti}, a'_{ti}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t & \left[\delta \int_{[\varepsilon_L, \varepsilon_H]} \varepsilon y a'_{ti}(d\varepsilon) + (1 - \delta) \bar{\varepsilon} y a_{ti} \right] \\ \text{s.t. (26) and } \delta \int_{[\varepsilon_L, \varepsilon_H]} & a'_{ti}(d\varepsilon) \leq v a_{td} + \delta a_{ti}. \end{aligned}$$

Let W^* denote the maximum value of this problem. Then clearly, $W^* \leq \bar{W}^*$, where

$$\begin{aligned} \bar{W}^* &= \max_{\{a_{td}, a_{ti}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \{ \varepsilon_H y v a_{td} + [\delta \varepsilon_H + (1 - \delta) \bar{\varepsilon}] y a_{ti} \} \text{ s.t. (26)} \\ &= \frac{1}{1 - \beta} \varepsilon_H y A. \end{aligned}$$

The allocation rule $a_{td} = A/v$ and $a_{ti} = 0$ together with the Dirac measure defined in the statement of the proposition, achieve $\bar{W}^*$ and therefore solve the Planner's problem. ■