Currency Wars

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Abstract

Many central banks manage the stochastic behavior of their currencies' exchange rates by imposing pegs relative to a target currency. We study the effects of such currency manipulation in a multi-country model of exchange rate determination with endogenous capital accumulation. We find that the imposition of an exchange rate peg relative to a given target currency increases the volatility of consumption in the target country and decreases the volatility of the target currency's exchange rate relative to all other currencies in the world. In addition, currency pegs affect the formation of capital across sectors and countries. For example, an economically smaller country (such as Saudi Arabia) pegging its currency to an economically large country (such at the U.S.) decreases capital accumulation in the larger country and increases its real and nominal interest rate.

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1 Introduction

Many central banks manage the stochastic behavior of their currencies' exchange rates by imposing pegs relative to a target currency (Reinhart and Rogoff (2004)). A large literature studies the costs and benefits of such exchange rate management from the perspective of the country that adopts the peg (e.g. Rebelo and Vegh (1995), Helpman and Razin (1987), and Aghion et al. (2009)). In contrast, very little is known about the effect of exchange rate management on the country that is the target of the peg and about the effect on countries outside the peg. In this paper we study these "external" effects of exchange rate management.

Consider a world with many countries where a "pegging" country (say Saudi Arabia) imposes a real exchange rate peg on a "target" country (say the U.S.). Suppose households in all countries consume a mixture of traded and non-traded goods. An immediate effect of a real exchange rate peg is a reallocation of tradable goods between all countries in the world. Hence, we expect a peg to not only affect the relationship between the pegging and target countries, but the exchange rates between all pairs of countries.

This paper develops an international asset pricing model where we introduce a real exchange rate peg and characterize the effects of the peg on consumption, asset prices, and capital accumulation. We show that the exchange rate peg will affect tradable good consumption in both the pegging and target countries, and this impacts many economic relationships between the target country and all other countries in the world. For example, the exchange rate peg decreases the volatility of the exchange rate between the target country and any other country in the world. In addition, a smaller country pegging its exchange rate to a larger country will lower interest rates and increase capital accumulation in its own country but raise the interest rate and lower capital accumulation in the target country.

We first show the main implications of the peg in a simplified model in which asset markets are complete and the world is populated by a continuum of households that receive stochastic endowments of a traded and of a non-traded consumption good. All households within a given country receive the same per capita endowments. Households share all the risk from their traded endowment and some of the risk from their non-traded endowment by shipping traded goods across countries.

In this model, the pegging country ships tradable goods to and from the target country depending on the relative endowments of the non-traded goods. This allows the pegging country to control relative price levels between the two countries and to reduce the volatility of its bilateral exchange rate with the target country. Assets that make payments partially fixed in terms of the pegging or target country's consumption bundles become more similar, and hence we see that bond spreads between the pegging and target countries decrease. This, in turn, impacts the bond spreads and stock spreads between the target country and all other countries in the world.

We then analyze the full model in which markets are incomplete and the economy is affected by monetary shocks in addition to endowment shocks. Introducing money allows us to break the tight link between exchange rates and aggregate consumption growth. In the full model, a subset of households in each country are precluded from trading in financial markets and can only hold nominal bonds that make fixed payments in terms of the currency used in the country in which they reside. When inflation rises, these "inactive" households pay an inflation tax that goes to the benefit of "active" households who consume proportionately more whenever inflation is high. We characterize the impact of the peg on bond spreads and spreads in stocks in the non-traded sector. The results in this section are qualitatively similar to those from the simplified model.

Finally, we complete the model by introducing capital accumulation in the non-traded sector, characterize the differences in capital investment between countries and the impact of the exchange rate peg on these differences.

To our knowledge, this paper is the first to characterize the effects of exchange rate management within a multi-country model of exchange rate determination. It contributes to the existing literature in three ways.

First, our model characterizes the effects of currency pegs on the correlation structure of multilateral exchange rates, on the cross-section of currency returns, and on capital accumulation. It relates closely to an emerging literature that uses factor models to characterize these objects in the data (Lustig et al., 2011; Menkhoff et al., 2012; Verdelhan, 2013; Hassan and Mano, 2013) and to the existing theoretical literature on cross-sectional differences in interest rates (Hassan, 2013; Maggiori, 2013; Martin, 2012).

Second, by focusing on the effects of currency manipulation on the target country and on countries outside the peg, the model provides the basis for a welfare-based analysis of cooperative and non-cooperative policies of exchange rate management. In this sense, the paper relates to a large literature that analyzes the effects and the political economy of cooperative and non-cooperative trade policy (Johnson, 1954; Grossman and Helpman, 1997; Maggi and Rodriguez-Clare, 1998; Bagwell et al., 2002; Ossa, 2011) and to the existing literature that focuses on the costs and benefits of exchange rate pegs for the pegging country (Krugman, 1979; Flood and Garber, 1984; Casella, 1992).

Third, by endogenizing capital accumulation within a framework that allows us to solve for equilibrium exchange rates with a large number of countries, our paper paves the way for a quantitative estimation of the real effects of phenomena such as the Chinese peg to the dollar, the wide-spread dollarization of emerging economies, and the establishment the European Monetary System.

The model in this paper nests the canonical real model of exchange rate determination in Backus and Smith (1993) and a simplified version of the monetary model by Alvarez et al. (2002). To that end, this paper relies on the consumption of non-traded goods to generate movements in the real exchange rate. This approach has a long tradition in international macroeconomics, and includes papers such as Stulz (1987) and Tesar (1993).

2 Model Setup

In this section, we set up an international asset pricing model where stochastic properties of the real exchange rate arise endogenously as a function of real and monetary shocks. This set up closely follows the model from Hassan (2013). Households consume a freely traded good and a country specific non-traded good. The non-traded component in consumption allows the consumption price index to differ across countries and induces variation in the real exchange rate. A subset of households in each country has access to complete international asset markets, while the remainder of households can only hold nominal bonds that make fixed payments denominated in the currency of their country of residence. In equilibrium, the real exchange rate responds to real shocks to the endowments of non-traded goods across countries and to monetary shocks that affect the inflation rates of national currencies. One country, labelled the "pegging" country, imposes a soft peg on its exchange rate with a second country, labelled the "target" country. This soft peg is achieved by manipulating the consumption of traded goods in the pegging country. Let's call all other countries in the economy "outside" countries.

There are two discrete time periods in the model economy. Let time $t = 1, 2$. There exists a unit measure of households. These households are partitioned into N subsets (countries). Denote the measure of country n by θ^n. Households within a country are symmetric. Let $Y^n_{T,2}$ represent a country n household's endowment of the traded good in the second period. Let $Y^n_{N,2}$ be the endowment of the country specific non-traded good. Endowments are log-normally distributed

$$y^n_{T,2}, y^n_{N,2} \sim N\left(-\frac{1}{2}\sigma^2_n, \sigma^2_n\right) \tag{1}$$

we allow the variance of endowments, σ^2_n, to differ cross countries but we assume that the variance of the endowment of traded and non-traded good is the same within each country. Throughout this paper, lowercase variables stand for logs and uppercase variables stand for levels. For simplicity, the endowments in the first-period are not stochastic. In the first period, households receive one unit of the traded and one unit of the non-traded consumption good. There is no storage technology in this economy. International trade in the traded good is costless. Throughout the paper, we use the traded consumption good as the numeraire, such that all prices and returns are accounted for in the same units.

Households exhibit constant relative risk aversion according to

$$U(i) = \frac{1}{1-\gamma}C_1(i)^{1-\gamma} + \beta\frac{1}{1-\gamma}\mathbb{E}\left[C_2(i)^{1-\gamma}\right] \tag{2}$$

where β is a discount factor and $C_t(i)$ is the consumption index for household i at time t. we assume that households are risk-averse with $\gamma > 0$. The consumption index is defined as

$$C_t(i) = \left[\tau C_{T,t}(i)^\alpha + (1-\tau)C_{N,t}(i)^\alpha\right]^{\frac{1}{\alpha}} \tag{3}$$

where C_N is consumption of the country-specific non-traded good, C_T stands for consumption of the traded good, $\tau \in (0, 1)$ is the wright of the traded good in the consumption index, and ε_α is the elasticity of substitution between traded and non-traded goods.

In the first period, a fixed proportion ϕ of households within each country trade a complete set of state-contigent securities in international markets. Label these households as "active". The remaining $1-\phi$ fraction of households within each countries is excluded from trading state-contigent securities. Label these households as "inactive". Instead, inactive households cede the claims to their second period endowments to active households in return for a nominal bond, which makes a payment equal to the number of traded goods needed to buy one unit of the country-specific final consumption bundle fixed at first-period prices of the national currency. we write this payment as $P_2^n e^{-\mu^n}$ where P_2^n is country n's consumer price index in period 2 and μ^n is a monetary shock to the growth rate of the price of one unit of the traded good in terms of the currency of country n,

$$\mu^n \sim N\left(-\frac{1}{2}\tilde{\sigma}_n, \tilde{\sigma}_n\right) \tag{4}$$

In order to simplify notation, let ω the realization of second period endowments and monetary shocks and let $g(\omega)$ be the associated density. Before trading commences, active households receive a country-specific transfer, κ^n, that de-centralizes the Social Planners allocation with unit Pareto weights.

All households take prices as given. Active households maximize their lifetime utility (2) subject to their intertemporal budget constraint

$$P_1^n C_1(i) + \int_\omega Q(\omega)\left(P_2^n(\omega) C_2(\omega, i)\right) d\omega + \frac{1-\phi}{\phi} P_2^n e^{-\mu^n} \tag{5}$$

$$\leq Y_{T,1}^n + P_{N,1}^n Y_{N,1}^n + \int_\omega Q(\omega) \frac{1}{\phi}\left(Y_{T,2}^n(\omega) + P_{N,2}^n(\omega) Y_{N,2}^n(\omega)\right) d\omega + \kappa^n \tag{6}$$

where $Q(\omega)$ is the first period price of a state contingent security that pays one unit of the traded good if state ω occurs in the second period, P_N^n denotes the price of the non-traded good in country n, and $\frac{1-\phi}{\phi}$ is the number of inactive households per active household in each country. Inactive households also maximize (2), but subject to the constraints $P_1^n C_1(i) \leq Y_{T,1}^n + P_{N,1}^n Y_{N,2}^n$ and $P_2^n C_2(i) \leq P_2^n e^{-\mu^n}$.

In our model, a country manipulates its exchange rate by changing the consumption of tradable goods of the households in the country. This is in contrast to other models in which the exchange rate manipulation is modelled through monetary policy, such as Gali and Monacelli (2005). We choose this form of exchange rate manipulation because it allows us to produce tractable analytic results.

Assume there are at least two countries in the model economy. Let the first country be the pegging country, denoted by p, and let the second country be the target country, denoted by t. The objective of the pegging country is to decrease the volatility of its exchange rate with the target country. Denote the variance of the log of the real exchange rate between the pegging and target country without an exchange rate peg is $var(p^t - p^p)^*$, and denote the variance of the log exchange rate between the pegging and target country in the economy with an exchange rate peg by $var(p^t - p^p)$. Then, the pegging country would like to constrain the consumption of traded goods by households in the pegging country such that

$$\frac{var(p^t - p^p)}{var(p^t - p^p)^*} = (1 - \zeta)^2 \tag{7}$$

for some $\zeta \in (0, 1)$. In Appendix A, we show how one can derive the desired constraint for a hard currency peg. In this paper, we generalize the hard peg constraint ($\zeta = 1$) to the soft peg constraint described by (7). This constraint is

$$1 = \frac{C_2^t}{C_2^p} \left(\frac{C_{N,2}^t}{C_{N,2}^p} \right)^{-\frac{(1-\zeta)(1-\alpha)(1-\tau)}{(1-\tau)(1-\alpha)+\gamma\tau}} \tag{8}$$

where $\hat{\zeta} = (1 - \zeta)^2$.

The economy is at an equilibrium when all households maximize utility subject to their constraints and goods markets clear.

3 Solution Method

The active household's problem is characterized by the Euler equation

$$Q(\omega) = \beta \frac{\Lambda_{T,2}}{\Lambda_{T,1}} g(\omega) \tag{9}$$

where $\Lambda_{T,t} = C_t(i)^{1-\gamma-\alpha} C_{T,t}(i)^{\alpha-1}$ is the active households' marginal utility of traded good consumption at time t. In equilibrium, this marginal utility is equalized across all active households in all countries. Thus, all active households price assets using the ratio of marginal utilities from tradable consumption as the unique stochastic discount factor. This is due to the fact that we chose the traded good as the numeraire and any other choice of units would result in a different stochastic discount factor for residents of each country. Based on this result, we can make a general statement about the spread on any two international assets. This result was derived in **?**:

Proposition 3.1. *The difference in log expected returns between two arbitrary assets with log-normally distributed payouts X and Z equals the covariance of the difference of their log payouts with the log of the marginal utility of tradable consumption.*

$$\log ER[X] - \log ER[Z] = cov(\lambda_{T,2}, z - x) \tag{10}$$

where $\lambda_{T,2}$ is normally distributed and $R[\cdot]$ refers to the gross return of an asset paying $\cdot$.

Proof. See Appendix B.1. □

Households prefer assets that pay off well when the marginal utility from tradable consumption is high. Whichever asset has the higher covariance with the marginal utility of tradable consumption must therefore pay a lower expected return in equilibrium. Throughout the paper, we work with differences in log expected returns to economize on notation. All results for differences in expected returns are qualitatively the same.

Solving for the spread on any pair of international assets thus requres us to solve for the active household's equilibrium consumption of traded goods and for the log difference of the assets' equilibrium payoffs. As asset markets are complete for the subset of active households, we can calculate the equilibrium consumption and the equilibrium payoffs of all assets by solving the Social Planner's problem, subject to the constraints for the inactive households' optimal consumption and the currency peg constraint.

Since inactive households cannot transfer resources across periods, their optimal program reduces to weighing their consumption between traded and non-traded goods in each period. They maximize their utility by consuming the bundle $\hat{C}^n_{T,1} = (\tau P^n_2)^{\frac{1}{1-\alpha}}, \hat{C}^n_{N,1} = \left((1-\tau)P^n_1/P^n_{N,1}\right)^{\frac{1}{1-\alpha}}$ in the first period and

$$\hat{C}^n_{T,2} = \exp(-\mu^n)(\tau P^n_2)^{\frac{1}{1-\alpha}},\ \hat{C}^n_{N,2} = \exp(-\mu^n)\left(\frac{(1-\tau)P^n_2}{P^n_{N,2}}\right)^{\frac{1}{1-\alpha}} \tag{11}$$

in the second period, where $\hat{C}_T^n$ and $\hat{C}_N^n$ refer to the conusmption of traded and non-traded goods by inactive agents in country n, respectively.

Endowments cannot be carried over from the first period to the second. Therefore, we can solve the Social Planner's problem for period two only and we omit the time subscript going forward. Since all active households within a given vountry are identical and receive the same endowments, they must also consume the same bundle $(C_T^n(\omega), C_N^n(\omega))$ in equilibrium. The Social Planner's problem can therefore be written as

$$\max \phi \sum_{n=1}^{3} \theta^n \frac{1}{1-\gamma} \left[\tau (C_T^n)^\alpha + (1-\tau)(C_N^n)^\alpha\right]^{\frac{1-\gamma}{\alpha}} \tag{12}$$

subject the following resource constraints

$$\theta^n \left(\phi C_N^n + (1-\phi)\hat{C}_N^n\right) = \theta^n Y_N^n,\ \forall n \tag{13}$$

$$\phi \left(\sum_{n=1}^{N} \theta^n C_T^n\right) + (1-\phi)\left(\sum_{n=1}^{N} \theta^n \hat{C}_T^n\right) = \sum_{n=1}^{N} \theta^n Y_T^n \tag{14}$$

as well the pegging constraint given by (8) and the behavior of the inactive households given by (11). In order to provide closed-form solutions, we log-linearize the first order conditions and resource constraints around the deterministic solution, i.e. the point at which $\sigma_n, \tilde{\sigma}_n = 0\ \forall n$. The log-linearized first order conditions and resource constraints yirld a system of $3n+1$ equations, which can then be solved for the unknowns $\left\{\{c_T^n, c_N^n, p_N^n\}_n, \lambda_T\right\}$

4 Complete Markets

we first analyze the case in where all households are active, $\phi = 1$. In this case, monetary shocks have no effect on the equilibrium allocation, and the real exchange rate is determined exclusively by real shocks.

Equation (15) shows the equilibrium consumption of the traded good in an arbitrary outside country m.

$$c_{T,2}^m = \bar{y}_{T,2} + \frac{(\alpha+\gamma-1)(1-\tau)}{(1-\alpha)(1-\tau)+\tau\gamma}(\bar{y}_{N,2} - y_{N,2}^m) \tag{15}$$

Where $\bar{y}_N = \sum_{n=1}^{N} \theta^n y_N^n$ is the average log endowment of non-tradable goods across countries, and $\bar{y}_T$ is the average log endowment of tradable goods. We can see that the consumption of traded goods by an outside country is not affected by the introduction of the currency peg, because ζ does not enter into the expression. Consumption of the traded good moves one for one with the world supply. Since the traded good can be freely shipped around the globe, households perfectly share risk when it comes to endowments of the traded good. The second term of (15) shows that households also use the traded good to insure against risk in their non-tradable endowments. Households compensate for low endowments of the non-tradable good through purchases of the tradable good if the following condition holds:

Condition 1. *Households are sufficiently risk-averse such that* $\gamma\epsilon_\alpha > 1$

The consumption of the tradable good in the target country is

$$c_{T,2}^t = \bar{y}_{T,2} + \frac{(\gamma+\alpha-1)(1-\tau)}{(1-\alpha)(1-\tau)+\tau\gamma}(\bar{y}_{N,2} - y_{N,2}^t) + \zeta \frac{(1-\alpha)(1-\tau)}{\tau((1-\alpha)(1-\tau)+\gamma\tau)} \frac{\theta^p}{\theta^p+\theta^t}(y_{N,2}^p - y_{N,2}^t)$$

and the consumption of the traded good in the pegging country is

$$c_{T,2}^p = \bar{y}_{T,2} + \frac{(\gamma+\alpha-1)(1-\tau)}{(1-\alpha)(1-\tau)+\tau\gamma}(\bar{y}_{N,2} - y_{N,2}^p) + \zeta \frac{(1-\alpha)(1-\tau)}{\tau((1-\alpha)(1-\tau)+\gamma\tau)} \frac{\theta^t}{\theta^p+\theta^t}(y_{N,2}^t - y_{N,2}^p)$$

The expressions for the consumption of the traded good in the pegging and target countries contain a third term. This term reflects the pegging country's currency manipulation by controlling the consumption of the traded good. If the pegging country's endowment of the non-traded good is low compared to the target country's endowment, then the pegging country must increase consumption of the traded good to bring it's price level back down in line with the target country's price level.

The peg not only affects the level of consumption in the pegging and target countries, but it also the volatility of consumption. This impact can be described by the following proposition:

Proposition 4.1. *The introduction of the exchange rate peg raises the volatility of consumption in* *both the target and pegging countries.*

$$var(c^p_{T,2}) > var(\hat{c}^p_{T,2}) and var(c^t_{T,2}) > var(\hat{c}^t_{T,2})$$

The exchange rate peg has no effect on the volatility of consumption of the outside countries.

$$var(c^o_{T,2}) = var(\hat{c}^o_{T,2})$$

where $\hat{c}^i_{T,2}$ denotes the analogous consumption in an economy without currency pegs.

Proof. The difference between the volatility of log consumption in the target country when there is an exchange rate peg and the volatility of log consumption without the exchange rate peg reduces to

$$\frac{\theta^t\zeta(1-\alpha)(1-\tau)^2(\zeta(1-\alpha)\theta^t(\sigma_t^2+\sigma_p^2)+2(\theta^t+\theta^p)((1-\theta^p)\sigma_p^2+\theta^t\sigma_t^2)(\gamma+\alpha-1)\tau)}{(\theta^p+\theta^t)^2\tau^2\left((1-\alpha)(1-\tau)+\gamma\tau\right)^2}$$

which is always positive when Condition 1 holds. The difference between the volatility of log consumption in the pegging country when there is an exchange rate peg and the volatility of log consumption without the exchange rate peg reduces to

$$\frac{\theta^p\zeta(1-\alpha)(1-\tau)^2(\zeta(1-\alpha)\theta^p(\sigma_t^2+\sigma_p^2)+2(\theta^t+\theta^p)((1-\theta^p)\sigma_p^2+\theta^t\sigma_t^2)(\gamma+\alpha-1)\tau)}{(\theta^p+\theta^t)^2\tau^2\left((1-\alpha)(1-\tau)+\gamma\tau\right)^2}$$

which is also always positive when Condition 1 holds. □

Proposition 4.1 states that when a pegging country imposes an exchange rate peg with a second country, the consumption of tradable goods and hence the aggregate consumption within both countries becomes more volatile.

In equilibrium, the peg acts to transfer tradable consumption between the target and pegging countries without any involvement from outside countries. This result comes from the fact that the peg constraint enters the objective functions of the households in the pegging and target countries, only. Moreover, the marginal utility of consumption of households in the outside country is unaffected by the exchange rate peg, so their optimizing behavior is also unchanged. This fact is reflected in the equilibrium marginal utility of tradable consumption,

$$\lambda_T = -((1-\tau)(1-\alpha) + \tau\gamma)\sum_{n=1}^{N}\theta^n y_T^n - (1-\tau)(\alpha+\gamma-1)\sum_{n=1}^{N}\theta^n y_N^n \tag{16}$$

First, this equation shows us that marginal utility of tradable consumption behaves exactly the same in the economy with an exchange rate peg as it did in the economy without the exchange rate peg from Hassan (2013). The marginal utility from tradable consumption falls with the world supply of tradable goods as well as non-tradable goods. Thus, λ_T tends to be low in "good" states of the world.

4.1 Exchange Rates and Spreads on International Bonds

Definition 1. *A country n risk-free bond is an asset which is risk-free in terms of the utility of the residents of country n. It pays P^n traded goods in the second period.*

From the perspective of the households of each country, the risk free asset pays the exact number of units of the traded good required to buy one unit of the country specific final consumption bundle, P^n. The value of this asset in terms of traded goods depends on the state of the world which is realized ex-post. We can obtain the equilibrium price of each country's final consumption bundle using the Lagrange multipliers associated with the Social Planner's problem.

$$p^n = -\gamma c^n - \lambda_T \tag{17}$$

The ratio of the price of the final consumption bundles of two countries is their real exchange rate. The log real exchange rate between two countries f and h is

$$s^{f,h} = p^f - p^h \tag{18}$$

we can characterize all exchange rates in this world by only calculating the exchange rates between three countries. These countries are the pegging country (p), the target country (t) and an arbitrary outside country (o). The log real exchange rate for each pair of countries in the economy is

$$s^{p,t} = p^p - p^t = \frac{(1-\alpha)(1-\tau)(1-\zeta)\gamma}{(1-\alpha)(1-\tau)+\gamma\tau}\left(y^t_{N,2} - y^p_{N,2}\right) \tag{19}$$

$$s^{p,o} = \frac{(1-\alpha)(1-\tau)\gamma}{(1-\alpha)(1-\tau)+\gamma\tau}\left(y^o_{N,2} - y^p_{N,2}\right) + \zeta\frac{(1-\alpha)(1-\tau)\gamma}{(\gamma\tau+(1-\alpha)(1-\tau))}\frac{\theta^t}{\theta^t+\theta^p}\left(y^p_{N,2} - y^t_{N,2}\right) \tag{20}$$

$$s^{t,o} = \frac{(1-\alpha)(1-\tau)\gamma}{(1-\alpha)(1-\tau)+\gamma\tau}\left(y^o_{N,2} - y^t_{N,2}\right) + \zeta\frac{(1-\alpha)(1-\tau)\gamma}{(\gamma\tau+(1-\alpha)(1-\tau))}\frac{\theta^p}{\theta^p+\theta^t}\left(y^t_{N,2} - y^p_{N,2}\right) \tag{21}$$

As desired, when $\zeta = 1$, the endowments of non-tradable goods have no impact on the deviation of the exchange rate between the pegging and target countries from steady state. Because the exchange rate peg distorts the aggregate price level in the pegging and target countries, we expect the exchange rate peg to affect the exchange rate between an outside country and any country in the exchange rate peg. This is indeed the case. The first term in expressions (20) and (21) is the exchange rate between any pair of outside countries. The second term in each of these expressions reflects the distortion caused by the exchange rate peg and is proportional to the exchange rate between the pegging and target countries.

The peg also has a well defined impact on the volatility of the exchange rate for the target and pegging countries. The variance of the log real exchange rate between any two outside countries h and f is

$$var(p_f - p_h) = \left[\frac{\gamma(1-\alpha)(1-\tau)}{(1-\alpha)(1-\tau)+\gamma\tau}\right]^2\left(\sigma_f^2 + \sigma_h^2\right) \tag{22}$$

Equation (23) shows that the currency peg successfully reduces the volatility of the exchange rate between the pegging and target countries.

$$var(p_t - p_p) = (1-\zeta)^2\left[\frac{\gamma(1-\alpha)(1-\tau)}{(1-\alpha)(1-\tau)+\gamma\tau}\right]^2\left(\sigma_t^2 + \sigma_p^2\right) \tag{23}$$

We can see that the pegging country faces a trade off between lowering the volatility of its exchange rate with the target country and increasing its volatility of consumption. The proof of proposition 4.1 shows that increase in the volatility of tradable good consumption is increasing in ζ, and the variance of the log real exchange rate between the pegging and target countries is decreasing in ζ.

The exchange rate peg will also affect the volatility of the exchange rate between either country in the exchange rate peg and another outside country. The variance of the exchange rate between an arbitrary outside country o and the countries in the exchange rate peg is

$$var(p_o - p_p) = \left[\frac{\gamma(1-\alpha)(1-\tau)}{(1-\alpha)(1-\tau)+\gamma\tau}\right]^2 (\sigma_o^2 + \sigma_p^2) + \frac{\theta^t(1-\alpha)^2(1-\tau)^2\gamma^2(\theta^t(\sigma_p^2+\sigma_t^2)\zeta^2 - 2(\theta^p+\theta^t)\sigma_p^2\zeta)}{(\theta^p+\theta^t)^2((1-\alpha)(1-\tau)+\gamma\tau)^2} \tag{24}$$

$$var(p_o - p_t) = \left[\frac{\gamma(1-\alpha)(1-\tau)}{(1-\alpha)(1-\tau)+\gamma\tau}\right]^2 (\sigma_o^2 + \sigma_t^2) + \frac{\theta^p(1-\alpha)^2(1-\tau)^2\gamma^2(\theta^p(\sigma_p^2+\sigma_t^2)\zeta^2 - 2(\theta^p+\theta^t)\sigma_t^2\zeta)}{(\theta^p+\theta^t)^2((1-\alpha)(1-\tau)+\gamma\tau)^2} \tag{25}$$

Each of these expressions for the variance of the real log exchange rate is made up of two parts. The first part is the normal variance of the real exchange rate between two countries. The second part is an adjustment that reflects the fact that one country's price level is being manipulated by the pegging country. To describe the effect of the exchange rate peg, we first define the following condition

Condition 2. *Suppose the size the the pegging and target countries and the volatility of endowments in the pegging and target countries satisfy*

$$\frac{\theta^t}{\theta^p+\theta^t}\frac{\sigma_p^2+\sigma_t^2}{\sigma_p^2}\zeta < 2 \text{ and } \frac{\theta^p}{\theta^p+\theta^t}\frac{\sigma_p^2+\sigma_t^2}{\sigma_t^2}\zeta < 2$$

Since $\zeta \in [0,1]$, if and if endowments in both the pegging and target countries are drawn from distributions with equal variance, then Condition 2 will be met. Now, we can derive the following results about the exchange rate peg.

Proposition 4.2. *If Condition 2 holds then the exchange rate peg decreases the volatility of the exchange rate between the pegging country and an outside country as well as the volatility of the exchange rate between the target country and an outside country.*

Proof. In each of the expressions for the variance of the log real exchange rate, the sign of the second part is determined by the sign of $\theta^t(\sigma_p^2+\sigma_t^2)\zeta - 2(\theta^p+\theta^t)\sigma_p^2$ or $\theta^p(\sigma_p^2+\sigma_t^2)\zeta - 2(\theta^p+\theta^t)\sigma_t^2$. □

In general, an exchange rate would likely decrease the volatility of the exchange rate between either of the countries in the peg and any outside country. or the rest of this section suppose the following condition holds

Condition 3. *The variance adjusted measure of difference in country size* $\theta^t\sigma_t^2 - \theta^p\sigma_p^2$ *is monotonic in the actual difference in country size* $\theta^t - \theta^p$*, i.e.* $\theta^t\sigma_t^2 > \theta^p\sigma_p^2$ *iff* $\theta^t > \theta^p$*.*

This restriction on the variances of endowments requires that σ^2 decreases less than linearly with country size. Such a linear relationship would arise in a model in which there are no country-specific shocks and endowments to each individual are i.i.d. As long as there is some country specific element to shocks faced by households, Condition 3 will generally hold.

The difference in log expected returns of two outside countries' risk free bonds is given by

$$\log ER[P^f] - \log ER[P^h] = cov(\lambda_T, p^h - p^f) = \frac{(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2(\theta^h\sigma_h^2 - \theta^f\sigma_f^2) \quad (26)$$

Where r^n is the country n risk free interest rate and $\Delta\mathbb{E}s^{f,h} = \log\left(\mathbb{E}p_2^f/\mathbb{E}p_2^h\right) - s_1^{f,h}$ is the expected change in the real exchange rate between countries f and h. Given conditions 1 and 2, the larger country's risk-free bond pays lower expected returns. This is a result taken directly from Proposition 1 of Hassan (2013).

The spread on international bonds between the pegging and target country is

$$cov(\lambda_T, p^p - p^t) = \frac{(1-\zeta)(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2(\theta^p\sigma_p^2 - \theta^t\sigma_t^2) \quad (27)$$

which can easily be characterized by the following proposition.

Proposition 4.3. *An exchange rate peg decreases the magnitude of the spread in international bonds between the pegging and target countries defined by*

$$\left\|\frac{(1-\zeta)(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2(\theta^p\sigma_p^2 - \theta^t\sigma_t^2)\right\|$$

This result should be obvious from examining the equation for the spread in international bonds between the target and pegging countries. Since the goal of the peg is to make the price level in the two countries closer to each other, it makes sense that the expected returns on international bonds becomes closer as well. Since the larger country earns lower expected returns on bonds, this means that the peg increases the expected log return on bonds in the larger of the two countries and decreases the expected log return on bonds in the smaller of the two countries. We should be able to see this effect in bond spreads between countries in the peg and outside countries. These bond spreads are given by:

$$
\begin{aligned}
cov(\lambda_T, p^t - p^o) =& \frac{(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2\left(\theta^t\sigma_t^2 - \theta^o\sigma_o^2\right) \qquad (28)\\
&+ \zeta\frac{\theta^p}{\theta^p+\theta^t}\frac{(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2\left(\theta^p\sigma_p^2 - \theta^t\sigma_t^2\right)\\
cov(\lambda_T, p^p - p^o) =& \frac{(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2\left(\theta^p\sigma_p^2 - \theta^o\sigma_o^2\right) \qquad (29)\\
&+ \zeta\frac{\theta^t}{\theta^p+\theta^t}\frac{(1-\alpha)(\alpha+\gamma-1)}{(1-\alpha)(1-\tau)+\tau\gamma}\gamma(1-\tau)^2\left(\theta^t\sigma_t^2 - \theta^p\sigma_p^2\right)
\end{aligned}
$$

The bond spread between any country in the exchange rate peg and an outside country is composed of two parts. The first part is simply the bond spread in an economy with no exchange rate peg. The second part is an adjustment proportional the bond spread between the pegging and target country. These expressions can be summarize qualitatively with the following proposition:

Corollary 4.4. *Suppose the pegging country is larger than the target country. Then, the exchange rate peg increases the log expected return of the bond in the pegging country relative to the log expected return of a bond in an outside country. At the same time, the peg decreases the log expected return of a target country bond relative to the log expected return of an outside country bond.*

Proof. Manipulating the definition of the spread in international bonds, we get

$$\log ER[P^t] = \log ER[P^o] - cov(\lambda_T, p^t - p^o)$$

$$\log ER[P^p] = \log ER[P^o] - cov(\lambda_T, p^p - p^o)$$

Then if the pegging country is the larger of the two countries, then $cov(\lambda_T, p^t - p^o)$ is increasing in ζ and $cov(\lambda_T, p^p - p^o)$ is decreasing in ζ. Hence, $\log ER[P^t]$ is decreasing and $\log ER[P^p]$ is increasing. Additionally, we know from the previous section that the exchange rate peg does not impact the marginal utility of consumption of the outside country or the marginal utility of consumption of the traded good. Therefore, the effects of the exchange rate peg on bond spreads must enter through the impact of the exchange rate peg on the expected return on bonds in the pegging and target countries. □

4.2 Spread on International Stocks

For the same reasons that the exchange rate peg affects returns on international bonds, we expect the exchange rate peg to affect stocks in the pegging and target countries. Hassan (2013) was able to directly link differences in country size to differences in stock returns between countries. The main result from this section shows that under the presence of exchange rate manipulation, this relationship may no longer hold.

Definition 2. *A country n stock in the non-traded (traded) sector is a claim to one household's second period endowment of the non-traded (traded) good. Stock in the non-traded sector thus pays $P_N^n Y_N^n$ units of the traded good and stock in the traded sector pays Y_T^n units.*

We derive the the spread between stocks in the traded and non-traded sectors between each pair of countries. The log spread between stocks in the non-traded sector is given by

$$\log ER\left[P_N^f Y_N^f\right] - \log ER\left[P_N^h Y_N^h\right] = cov\left(\lambda_T, (p_N^h + y_N^h) - (p_N^f + y_N^f)\right) \tag{30}$$

Recall $\varepsilon_\alpha = (1-\alpha)^{-1}$ is the elasticity of substitution between traded and non-traded goods. Given two arbitrary outside countries h and f, the difference in log expected returns of two countries' stocks in the non-traded sector is given by

$$\log ER\left[P_N^f Y_N^f\right] - \log ER\left[P_N^h Y_N^h\right] = \frac{(1-\tau)(\gamma\epsilon_\alpha - 1)(\gamma + \tau - \tau\gamma\epsilon_\alpha - 1)}{\epsilon_\alpha(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^h\sigma_h^2 - \theta^f\sigma_f^2\right) \tag{31}$$

The spread on international stocks in the non-traded sector between the pegging or target country and an arbitrary outside country o in the economy with the exchange rate peg is given by:

$$\begin{aligned}\log ER\left[P_N^o Y_N^o\right] - \log ER\left[P_N^t Y_N^t\right] =& \frac{(1-\tau)(\gamma\epsilon_\alpha - 1)(\gamma+\tau-\tau\gamma\epsilon_\alpha - 1)}{\epsilon_\alpha(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^t\sigma_t^2 - \theta^o\sigma_o^2\right) \\ &+ \frac{\zeta^2\theta^p(\gamma\epsilon_\alpha-1)(1-\tau)^2}{(\theta^p+\theta^t)\epsilon_\alpha^2\tau(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^t\sigma_t^2 - \theta^p\sigma_p^2\right) \qquad (32) \\ \log ER\left[P_N^o Y_N^o\right] - \log ER\left[P_N^p Y_N^p\right] =& \frac{(1-\tau)(\gamma\epsilon_\alpha - 1)(\gamma+\tau-\tau\gamma\epsilon_\alpha - 1)}{\epsilon_\alpha(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^p\sigma_p^2 - \theta^o\sigma_o^2\right) \\ &+ \frac{\zeta^2\theta^t(\gamma\epsilon_\alpha-1)(1-\tau)^2}{(\theta^p+\theta^t)\epsilon_\alpha^2\tau(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^p\sigma_p^2 - \theta^t\sigma_t^2\right) \qquad (33)\end{aligned}$$

The spread on stocks in the non-tradable sector involving countries affected by the exchange rate peg can also be described in two parts. The first part gives us the spread on stocks in the non-tradable sector without the presence of the peg. The second part is the adjustment due to the exchange rate peg that is proportional to the spread on stocks in the non-tradable sectors of the pegging and target countries, under the scenario with no peg. The spread on stocks in the non-tradable sectors of the pegging and target countries takes on a similar form.

$$\begin{aligned}\log ER\left[P_N^t Y_N^t\right] - \log ER\left[P_N^p Y_N^p\right] =& \frac{(1-\tau)(\gamma\epsilon_\alpha - 1)(\gamma+\tau-\tau\gamma\epsilon_\alpha - 1)}{\epsilon_\alpha(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^p\sigma_p^2 - \theta^t\sigma_t^2\right) \\ &+ \frac{\zeta^2(\gamma\epsilon_\alpha-1)(1-\tau)^2}{\epsilon_\alpha^2\tau(1-\tau+\tau\gamma\epsilon_\alpha)}(\theta^p\sigma_p^2 - \theta^t\sigma_t^2) \qquad (34)\end{aligned}$$

Condition 4. *The elasticity of* P_N^f/P_N^h *with respect to* Y_N^h *is greater than one,* $\gamma+\tau-\tau\gamma\varepsilon_\alpha > 1$

Under condition 3, Hassan (2013) shows that stock in the larger country's non-traded sector pays lower log expected returns. This result does not necessarily hold if one of the countries is being affected by the exchange rate peg. Consider equation (33). Suppose the outside country is larger than the pegging country. Without the exchange rate peg, we expect the spread to be negative. With the exchange rate peg, if the pegging country is much larger than the pegging country, then the spread on stocks in the non-traded sector can turn positive.

The log spread between stocks in the traded sector is given by

$$\log ER\left[Y_T^f\right] - \log ER\left[Y_T^h\right] = cov\left(\lambda_T, y_T^h - y_T^f\right) \qquad (35)$$

Since the marginal utility of tradable good consumption is not affected by the exchange rate peg, we expect the log spread between stocks in the traded sector to remain unchanged. That is

$$\log ER\left[Y_T^f\right] - \log ER\left[Y_T^h\right] = -\left((1-\tau)(1-\alpha)+\tau\gamma\right)\left(\theta^h\sigma_h^2 - \theta^f\sigma_f^2\right) \tag{36}$$

The impact of the exchange rate peg on the spread on stocks can be summarized by the following two propositions:

Proposition 4.5. *Given conditions 1 and 2, imposing an exchange rate peg decreases the expected return of stock in the non-tradable sector of the larger of the two countries in the peg relative to an outside country.*

Proof. This is evident by examining equations (32) and (33). Re-write these equations as

$$\begin{aligned}
\log ER\left[P_N^t Y_N^t\right] =& \log ER\left[P_N^o Y_N^o\right] - \frac{(1-\tau)(\gamma\epsilon_\alpha - 1)(\gamma+\tau-\tau\gamma\epsilon_\alpha - 1)}{\epsilon_\alpha(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^t\sigma_t^2 - \theta^o\sigma_o^2\right) \\
&- \frac{\zeta^2\theta^p(\gamma\epsilon_\alpha-1)(1-\tau)^2}{(\theta^p+\theta^t)\epsilon_\alpha^2\tau(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^t\sigma_t^2 - \theta^p\sigma_p^2\right) \\
\log ER\left[P_N^p Y_N^p\right] =& \log ER\left[P_N^o Y_N^o\right] - \frac{(1-\tau)(\gamma\epsilon_\alpha - 1)(\gamma+\tau-\tau\gamma\epsilon_\alpha - 1)}{\epsilon_\alpha(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^p\sigma_p^2 - \theta^o\sigma_o^2\right) \\
&- \frac{\zeta^2\theta^t(\gamma\epsilon_\alpha-1)(1-\tau)^2}{(\theta^p+\theta^t)\epsilon_\alpha^2\tau(1-\tau+\tau\gamma\epsilon_\alpha)}\left(\theta^p\sigma_p^2 - \theta^t\sigma_t^2\right)
\end{aligned}$$

Since the exchange rate peg does not affect the consumption of households in the outside country, it is clear that the (log) expected return of stock in the non-tradable sector of the larger country in the peg is decreasing in ζ □

Proposition 4.6. *Given conditions 1, 2 and 3 the spread in international stocks between the two countries in the currency peg increases in magnitude.*

Proof. This is evident by examining equation (34) and noticing that conditions 1, 2, and 3 imply the coefficients on $(\theta^p\sigma_p^2 + \theta^t\sigma_t^2)$ are both positive. □

These results are qualitatively similar to the effect of the peg on international bond spreads.

5 Incomplete Markets

In the previous section we established the impact of the exchange rate peg on the equilibrium allocation and volatility of resources in the world. In this section, we re-introduce market segmentation and consider the impact of the exchange rate peg on the impact of monetary shocks in the economy. To build intuition for these impacts we start off by shutting down the real shocks, $\sigma_n = 0$. This allows me to write down analytic results.

Inactive households hold nominal bonds denominated in their national currencies and are thus vulnerable to monetary shocks. A positive monetary shock acts as an "inflation tax" on inactive households. The higher the inflation, the less their nominal bonds are worth and the less they are able to consume. However, since monetary shocks have no bearing on the real endowments available for consumption, this reduction of consumption on the part of the inactive households goes to the benefit of active households in equilibrium.

This shift in consumption from inactive to active households has implications for asset prices because only active households trade in asset markets and it is their marginal utility that determines asset prices. Absent real shocks, active households' marginal utility from traded goods becomes

$$\lambda_T = -\left(\frac{1-\phi}{\phi}\right)\sum_{n=1}^{N}\theta^n\mu^n \tag{37}$$

As before, the exchange rate peg has not affected the marginal utility of tradable consumption. Inflationary shocks lower λ_T, essentially by increasing the endowment of tradable goods for active households.

Equilibrium consumption of tradable goods in an outside country is

$$c_T^o = \frac{1-\phi}{\phi}\mu^o + \frac{\gamma(\tau(1-\phi)+\phi)(1-\phi)}{(\gamma\tau+(1-\alpha)(1-\tau)\phi)\phi}\left(\bar{\mu}-\mu^o\right) \tag{38}$$

Where $\bar{\mu}$ is weighted sum of monetary shocks in all countries. The first term on the right hand side reflects the immediate rise in active households' consumption,which is proportional to the number of inactive households per active household. The second term reflects the risk-sharing among active households of different countries. Some of the initial rise in consumption is shared internationally.

The consumption of the traded good in the target country

$$
\begin{aligned}
c^t_{T,2} =& \frac{1-\phi}{\phi}\mu^t + \frac{\gamma(\tau(1-\phi)+\phi)(1-\phi)}{(\gamma\tau+(1-\alpha)(1-\tau)\phi)\phi}\left(\bar{\mu}^n - \mu^t\right) \\
&+ \zeta\frac{\kappa_T}{\kappa_N}\left[\frac{(1-\tau)(1-\alpha)(1-\phi)}{\tau(\gamma\tau+(1-\alpha)(1-\tau)\phi)}\right]\frac{\theta^p}{\theta^p+\theta^t}\left(\mu^p - \mu^t\right)
\end{aligned}
$$

and the consumption of the traded good in the pegging country

$$
\begin{aligned}
c^p_{T,2} =& \frac{1-\phi}{\phi}\mu^p + \frac{\gamma(\tau(1-\phi)+\phi)(1-\phi)}{(\gamma\tau+(1-\alpha)(1-\tau)\phi)\phi}\left(\bar{\mu}^n - \mu^p\right) \\
&+ \zeta\frac{\kappa_T}{\kappa_N}\left[\frac{(1-\tau)(1-\alpha)(1-\phi)}{\tau(\gamma\tau+(1-\alpha)(1-\tau)\phi)}\right]\frac{\theta^t}{\theta^p+\theta^t}\left(\mu^t - \mu^p\right)
\end{aligned}
$$

Where

$$
\kappa_T = \gamma\tau(1-\zeta(1-\tau)(1-\phi)) + (1-\alpha)(1-\tau)(\zeta\tau(1-\phi)+\phi) \tag{39}
$$

$$
\kappa_N = \gamma\tau + \zeta^2(1-\alpha)(1-\tau)(1-\phi) + (1-\alpha)(1-\tau)\phi \tag{40}
$$

Each of these coefficients are unambiguously positive. The expressions for tradable consumption in the pegging and target countries contain an additional term that adjusts consumption according to the currency peg. If inflation in the pegging country is high relative to the target country, then without the additional adjustment term, the price level in the pegging country would be too low relative to the target country's. The third term tells us the pegging country ships traded goods to the target country, which increases the price level in the pegging country and decreases the price level in the target country.

This behavior is reflected in the exchange rates between the pegging and target country. For a pair of outside countries, denoted h and f, the log real exchange rates is

$$
s^{f,h} = \frac{(1-\alpha)(1-\tau)(1-\phi)\gamma}{\gamma\tau+(1-\alpha)(1-\tau)\phi}\left(\mu^h - \mu^f\right) \tag{41}
$$

Absent real shocks, the log real exchange rate between the pegging and target country is

$$
s^{p,t} = \frac{(1-\alpha)(1-\zeta)(1-\tau)(1-\phi)\gamma}{\gamma\tau+(1-\alpha)(1-\tau)(1-\phi)\zeta^2+(1-\alpha)(1-\tau)\phi}\left(\mu^t - \mu^p\right) \tag{42}
$$

As in the case with complete markets, the exchange rate peg successfully reduces the difference between the price level of the two countries and brings the exchange rate between the pegging and target countries closer to unity. This can be seen from equation (42), because the numerator of the expression decreased and the denominator of the expression increased when compared to equation (41).

Additionally, the peg affects the exchange rate between countries in the peg and outside countries. Their log real exchange rates are given by

$$s^{p,o} = \frac{(1-\alpha)(1-\tau)(1-\phi)\gamma}{\gamma\tau + (1-\alpha)(1-\tau)\phi}\left(\mu^o - \mu^p\right) + \zeta\frac{(1-\alpha)(1-\tau)(1-\phi)\gamma\kappa_P}{(\gamma\tau + (1-\alpha)(1-\tau)\phi)\kappa_N}\frac{\theta^t}{\theta^p + \theta^t}\left(\mu^p - \mu^t\right) \quad (43)$$

$$s^{t,o} = \frac{(1-\alpha)(1-\tau)(1-\phi)\gamma}{\gamma\tau + (1-\alpha)(1-\tau)\phi}\left(\mu^o - \mu^t\right) + \zeta\frac{(1-\alpha)(1-\tau)(1-\phi)\gamma\kappa_P}{(\gamma\tau + (1-\alpha)(1-\tau)\phi)\kappa_N}\frac{\theta^p}{\theta^p + \theta^t}\left(\mu^t - \mu^p\right) \quad (44)$$

where

$$\kappa_P = \gamma\tau + (1-\alpha)(1-\tau)(1-\phi)\zeta + (1-\alpha)(1-\tau)\phi \quad (45)$$

and is unambiguously positive. The additional term in these expressions reflects the risk sharing between the pegging and target countries as a result of the exchange rate peg. For example, equation (44) tells us the currency in the target country depreciates relative to outside countries if the pegging country is hit with a large inflation shock. This occurs when the pegging country transfers tradable goods to the target country to maintain it's exchange rate.

5.1 Spread on International Bonds

Because the peg affects the price level in both the pegging and target countries, we should expect the peg to affect the expectation of real and nominal interest rates in each country as well. This impacts the spreads on international bonds between the countries in the peg as well as between countries in the peg and outside countries. The spreads on international bonds with outside countries is

$$\begin{aligned} cov(\lambda_T, p^t - p^o) = & \frac{(1-\tau)\gamma^2(1-\phi)^2}{\phi((1-\tau)\phi + \gamma\epsilon_\alpha\tau)}\left(\sigma_t^2\theta^t - \sigma_o^2\theta^o\right) \\ & + \zeta\frac{(1-\alpha)(1-\tau)(1-\phi)^2\gamma^2\kappa_P}{(\gamma\tau + (1-\alpha)(1-\tau)\phi)\phi\kappa_N}\frac{\theta^p}{\theta^p + \theta^t}\left(\theta^p\sigma_p^2 - \theta^t\sigma_t^2\right) \end{aligned} \quad (46)$$

$$
\begin{aligned}
cov(\lambda_T, p^p - p^o) = & \frac{(1-\tau)\gamma^2(1-\phi)^2}{\phi((1-\tau)\phi + \gamma\epsilon_\alpha\tau)} \left(\sigma_p^2\theta^p - \sigma_o^2\theta^o\right) \\
& + \zeta \frac{(1-\alpha)(1-\tau)(1-\phi)^2\gamma^2\kappa_P}{(\gamma\tau + (1-\alpha)(1-\tau)\phi)\phi\kappa_N} \frac{\theta^t}{\theta^p + \theta^t} \left(\theta^t\sigma_t^2 - \theta^p\sigma_p^2\right)
\end{aligned} \tag{47}
$$

and the spreads on international bonds between the pegging and target country is

$$
cov(\lambda_T, p^p - p^t) = \frac{(1-\alpha)(1-\zeta)(1-\tau)(1-\phi)^2\gamma^2}{\phi\kappa_N} \left(\theta^p\sigma_p^2 - \theta^t\sigma_t^2\right) \tag{48}
$$

The impact of the peg on these spreads can be summarized by the following propositions

Proposition 5.1. *A currency peg decreases the magnitude of the spread in (real) international bonds between the pegging and target countries.*

Proposition 5.2. *Suppose conditions 1 and 2 hold. A currency peg increases the level of the spread in (real) international bonds between an outside country and the larger of the two countries in the currency peg. A currency peg decreases the level of the spread between (real) international bonds between the country outside of the currency peg and the smaller of the two countries in the currency peg.*

Both of these results can be obtained by simply looking at the expressions for the spread on international bonds. These results are the same as the results from the section where markets are complete.

6 Endogenous Capital Accumulation

This model generalizes easily the case where goods are produced using labor and capital rather than endowed exogenously. This extension allows us to analyze the household's capital accumulation decision as a function of the exchange rate peg and country size. The relevant result from Hassan (2013) is that larger countries accumulate a larger per capital stock in their non-traded sectors. Since only differences in the endowment in the non-tradable sector matter for exchange rates and spreads on bonds, we will introduce production and endogenous capital accumulation in the non-tradable sector and continue to assume exogenous endowments in the traded good.

Output of the non-traded good in each country is produced using a Cobb-Douglass production function. Therefore, output per capital of the non-traded good in each country becomes

$$Y_{N,2}^{n} = \exp(\eta^{n}) \left(\frac{K^{n}}{\theta^{n}} \right)^{\nu} \tag{49}$$

where $\nu \in (0,1)$ and the total supply of labor per country is equal to the measure of households in the country, θ^n. Each country experiences a shock to total factor producivity in the second period

$$\eta^{n} \sim N\left(-\frac{1}{2}\sigma_{n}^{2}, \sigma_{n}^{2}\right) \tag{50}$$

Households in receive an endowment in the first period and make a decision to consume some portion of the endowment and to put the remaining portion of the endowment towards capital investment. For country n, capital accumulates according to

$$\theta^{n} Y_{1} - \int_{i \in n} C_{1}(i) di = K^{n} \tag{51}$$

where Y_1 is the deterministic per-capita endowment of each country's final consumption bundle in period one. Capital has no value after production in the second period. In this sense, we can say that capital fully depreciates after production takes place.

We solve the model using the perturbation methods developed in Judd (1998). Perturbation methods deliver the coefficients to a Taylor series approximation of the equilibrium variables. Households in the economy make investment decisions in the first period. The only information they have is their expectation of future shocks, which is defined by the standard deviation of the shocks for each country. A log-linear approximation of equilibrium capital investment ignores the second order effects of the standard deviation of shocks on capital investment. Thus, this approximate solution would be too inaccurate to provide us with any insight into investment behavior. Perturbation methods allow us to achieve a much closer approximation to the true investment function.

The goal of this exercise is to write all equilibrium variables as a Taylor series in the state variables. In order to show analytical expressions, we focus on the case of three countries ($n = p, t, o$), complete markets ($\phi = 1$) and an elasticity of substitution between traded and non-traded goods of one ($\alpha = 0$). With these restrictions, the household utility function reduces to Cobb-Douglass form with parameter τ. The numerical solution for the full model is straight-forward, but the analytical expressions are too complicated to be displayed here.

Using the fact that all households in each country must consume the same bundle (C_T^n, C_N^n) in each period, we can write the Social Planner's problem as

$$\max \mathbb{E}_1 \sum_{n=p,t,o} \theta^n \left[(C_1^n)^{1-\gamma} + \beta \, (C_2^n)^{1-\gamma} \right] \tag{52}$$

subject to the utility function defined by (3), economy's resource constraints given by (13), (14), (49), (51) and the exchange rate peg (8). We plug in the exchange rate peg into the objective function and derive the first order conditions. The equilibrium is defined by eight first order conditions and four budget constraints.

The first five first order conditions and the four budget constraints relate the allocation of goods in the second period across countries to the realization of endowments and the level of capital accumulation. We can denote this system by

$$G(\eta^p, \eta^t, \eta^o, y_T^p, y_T^t, y_T^o, K_N^p, K_N^t, K_N^o) = 0 \tag{53}$$

This condition implicitly defines the equilibrium allocation of traded goods aross the three countries and we can derive the Taylor approximations to the functions $C_{T,2}^p(\omega)$, $C_{T,2}^t(\omega)$ and $C_{T,2}^o(\omega)$, where ω represents the vector of TFP shocks, endowment shocks and levels of capital accumulation.

The remaining three first order conditions relate the level of capital accumulation in the first period to the expected allocation in the second period. These first order conditions take on the form

$$\left(Y_1 - \frac{K^n}{\theta^n} \right)^{-\gamma} = \mathbb{E} \left[\beta(1-\tau)\nu \frac{\theta^n}{K^n} \left(\left(\left(\frac{K^n}{\theta^n} \right)^{\nu} \eta^n \right)^{1-\tau} (C_{T,2}^n)^{\tau} \right)^{1-\gamma} \right] \tag{54}$$

for $n = p, t, o$. They implicitly define the policy functions governing capital accumulation, $K_N^n(\sigma_p, \sigma_t, \sigma_o)$.

We choose the endowment in the first period as

$$Y_1 = 1 + (\beta(1-\tau)\nu)^{-\frac{1}{\gamma}}$$

such that the deterministic solution of the extended model coincides with the deterministic solution of the model in the main part of the paper. That is to say consumption of the traded and non-traded good in the first and second period equals 1 for all households in all countries. In order for non-traded good output to equal 1 per household, we set $K_N^n(0,0,0) = \theta^n$ for country n in the deterministic equilibrium.

We start by taking a series of derivatives of the conditions in $G(\eta^p, \eta^t, \eta^o, y_T^p, y_T^t, y_T^o, K_N^p, K_N^t, K_N^o) = 0$ and solve for the coefficients in a Taylor expansion of $C_{T,2}^n(\cdot)$, for each country n in the economy, around the deterministic solution of the model. For example, the equation given by

$$\frac{\partial G(\eta^p, \eta^t, \eta^o, y_T^p, y_T^t, y_T^o, K_N^p, K_N^t, K_N^o)}{\partial \eta^n} = 0$$

allows me to solve for the coefficient

$$\frac{\partial C_{T,2}^n(\eta^p, \eta^t, \eta^o, y_T^p, y_T^t, y_T^o, K_N^p, K_N^t, K_N^o)}{\partial \eta^n}|_{\text{Deterministic Solution}}$$

and hence for the Taylor expansions for $C_{T,2}^p(\cdot)$, $C_{T,2}^t(\cdot)$ and $C_{T,2}^o(\cdot)$

The next step is to use this solution for second period consumption to solve for capital investment in the first period. We take the remaining equilibrium conditions given by (54) and substitute in the implicit functions for K_N^n and $C_{T,2}^n$. Without loss of generality, let us re-define

$$\sigma_n = \bar{\omega}_n \sigma, \eta^n = \tilde{\eta}^n \bar{\omega}_n \sigma \text{ and } y_T^n = \tilde{y}_T^n \bar{\omega}_n \sigma \tag{55}$$

for $n = p, t, o$. $\bar{\omega}_n$ are positive constants and $\tilde{\eta}^n, \tilde{y}_T^n \sim N(0,1)$. We can solve for $\frac{\partial^j K_N^n(\sigma)}{\partial^j \sigma}$ in two steps. First, take the partial derivative of (54) on both sides of the equation. What we are left with is a deterministic function of $\frac{\partial^j K_N^n(\sigma)}{\partial^j \sigma}$ on the left hand side and the expectation of a stochastic function of $\frac{\partial^j K_N^n(\sigma)}{\partial^j \sigma}$ on the right hand side. Second, we take expectations of the right hand side by integrating over the shocks. This step has been made easy as all re-defined shocks have a standard normal distribution. We can then repeat this procedure for all terms in the Taylor expansion

$$K_N^n \sim \sum_j \frac{1}{j!}\left(\frac{\partial^j K_N^n(\sigma)}{\partial^j \sigma}|_{\text{Deterministic Solution}}\right)\sigma^j$$

for $n = p, t, o$ by taking higher-order derivatives of both sides of (54) and solving for the implicitly defined coefficients.

We can now solve for investment in the first period as well as the realization of consumption in the second period. Moreover, we can solve for the first period expectations of second period variables by applying the approximations of second period consumption and performing a non-linear change of variables. For example, we get the expected payoff of the country n risk-free bond by plugging $C^p_{T,2}(\cdot)$, $C^t_{T,2}(\cdot)$ and $C^o_{T,2}(\cdot)$ into the expression for the price level such that

$$\mathbb{E}P^n_2 = \mathbb{E}\left[\frac{\left(C^n(\eta^p, \eta^t, \eta^o, y^p_T, y^t_T, y^o_T, K^p_N, K^t_N, K^o_N)\right)^{-\gamma}}{\Lambda_{T,2}\left(\eta^p, \eta^t, \eta^o, y^p_T, y^t_T, y^o_T, K^p_N, K^t_N, K^o_N\right)}\right]$$

For the following propositions, we use a second order expansion. In a world with no exchange rate peg, we are able to easily characterize the differences in capital investment between different countries.

Proposition 6.1. *Given conditions 1 and 2, in a world with no exchange rate pegs, the larger country accumulates more capital per capita in the non-traded sector.*

Proof. Consider the difference in capital investment between two arbitrary outside countries h and f. Dividing the second-order Taylor expansion for capital accumulation with θ^n and taking the difference in per capita capital investment yields

$$\frac{K^f_N}{\theta^f} - \frac{K^h_N}{\theta^h} = \frac{(\gamma-1)^3(1-\tau)^2\tau^2(\theta^f - \theta^h)}{(1+(\gamma-1)\tau)\left((1-\tau)\tau\nu(\gamma-1) + \tau(1+(\gamma-1)\tau)\left(1+\gamma\left(\beta(1-\tau)\nu\right)^{\frac{1}{\gamma}}\right)\right)}$$

The denominator of the above expression is strictly positive. Thus, the sign of the difference in capital accumulation is determined by $\theta^t - \theta^p$. □

In contrast to this proposition, we find that if a pegging country imposes a hard exchange rate peg with $\zeta = 1$, then even if the target country is larger than the pegging country, the pegging country can and usually will accumulate more capital than the target country. Under the case where $\zeta = 1$, the difference in per capita capital accumulation is equal to

$$\frac{K^t_N}{\theta^t} - \frac{K^p_N}{\theta^p} = \frac{\left(\left(\theta^t + \theta^p\right)(\gamma-1)^2\tau^2 - 1\right)(1-\tau)^2\tau^2(\theta^t - \theta^p)}{(\theta^t + \theta^p)\tau(1+(\gamma-1)\tau)(\tau(1+\gamma\left(\beta(1-\tau)\nu\right)^{\frac{1}{\gamma}} + \nu(1-\tau)))}$$

The sign of this expression depends on $\left(\theta^t + \theta^p\right)(\gamma - 1)^2\tau^2 - 1$, which can very easily be negative, depending on the share of household expenditures that goes towards tradable goods, τ.

7 Conclusion

This paper solves an international asset pricing model which endogenizes the stochastic properties of, exchange rates, international asset prices, and the level of capital accumulation across countries. It explores the effects of exchange rate pegs on the economies of the target country and of countries outside the peg. We are able to characterize the impact of the peg on the consumption of households in each country, the exchange rates between countries and the spreads on bonds and stocks in the world. Additionally, we solve for the impact of exchange rate pegs on differences in capital investment between countries.

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A Deriving the Exchange Rate Peg Constraint

The real exchange rate is given by

$$S_2^{t,p} = \frac{\frac{\Lambda_t}{\Lambda_{T,2}}}{\frac{\Lambda_p}{\Lambda_{T,2}}} = \left(\frac{C_2^t}{C_2^p}\right)^{-\gamma}$$

We will start by defining a hard exchange rate peg. Afterwards, we try to derive a constraint that will emulate a soft exchange rate peg. Suppose the pegging country wants to fix the real exchange rate to the target country at its period 1 level. Then, the utility maximization has an additional constraint

$$\frac{(C_1^t)^{-\gamma}}{(C_1^p)^{-\gamma}} = \frac{(C_2^t(\omega))^{-\gamma}}{(C_2^p(\omega))^{-\gamma}}$$

Using the functional form of the consumption function tells us the desired constraint is

$$C_{T,2}^p(\omega) = \left[\frac{\left(\frac{C_2^t(\omega)C_1^h}{C_1^t}\right)^{\alpha} - (1-\tau)\left(C_{N,2}^p(\omega)\right)^{\alpha}}{\tau}\right]^{\frac{1}{\alpha}}$$

In our set up, consumption for all households in the first period is 1. Therefore, the above expression reduces to

$$C_{T,2}^p(\omega) = \left[\left(C_{T,2}^t(\omega)\right)^{\alpha} + \frac{1-\tau}{\tau}\left[\left(C_{N,2}^t(\omega)\right)^{\alpha} - \left(C_{N,2}^p(\omega)\right)^{\alpha}\right]\right]^{\frac{1}{\alpha}}$$

Using this peg constraint and solving the model gives us the following result for the consumption of the traded good in the pegging country gives us

$$\begin{aligned} c_{T,2}^p = \bar{y}_{T,2} &+ \frac{(\gamma+\alpha-1)(1-\tau)}{(1-\alpha)(1-\tau)+\tau\gamma}(\bar{y}_{N,2} - y_{N,2}^p) \\ &+ \left(\frac{(1-\alpha)(1-\tau)}{\tau((1-\alpha)(1-\tau)+\gamma\tau)}\frac{\theta^t}{\theta^p+\theta^t}\right)(y_{N,2}^t - y_{N,2}^p) \end{aligned} \tag{56}$$

and the variance in the exchange rate between the pegging and target country was 0. We looked for a constraint under which the volatility of the exchange rate between the pegging and target country is between the variance in the exchange rate without a peg and 0.

B Details on Solving the Model

B.1 Proof of Proposition 3.1

Consider an arbitrary asset with a stochastic payout of X units of the traded good in period 2 and a period 1 price of V_X. Given the state contigen seurities prices from the households' first order conditions,

$$V_X = \beta \mathbb{E}\left[\frac{\Lambda_{T,2}}{\Lambda_{T,1}} X\right]$$

Taking logs on both sides gives

$$v_X = \log \beta + \log \mathbb{E}\left[\Lambda_{T,2} X\right] - \lambda_{T,1}$$

If asset returns and marginal utilities are log-normally distributed,

$$v_X = \log \beta + \mathbb{E}\lambda_{T,2} + \mathbb{E}x + \frac{1}{2} var(\lambda_{T,2}) + \frac{1}{2} var(x) + cov(\lambda_{T,2}, x) - \lambda_{T,1}$$

Therefore,

$$\log ER\left[X\right] = \lambda_{T,1} - \log \beta - \mathbb{E}\lambda_{T,2} - \frac{1}{2} var(\lambda_{T,2}) - cov(\lambda_{T,2}, x)$$

Suppose another asset has payout Z. Then, expected log returns of this second asset are

$$\log ER\left[Z\right] = \lambda_{T,1} - \log \beta - \mathbb{E}\lambda_{T,2} - \frac{1}{2} var(\lambda_{T,2}) - cov(\lambda_{T,2}, z)$$

Differencing these two gives us

$$\log ER\left[X\right] - \log ER\left[Z\right] = cov(\lambda_{T,2}, z - x)$$

B.2 Deriving the Price Index

The cost of one unit of consumption in country n is defined as

$$P^n = \arg\min C_T(i) + P_N C_N(i) \text{ subject to } C(i) = 1$$

for household i. Using the constant elasticity of substitution form for the aggregate consumption bundle, the first order conditions imply

$$C_N(i) = (P_N)^{\frac{1}{\alpha-1}} \left(\frac{\tau}{1-\tau}\right)^{\frac{1}{\alpha-1}} C_T(i)$$

The objective function and the constraint imply

$$P^n = \frac{C_T(i) + P_N C_N(i)}{(\tau C_T(i)^\alpha + (1-\tau) C_N(i)^\alpha)^{1/\alpha}}$$

Combining our equation for $C_N(i)$ and P^n, we find that $C_T(i)$ cancels out of the fraction. Multiplying the numerator and denominator by $\tau^{\frac{1}{1-\alpha}}$, defining $\varepsilon_\alpha = (1-\alpha)^{-1}$ and simplifying gives us

$$P_t^n = \left(\tau^{\varepsilon_\alpha} + (1-\tau)^{\varepsilon_\alpha} \left(P_{N,t}^n\right)^{1-\varepsilon_\alpha}\right)^{\frac{1}{1-\varepsilon_\alpha}}$$

B.3 Equilibrium Consumption of Inactive Households

Inactive households maximize (2) subject to the constraints $P_1^n C_1(i) \leq Y_{T,1} + P_{N,1}^n Y_{N,1}^n$ and $P_2^n C_2(i) \leq P_2^n e^{-\mu^n}$. Taking the derivative of the objective function with respect to C_T and C_N and equating them yields the following optimality condition

$$(P_N^n)^{\frac{1}{1-\alpha}} = \frac{C_N(i)}{C_T(i)} \left(\frac{1-\tau}{\tau}\right)^{\frac{1}{\alpha-1}}$$

for all households i. Plugging this ocndition into the constraints solving for the consumption of the inactive households yields the desired results.