Equilibrium Labor Turnover, Firm Growth and Unemployment

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Abstract

This paper considers a dynamic economy in which employees do not observe firm productivity, which is subject to shocks, and employers cannot commit to future wages. Workers search on the job for better paid employment and firms pay hiring costs. A signalling equilibrium is characterized where more productive firms pay higher wages and workers transit from less to more productive employers as they climb the wage ladder. There is firm turnover: new small start-up firms are created while some existing firms die. Consistent with Gibrat’s law, firm growth rates are size independent but increase with firm productivity (which evolves stochastically). With endogenous aggregate job creation rates and job-to-job transitions, the model provides a rich, coherent, non-steady state framework of equilibrium wage formation and worker flows. Existence of a steady state equilibrium for any finite number of firm productivity types is established. Steady state is unique when firm productivity is permanent and there are many firm types.

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A unique non-steady state equilibrium exists in the case of one type. In the general case, a unique equilibrium can be established when the elasticity of the hire rate with respect to productivity is sufficiently small.

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1 Introduction

This paper describes a tractable dynamic variant of the Burdett and Mortensen (1998) model [BM from now on] that can be used for both macro policy applications and micro empirical analysis. The framework allows rich firm size dynamics. Innovative start-up companies are born small but the more productive grow quickly over time. Conversely large existing firms may experience adverse shocks and so enter periods of decline. Fast growing firms are characterized by high productivity, high wages, low quit rates and high recruitment rates. As a firm’s productivity is a persistent process, firm size is correlated with productivity and thus with wages, but firm size has no direct (causal) impact on productivity or wages. With endogenous aggregate job creation rates and job-to-job transitions (via on-the-job search), this paper identifies a new, coherent framework of equilibrium wage formation and labor force adjustment outside of steady state.

In this framework firms are occasionally hit by firm specific productivity shocks, regarded as private information to the firm. In contrast to the wage bargaining literature with asymmetric information, however, we consider a market environment in which employees may obtain outside job offers through on-the-job search. Assuming, as in BM, that firms have all the bargaining power (i.e. firms post current wages) but cannot commit to future wages, then equilibrium finds each firm’s posted wage is a trade-off between paying a higher wage and reducing its employee quit rate. As the loss in profit through a quit is greater for higher productivity firms, a fully revealing signalling equilibrium exists where higher productivity firms pay strictly higher wages and enjoy strictly lower quit rates. This wage structure remains tractable outside of steady state where workers anticipate future wages given current information and the state of the economy.

An investment margin is also incorporated into the model reflecting the fact that recruiting new employees is a costly process. The market equi-
librium is not efficient as firms choose their wage and hiring strategies to maximize own profit, not taking into account that a worker who quits reduces the profit of his/her original employer. Thus in contrast to the recent non-steady state search literature, equilibrium is not constrained efficient and so turnover outcomes cannot be identified by solving the Planner’s problem. Equilibrium, however, is more interesting for, with incomplete markets, equilibrium wage formation has a direct impact on unemployment, turnover and investment.

The wage structure is that of an efficiency wage model, related to Weiss (1980), but set in a dynamic, non-steady state equilibrium framework. There is a small recent literature on the aggregate dynamics of labour markets with on-the-job search. Menzio and Shi (2010) consider a directed search framework but has no concept of firm size. Robin (2011) assumes ex-post Bertrand competition between firms when a worker receives an outside offer, but assumes exogenous hiring rates. In the most closely related paper, Moscarini and Postel-Vinay (2010) study a dynamic version of BM in which each firm precommits to a contract that specifies state-contingent wages over the entire future. The principal differences between their paper and ours are consequences of the following different specification assumptions: no commitment, asymmetric information, and a hiring cost that implies equilibrium wage and hiring strategies are independent of firm size.

2 The Model

Time is continuous and denoted as $t \in [0, \infty)$. Agents are risk neutral, equally productive, infinitely lived and discount the future at rate $r > 0$. Each agent is either participant in the labor market who is either employed earning some wage $w$ or unemployed and searching for a job earning the flow value of home production, $b \geq 0$. In addition, there are fixed number of non-participant entrepreneurs who are seeking ideas for new business ventures. These generate agents generate new firms at rate $\mu$. On the immediate transformation of a idea into a start-up, the entrant sells the firm to investors

\footnote{In BM the wage paid controls two margins, the firm’s hire and quit rate. Here with the introduction of a hiring margin, wages only target the employee quit rate, as in Weiss (1980). The directed search approach, as considered in Acemoglu and Shimer (1998), instead adopts the other polar case, that wages only target the hire rate.}
for its value and becomes the start-up’s first employee.²

Firms are risk neutral with constant marginal revenue product of labor $\pi$. Existing firms die at rate $\delta > 0$ and so the measure of firms equals $\mu / \delta$. $\Phi(p)$ denotes the stationary distribution of productivities across existing firms with support denoted $[p, \bar{p}]$ and $\bar{p} > b$. It is convenient, however, to describe each firm $p$ by its corresponding rank $x = \Phi(p)$, noting that productivity $p = p(x)$ is given by the inverse function of $\Phi(.)$.

At a start-up, the rank of a new firm is a random draw from the uniform c.d.f. $\Gamma_0(.) = x$. Continuing firms with rank $x$ are subject to a technology shock process characterized by a given arrival rate $\gamma \geq 0$ and a new rank $x'$ randomly drawn from c.d.f. $\Gamma_1(.|x)$. Throughout we require first order stochastic dominance in $\Gamma_1(.|x)$, so that higher rank firms $x$ are more likely to remain high rank into the future. If $[0, \bar{x}(x)]$ denotes the support of $\Gamma_1(.|x)$, we assume $\lim_{x \to 0^+} \bar{x}(x) = 0$ so that rank $x = 0$ is an absorbing state (till firm death).

For any date $t$, let $G_t(x)$ denote the number of unemployed workers plus those workers employed at firms with rank no greater than $x \in [0, 1]$. Hence $G_t(0) = U_t$ is the number unemployed and $G_t(1) = 1$. The state of any firm is $(x, n, G_t(.))$ where $n$ is its (integer) number of employees and $G_t(.)$ serves as the information relevant aggregate state variable for the following reason. When a continuing firm hires a new employee, that new employee is considered a random draw from the set of all workers who strictly prefer the job to their current status. As job offers are random, each unemployed worker receives a job offer according to a Poisson process with parameter $\lambda = \lambda(G)$ which depends on the market state $G(.)$. Randomness implies all employed workers also receive outside offers at this rate but, as job offers are heterogeneous, not all are preferred to the worker’s current job.³ Let $W$ denote the expected lifetime payoff of an employee at a given firm. We use $F(W, G)$ to describe the fraction of outside job offers which yield value to the worker no greater than $W$ in aggregate state $G(.)$. $\lambda(G)$ and $F(W, G)$ are endogenous objects which, in equilibrium, are determined by aggregating

²An alternative specification might assume it takes time to develop a business idea and so describe turnover in the stock of entrepreneurs.

³It is often assumed in the literature that employed workers receive offers at a lower rate than unemployed workers to match transition data. However, Christiansen et al. (2005) demonstrate that one can explain the data with a model that allows for endogenous search effort. As the qualitative properties of that model are the same as assuming equal offer arrival rates, the simplification here is justified.
over the wage and recruitment strategies of all firms.

There is asymmetric information: $x$ is private information to the firm. We only consider fully revealing equilibria. Specifically we consider Markov Perfect (Bayesian) equilibria where each firm posts a sequence of spot wages using an equilibrium wage strategy $w = w(x, n, G(.))$ where $w(.)$ is strictly increasing in $x$. Note this Markov structure assumes a firm cannot precommit on future wages. It also imposes no discrimination: that a firm must pay equally productive workers the same.

Hiring is costly here not because there are vacancy posting costs, rather because employee time is required to screen new applicants and to assimilate new hires into firm practice. Following Lucas (1967) and Merz and Yashiv (2007), if a firm with $n$ employees decides to recruit an additional worker at rate $H$, then the cost of hiring is $nc(h)$ where $h = H/n$ is the effort required per employee. $c(.)$ is continuously differentiable and strictly convex with Inada conditions $c(0) = c'(0) = 0$. Note that these recruitment costs exhibit constant returns: recruitment costs double as employment $n$ and hires $H$ double.

3 Markov Perfect (Bayesian) Equilibria.

3.1 Separating Equilibria with Firm Size Invariance.

For ease of exposition, we only consider Markov perfect (Bayesian) equilibria in which the set of equilibrium wage strategies $w = w(x, n, G(.))$ for $x \in [0, 1]$ is

(R1) fully revealing; i.e. $w(x,.)$ is continuous and strictly increasing in $x \in [0, 1]$ and,

(R2) firm size invariant; i.e. $w(.)$ does not depend on $n$.

That price strategies are continuous and strictly increasing in $x$ is a standard property both in the BM literature and in the first price auction literature with independent private values (e.g. McAfee and McMillan (1987)). Firm size invariance (R2) occurs as there are constant returns to production and

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4Recent empirical work suggests that "adjustment costs" of this form explain the data better than the more traditional "recruiting cost" specification. For example, see Sala et al. (2012) and Christiano (2013).
In any equilibrium satisfying \((R1), (R2)\) and for any announced wage \(w' \in [w(0, \cdot), w(1, \cdot)]\), Bayes rule implies the worker believes the firm’s rank is \(\hat{x}(w', \cdot)\) where \(\hat{x}\) uniquely solves \(w' = w(\hat{x}, G)\). If the firm posts wage \(w' < w(0, \cdot)\), existence of an equilibrium requires belief \(\hat{x}(w', \cdot) = 0\). If instead the firm posts wage \(w' > w(1, \cdot)\) the out-of-equilibrium beliefs do not play any material role. For this case we specify belief \(\hat{x}(w', \cdot) = 1\).

### 3.2 Characterization

We begin with a standard observation. If a firm pays wage \(w = b\), an employee does not quit into unemployment: remaining employed at his/her current employer has a positive option value (the firm may increase its wage tomorrow) while the worker can always quit tomorrow. Thus any firm with \(n \geq 1\) must make strictly positive profit (as \(p > b\)). Strictly positive profit further implies \(W(x, \cdot) \geq V_u(\cdot)\) for all \(x \in [0, 1]\) (otherwise all employees quit into unemployment which yields zero profit). Thus along the equilibrium path it cannot be optimal for a worker to quit into unemployment and, as wages are fully revealing, the value of employment \(W(\cdot)\) is identified recursively by

\[
\begin{align*}
    rW(x,) &= w(x, \cdot) + \delta \left[ \max (V_u(\cdot), V_u(\cdot)) - W(x, \cdot) \right] \\
    &\quad + \gamma \int_0^1 \left[ W(z, \cdot) - W(x, \cdot) \right] d\Gamma_1(z | x) \\
    &\quad + \lambda(\cdot) \int_0^W \max [W' - W(x, \cdot), 0] dF(W', \cdot) + \frac{\partial W}{\partial t}.
\end{align*}
\]

In words, the flow value of employment equals flow wage income plus the capital gains associated with the following possibilities: the arrival of (a) a firm destruction or an entry shock, (b) a firm specific productivity shock with new realization \(z \sim \Gamma_1(\cdot | x)\), and (c) an outside offer with value \(W' \sim F(\cdot)\). In a steady state the final term, denoted \(\partial W/\partial t\), would be zero. Outside of steady state, the state variable \(G_t\) evolves endogenously. This latter term describes the expected capital gain through the dynamic evolution of \(G\), a term which is made precise once we have described the equilibrium \(G\) dynamics.

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\(^5\)Wage strategies are not firm size invariant in BM, Coles (2001), Moscarini and Postel-Vinay (2012) as those papers do not allow for hiring effort.
It is immediate that the value of employment $W(.)$ must be strictly increasing in $x$: the wage paid $w(.)$ is strictly increasing in $x$ and there is first order stochastic dominance in $\Gamma_1(.|x)$. Thus along the equilibrium path, an employee at firm $x$, which pays wage $w = w(x, G)$, only quits to an outside offer from a higher productivity firm $x' > x$ which reveals its type by offering a strictly higher wage $w(x', G) > w$.

Now consider an employed worker in firm $(x, n, G)$. Given the firm announces wage $w$, (R1) implies the worker’s belief of the firm’s type $\hat{x}(w, G)$ is degenerate. As (R2) implies wages do not depend on firm size, and wages are persistent, the rank of current wage $w$ offered is a sufficient statistic for the rank of the lifetime wage, the expected discounted value of employment in the firm. Denote the latter as $W(x, G)$ where $x = \hat{x}(w, .)$. Given this fact, the value of unemployment solves

$$rV_u = b + \lambda(.)\lambda(.)\int_{W}^{\infty}[W - V_u(.), 0]dF(W, .).$$

The analogous specification of $V_e$ will be reported later. For now, we note in passing that

$$V_a(G) = V_e(G).$$

must hold in equilibrium. In other words, the number of agents who choose entrepreneurship rather the unemployed search must adjust so that those not employed are indifferent between the two activities.

### 3.3 Firm Strategies.

Let $\Pi(x, n, G)$ denote the expected discounted lifetime profit of firm $(x, n, G)$ using an optimal wage and recruitment strategy. Suppose the firm posts wage $w$. As each employee believes the firm’s type is $x = \hat{x}(w, G)$, this posted wage induces employee quit rate $q(\hat{x}, G)$ (determined below). Hence firm $(x, n, G)$ chooses wage $w$ and per employee recruitment effort $h = H/n$ to solve the Bellman equation:

$$(r + \delta)\Pi(x, n, .) = \max_{w, h \geq 0} \left( \begin{array}{l} n[p(x) - w] - nc(h) \\ + nh [\Pi(x, n + 1, .) - \Pi(x, n, .)] \\ + nq(\hat{x}, .) [\Pi(x, n - 1, .) - \Pi(x, n, .)] \\ + \gamma \int_0^1 [\Pi(z, n, .) - \Pi(x, n, .)] d\Gamma_1(z|x) + \frac{\partial \Pi}{\partial x} \end{array} \right).$$
In words, the flow value of the firm equals its flow profit less recruiting costs plus the capital gains associated with (i) a successful hire \((n \rightarrow n+1)\) (ii) the loss of an employee through a separation \((n \rightarrow n-1)\), and (iii) a firm specific productivity shock with new draw \(z \sim \Gamma_1(.|x)\). The last term captures the effect on \(\Pi(.)\) through the non-steady state evolution of \(G\).

The constant returns structure implies \(\Pi(x, n, G) = nv(x, G)\) solves this Bellman equation, where \(v(x, G)\), the value of each employee in firm \((x, n, G)\), is given by:

\[
(r + \delta + \gamma)v(x, .) = \max_{w, h \geq 0} \left\{ \begin{array}{ll}
p(x) - w - q(\tilde{x}(w, .), .)v(x, .) + hv(x, .) - c(h) \\
+ \gamma \int_0^1 v(z, .)d\Gamma_1(z|x) + \frac{\partial w}{\partial t} \end{array} \right\}. \tag{4}
\]

The following transversality condition is also necessary for a solution to this dynamic programming problem:

\[
\lim_{t \to \infty} E_0 \left[ e^{-rt}v(x_t, G_t)|x_0, G_0 \right] = 0. \tag{5}
\]

Define the hire function

\[
h^*(v) = \arg \max_{h \geq 0} [hv - c(h)]. \tag{6}
\]

The Bellman equation (4) and the definition of \(h^*(.)\) implies the firm’s optimal recruitment strategy \(h(x, n, .) = h^*(v(x, .))\). Optimal recruitment effort per worker is thus independent of firm size and so, consistent with Gibrat’s law, hiring inflow \(H = nh\) is proportional to firm size.

The Bellman equation (4) implies the optimal wage strategy minimizes the sum of the wage bill and turnover costs; i.e.

\[
w(x, .) = \arg \min_w \left[w + q(\tilde{x}(w, .), .)v(x, .) \right]. \tag{7}
\]

This wage structure is not unlike a first price auction with a stochastic number of bidders (McAfee and McMillan (1987)). In that literature, the buyer’s optimal bid depends on the likely best outside price received by the seller. Here this information is encapsulated by the worker’s equilibrium quit function

\[
q(\tilde{x}, .) = \lambda(.[1 - F(W(\tilde{x}, .))], \tag{8}
\]

where \(\lambda(.)\) describes the (Poisson) probability that the worker receives an outside offer and \(1 - F(.)\) is the probability the outside offer yields expected value \(W' > W(\tilde{x}, G)\) and the worker quits.
To close the model, one needs to specify the value of entrepreneurship, \( V_e \). As the entrepreneur both sells the firm and becomes its first employee on entry and \( W(x, .) - V_e = W(x, .) - V_u \geq 0 \) and \( v(x) \geq 0 \) for every \( x \) because \( p(x) > b \) by assumption

\[
rv_e = b + \frac{\mu}{m} \int_0^1 (v(x) + W(x, .) - V_e)d\Gamma_0(x). \tag{9}
\]

where \( m \) is the mass of entrepreneurs. In other words, analogous to the value of unemployed search, the flow value is equal to the flow of home production plus the expected gain from entrepreneurship, the product of the rate at which the activity will generate a new business idea per entrepreneur and the surplus value that the idea will generate.

### 3.4 Aggregation

Determining \( \lambda(.) \) and \( F(.) \), which by (8) are the parameters of each firm’s decision problem (7), requires aggregating over the equilibrium wage and hiring strategies of all firms. Recall that \( G(x) \) describes the fraction of workers who are either unemployed or employed at firms with rank no greater than \( x \). Consider then a firm \( x \) which hires at equilibrium rate \( H = nh \). As job offers are random, then each job offer by this firm is accepted with probability \( G(x) \) (as an equilibrium offer is accepted by workers employed at some firm \( x' < x \)). Thus an expected hiring rate of \( H = nh(x, G) \) at firm \( (x, n, G) \) requires the firm makes job offers at rate \( H/G(x) \). Aggregating across all firms implies the aggregate flow of job offers is

\[
\int_0^1 \frac{h(x, G)dG(x)}{G(x)}
\]

where \( dG(x) \) describes the measure of workers employed at type \( x \) firms. As there is a unit measure of workers and job offers are random, the rate any given worker receives a job offer must therefore be

\[
\lambda(G) = \int_0^1 \frac{h(x, G)dG(x)}{G(x)}. \tag{10}
\]

To determine the composition of those job offers, define \( \tilde{F}(x, G) = F(W(x, .)) \) as the fraction of job offers made by firms with type no greater than \( x \) in
aggregate state $G$. The same aggregation argument implies
\[
\lambda(G)[1 - \hat{F}(x, G)] = \int_x^1 \frac{h(z, G)dG(z)}{G(z)}
\]
(11)
where the right hand side is the aggregated flow of job offers by firms with type greater than $x$.

### 3.5 The Equilibrium Distribution of Wages

An equilibrium solution to the model is a value of worker $v(x, G)$ that satisfies (4) together with the optimal wage and hire rate choices $(w(x, G), h(x, G))$ for every employment c.d.f. $G$ such that the equilibrium quit rate function is
\[
q(\hat{x}, G) = \int_{\hat{x}}^1 \frac{h(z, G)dG(z)}{G(z)},
\]
(12)
noting that all firm types $z > \hat{x}$ offer wage $w(z, \cdot) > w(\hat{x}, \cdot)$ and wage function
\[
w(x, G) = \arg\min_w \left[ w + v(x, G) \int_{\hat{x}(w,G)}^1 \frac{h(z, G)dG(z)}{G(z)} \right].
\]
(13)

Consider first the lowest productivity firm $x = 0$ and recall that positive profit implies $W(0, G) \geq V_u(G)$. A contradiction argument now establishes equilibrium implies $W(0, G) = V_u(G)$.\footnote{If $W(0, G) > V_u(G)$, firm $x = 0$ deviates with wage $w' < w(0, G)$. As its employees believe $\hat{x} = 0$, (13) implies announcing wage $w' < w(0, G)$ is cost decreasing, which is the required contradiction.} The intuition is that the lowest productivity firm cuts its wage until $W(0, G) \geq V_u(G)$ binds. It cannot cut its wage further as its employees would then quit into unemployment. This equilibrium condition yields lemma 1. Furthermore, the value of Lemma 1. In equilibrium the lowest wage offer is $w(0, G) = b$.

**Proof.** Given that the state $x = 0$ is absorbing, the assertion follows directly by putting $x = 0$ and using equations (1), (2) and the equilibrium requirement $W(0, G) = V_u(G)$.

**Lemma 2:** In any equilibrium, the value of worker $v(x, G) > 0$ is unique and increasing in $x$.

**Proof.** In the online appendix.
For simplicity assume \( G(.) \) is differentiable with \( G' > 0 \) for all \( x \in [0,1] \) and \( G(0) = U > 0 \). As \( G \) is differentiable, (13) implies the necessary first order condition for optimality is:

\[
1 - v(x,.) \frac{h(\hat{x},.)G'(\hat{x})}{G(\hat{x})} \frac{\partial \hat{x}}{\partial w} = 0,
\]

where \( \hat{x}(w,.) \) solves \( w = w(\hat{x},.) \). As \( \partial \hat{x}/dw = [1/\partial w] \), (14) implies (15) below is a necessary condition for equilibrium. The proof of Proposition 1 establishes it is also sufficient.

**Proposition 1.** Equilibrium implies \( w(.) \) is the solution to the initial value problem:

\[
\frac{\partial w}{\partial x} = \frac{v(x,. )h(x,. )G'(x)}{G(x)} \quad \text{for all } x \in [0,1]
\]

with \( w(0,. ) = b \).

**Proof. In the online appendix.**

With random outside offers, this wage outcome has the structure of a first price auction with independent private values and a stochastic number of bidders (see McAfee and McMillan (1987) for the static case). Firm \( x \) does not “bid” \( w' < w(x,.) \) as it yields a too high quit rate, while \( w' > w(x,.) \) is not optimal as the corresponding fall in the quit rate is not worthwhile. As firms make strictly positive profit, Proposition 1 implies \( w(.) \) is indeed continuous and strictly increasing in \( x \). The firm’s optimal wage \( w = w(x,.) \) thus fully reveals its type.

## 4 Existence of Equilibria.

Characterizing non-steady state equilibria is potentially complex as the state variable \( G(.) \) evolves endogenously over time and is infinitely dimensional. However, the problem is tractable when there are a finite number of types. In this section, we fully characterize such equilibria for the (limiting) case that firms are equally productive \( p(x) = \overline{p} \), and then go on to provide results for the heterogenous firm case.

### 4.1 The Homogenous Firm Case.

Given equally productive firms, let \( v_t = v(G_t) \) denote the value of an employee in aggregate state \( G_t \). As optimal recruitment effort \( h^*(v_t) \) is the same
for all firms, (10) implies equilibrium \( \lambda_t = \lambda(G_t) \) given by:

\[
\lambda_t = \int_0^1 \frac{h^*(v_t) dG_t(z)}{G_t(z)} = -h^*(v_t) \ln U_t. \tag{16}
\]

Equilibrium \( \lambda_t \) thus depends on \( v_t \) and the unemployment rate \( U_t \) but is otherwise independent of \( G_t \). Putting \( x = 0 \) in (4) and using (16) now yields

\[
\dot{v}_t = (r + \delta + \mu - h^*(v_t) \ln U_t) v_t - \left( \overline{\nu} - b + \max_{h \geq 0} [h v_t - c(h)] \right). \tag{17}
\]

As flow of entrepreneurs become employees on entry,

\[
\dot{U}_t = \delta (1 - U_t) - \lambda_t U_t - \mu \tag{18}
\]

Equations (17) and (18) describe a pair of autonomous differential equations for \((v, U)\). The information relevant market state has reduced from the entire distribution \( G \) to a singleton, the unemployment rate \( U_t = G_t(0) \). (17) reveals the underlying economic structure of the model: each firm’s recruitment effort \( h \) is a best response to the recruitment strategies \( h^* \) of the other firms. In particular increased competition for a firm’s employees, via higher recruitment rates \( h^* \) of outside firms, makes employee retention more costly. Given equilibrium wages determined according to Proposition 1, (17) and (18) determine equilibrium recruitment rate \( h^*(v_t) \) where \( h_t = h^*(v_t) \) is the best response of each firm given all others choose \( h^* \).

Coles and Mortensen (2012) establish the system of differential equations composed of (17) and (18) has a single steady state which is a saddle point. Figure 1 depicts its phase diagram. Because \( v_t \) on a solution trajectory above the stable saddle path must eventually exceed \( \overline{\nu} \), while any one below the steady state ultimately yields zero \( v \) (which contradicts optimal firm behavior), the stable saddle path represents the only equilibrium solution.

The equilibrium unemployment dynamics are stable but adjustment is not immediate. Periods of high unemployment imply lower job offer rates \( \lambda_t \) which, by reducing competition for a firm’s employees, increases the value of an employee \( v_t \). As firms respond by increasing recruitment effort \( h^* \), this sustains a gradual recovery to the steady state.
Figure 1: Phase Diagram for Equations (17) and (18)
Proposition 1 implies the equilibrium wage is:

\[ w(x, t) = b + h^*(u_t) v_t \ln \left( \frac{G_t(x)}{U_t} \right). \]

In other words, wages are disperse even in the limiting one firm type case as in BM. Wages are tied down by \( w = b \) at \( x = 0 \) but are otherwise forward looking, noting that \( v_t = v(U_t) \) depends on how unemployment rates are expected to evolve in the future. Note that all wages generally increase with the value of a worker, \( v \), but decrease with the measure of unemployment along the adjustment path at each decile of the wage distribution. Hence, the model offers the possibility of rich wage dynamics in response to aggregate shocks.

As firm growth rates are independent of firm size, firm size evolution is consistent with Gibrat’s law. Coles and Mortensen (2012) provides a complete discussion of the micro-implications of this framework.

### 4.2 Heterogeneous Firms.

Consider now a finite number of firm productivity states \((p_1, \ldots, p_I)\) where \( p_1 > b \) and where \( p_i - p_{i-1} = \Delta_p > 0 \) for all \( i = 2, \ldots, I \). Corresponding to this grid is a partition \( (a_{i-1}, a_i] \subseteq [0, 1] \) with \( a_0 = 0, a_{i-1} < a_i \) and \( a_I = 1 \) such that firms with rank \( x \in (a_{i-1}, a_i] \) have productivity \( p(x) = p_i \). Assume \( \Gamma_1(.) \) satisfies \( \Gamma_1(a_i|x) - \Gamma_1(a_{i-1}|x) = \theta_{jk} \) for all \( x \in (a_{j-1}, a_j] \) so that the transition probability from productivity state \( j \) to state \( k \) is independent of the firm’s rank in state \( j \). \( \gamma_{ij} = \gamma \theta_{ij} \) denotes the transition rate from state \( i \) to \( j \) and \( \Gamma_{0i} \) is the probability that a start-up has type no greater than \( i \).\(^7\)

Let \( G_i = G(a_i) \) denote the number of unemployed workers plus those employees in firms of type \( i \) or less, so \( G_0 = U \) and \( G_I = 1 \). We now show that the relevant market state \( G(.) \) reduces to the finite state vector \( G = (G_0, G_1, G_2, \ldots, G_{I-1}) \). Let \( v_i(G) \) denote the value of an employee in an \( i \) firm in aggregate state \( G \) and \( v = (v_1, v_2, \ldots, v_I) \) the vector of employee values. Lemma 4 now solves the equilibrium conditions for wage \( w_i = w_i(v, G) \), which is the wage paid by firm \( x = a_i \), and its corresponding quit rate \( q_i = q_i(v, G) \). This yields an autonomous differential equation system for \( (\dot{v}, \dot{G}) \).

\(^7\)Note \( \gamma_{ij} = 0 \) for all \( j > 1 \) as state \( i = 1 \) is absorbing.
Lemma 3. Equilibrium implies:

\[
\dot{v}_i = (r + \delta + q_i(\cdot))v_i - \left[ p_i - w_i(\cdot) + \max_{h \geq 0} \{ hv_i - c(h) \} + \sum_j \gamma_{ij} (v_j - v_i) \right], \quad i = 1, \ldots, I
\]  

(19)

\[
\dot{G}_i = \mu (\Gamma_{0i} - 1) + \delta (1 - G_i) - q_i(\cdot)G_i
\]  

+ \sum_{k=i+1}^{I} \left[ \Sigma_{j \leq i \gamma_{kj}} (G_k - G_{k-1}) - \sum_{k=1}^{i} \left[ \Sigma_{j > i \gamma_{kj}} (G_k - G_{k-1}) \right] \right], \quad i = 0, \ldots, I - 1
\]

(20)

where for \(i = 0, \ldots, I\)

\[
q_i(v, G) = \sum_{j=i+1}^{I} h^*(v_j) \left[ \ln G_j - \ln G_{j-1} \right]
\]  

(21)

\[
w_i(v, G) = b + \sum_{j=1}^{i} v_j h^*(v_j) \left[ \ln G_j - \ln G_{j-1} \right].
\]  

(22)

Proof. In an online appendix.

The equilibrium vector value \(v(G) = (v_1(G), \ldots, v_I(G))\) is a particular solution to this differential equation system, given an arbitrary initial distribution of workers \(G^0\) and the transversality condition (5) for each type \(i\).

Theorem 1 A steady state equilibrium exists in the case of a finite number of types.

Proof. Any vector of steady state values satisfies

\[
v_i = \frac{p_i - w_i(v, G) + \max_{h \geq 0} \{ hv_i - c(h) \} + \sum_j \gamma_{ij} (v_j - v_i)}{r + \delta + \mu + q_i(v, G)}, \quad i = 1, \ldots, I
\]

(23)

for each \(i = 1, \ldots, I\). The steady state vector of employment measures solves

\[
\dot{G}_i = \frac{\delta + \mu + \mu_0 (\Gamma_{0i} - 1) + \sum_{k=i+1}^{I} \left[ \Sigma_{j \leq i \gamma_{kj}} (G_k - G_{k-1}) \right] - \sum_{k=1}^{i} \left[ \Sigma_{j > i \gamma_{kj}} (G_k - G_{k-1}) \right]}{\delta + \mu + q_i(v, G)}, \quad i < I
\]

(24)

\[
G_0 = \frac{\delta + \mu - \mu_0}{\delta + \mu + q_0(v, G)} \text{ and } G_I = 1.
\]
As \( q_i(v, G) \) and \( w_i(v, G) \), \( i = 1, \ldots, I \), defined in (21) and (22) are real continuous functions, the RHSs of (23) and (24) jointly form a continuous map from the bounded real vector space \([0, \overline{\gamma}]^I \times [0, 1]^{I+1}\) into itself. Hence a fixed point exists by Brouwer’s fixed point theorem.

Establishing that there is only one steady state equilibrium is a more difficult task. Nevertheless it is unique in the case of no productivity shocks, \( \gamma = 0 \), and a large number of types. As

\[
\lim_{p_j - p_{j-1} \to 0} \left[ \ln G_j - \ln G_{j-1} \right] = \frac{G'(p)}{G(p)} \text{,}
\]

the continuous analogues of the steady state conditions above when \( \gamma = 0 \) can be written as

\[
v(p) = \frac{p - w(p) + \max_{h \geq 0} \{ hv(p) - c(h) \}}{r + \delta + \mu + q(p)} ,
\]

\[
G(p) = \frac{\delta + \mu + \mu_0 (\Gamma(p) - 1)}{\delta + \mu + q(p)} ,
\]

for all \( p \in [\underline{p}, \overline{p}] \) where \( \Gamma_0(p) \) is the distribution of productivity over firms at birth and the quit rate function is

\[
q(p) = \int_{\underline{p}}^{\overline{p}} h^*(v(z)) \frac{dG(z)}{G(z)}
\]

and the wage rate function is

\[
w(p) = b + \int_{\underline{p}}^{p} v(z) h^*(v(z)) \frac{dG(z)}{G(z)}
\]

A steady state is a collection of functions, \( v(p), G(p), q(p), \) and \( w(p) \), that satisfy these equations.

**Theorem 2** In the case of a continuum of types, a unique steady state equilibrium exists when firm productivity is permanent.

**Proof.** In the online appendix.

The fact that the state is a finite vector of size equal to the number of types facilitates an empirical implementation of the model. In any numerical example, one can find steady state equilibria by iterating equations (21) and
(22) constrained by equations (23) and (24) and then compute the eigenvalues of ODE system at the steady state. If the computed equilibrium is a saddle, then the stable trajectories generated by a linear approximation to the system near the steady state is the locally unique equilibrium function $v(G)$.

Although a general analytic proof of existence of a non-steady state is not currently available, we can formally establish that a unique non-steady state equilibrium exists in the case in which the hire rate $h = \hat{h}$ is exogenous. This case holds exactly when the hire cost function takes the form $c(h) = 0$ for all $h \leq \hat{h}$ and $c(h) = \infty$ for all $h > \hat{h}$ which lies at the boundary of the set of convex cost functions of interest. Given $h_i = \hat{h}$ for all $i$, Lemma 3 then implies $q_i = -\hat{h} \ln G_i$ and thus

$$
\dot{G}_i = \mu (\Gamma_{0i} - 1) + \delta (1 - G_i) - \hat{h} \ln G_i G_i
+ \sum_{k=i+1}^{I} \left[ \sum_{\gamma_{kj} \leq i} (G_k - G_{k-1}) - \sum_{\gamma_{kj} > i} (G_k - G_{k-1}) \right]
$$

for $i = 0, \ldots, I - 1$ with $G_I = 1$. As one can easily show that $G_i \in (0, 1)$ for all $i < I$, the initial condition $G^0$ uniquely determines the equilibrium employment dynamic. The associated value vector $v(G)$ is the unique forward solution to the equations of (19) along the equilibrium employment path conditional on $G^0 = G$ for all of its feasible values. That the forward solution exists follows by Lemma 3. Note that this solution approximates the case in which the generic hire function $h^*(v)$ is highly inelastic.8

5 Conclusion.

This paper has identified a new tractable equilibrium model of job and labor flows that seems ideal for both macro-policy applications and micro-empirical analysis. Novel features include a hiring cost structure which exhibits constant returns to scale, heterogeneous firm productivity that is subject to shocks and a wage renegotiation mechanism consistent with private information. Firm size dynamics are correspondingly rich as small start-up companies enter the market over time, some older, existing firms exit and employees keep searching on-the-job for preferred employment prospects. Coles

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8This proof is essentially the same as the existence and uniqueness proofs in Robin (2011) and Moscarini and Postel-Vinay (2010).
and Mortensen (2012) discuss the new micro-empirical implications of this framework. Future research will extend the framework to stochastic aggregate shocks and endogenous start-ups.

References


6 Appendix [to be put online].

Proof of Lemma 3.

The forward solution to (4) that satisfies the transversality condition (5) along any arbitrary future time path for the state \(\{G_t\}_{t=0}^{\infty}\) is the fixed point of the following transformation

\[
(Tv)(x, G_0) = \int_0^{\infty} \max_{w_t, h_t \geq 0} \left( p(x) - w_t + h_t v(x, G_t) - c(h_t) + \gamma \int_0^1 v(z, G_t) d\Gamma_1(z|x) \right) \\
x \exp \left( -\int_0^t (r + \delta + \gamma + \mu + q(\hat{x}(w_z, G_z), G_z)) dz \right) dt.
\]

As \(q(\hat{x}(w_z, .), .)) \geq 0\) in general and \(w_t \geq b\) by Lemma 1, it follows by (??) that

\[
(Tv)(x, G_0) \leq \int_0^{\infty} \max_{h_t \geq 0} \left( p(x) - b + h_t \mathbf{\overline{\sigma}} - c(h_t) + \gamma \mathbf{\overline{\sigma}} \right) e^{-(r+\delta+\gamma+\mu)t} dt \\
\leq \frac{\max_{h_t \geq 0} \left( \mathbf{\overline{\sigma}} - b + h_t \mathbf{\overline{\sigma}} - c(h) + \gamma \mathbf{\overline{\sigma}} \right)}{r + \delta + \gamma + \mu} = \mathbf{\overline{v}}
\]

for any \(v(x, G) \leq \mathbf{\overline{v}}\). Because \(p(x)\) is increasing in \(x\) and \(\Gamma_1(., |x)\) is stochastically increasing in \(x\), \((Tv)(x, G_t)\) is increasing in \(x\) if \(v(x, G)\) is increasing in \(x\). Thus the transformation \(T\) maps the set of uniformly bounded functions that are increasing in \(x\) into itself. Further, the transformation \(T\) is
increasing and

\[ T(v(x, G_0) + k) = v(x, G_0) + |k| \int_0^\infty (h^*(v) + \gamma) \times \exp \left( - \int_0^t (r + \delta + \gamma + \mu + q(\tilde{x}(w_\lambda, G_\lambda), G_\lambda)) dz \right) dt \]

\[ \leq v(x, G_0) + \frac{h^*(\pi) + \gamma}{r + \delta + \mu + \gamma} |k| \text{ for all } G_0 \]

because \( q(\tilde{x}(w_\lambda, \cdot), \cdot) \geq 0 \) and \( h^*(v(x, s)) \leq h^*(\pi) \) for any \( v(x, s) \leq \pi \). As the proof of Lemma 2 establishes \( h^*(\pi) < r + \delta + \mu \), the map thus satisfies Blackwell’s condition for a contraction map and so guarantees that a unique fixed point exists in the set of bounded functions increasing in \( x \). This completes the proof of Lemma 3.

**Proof of Proposition 1.** To show (15) is sufficient, let \( w(., G) \) denote the solution to the initial value problem defined in Proposition 1. As \( G(0) = U > 0 \), this solution is continuous and strictly increasing in \( x \).

Now consider any firm \( x \in (0, 1] \) and let

\[ C(w, G) = w + v(x, G) \int_{\tilde{x}(w, G)}^1 \frac{h(z, G) dG(z)}{G(z)} \]

describe the mininum in (13). If the firm sets a lower wage \( w' = w(x', .) < w \) with \( x' \in [0, x) \), its employees believe \( \tilde{x} = x' < x \). Hence

\[ \frac{\partial C}{\partial w'} = 1 - v(x, G) \frac{h(x', G) G'(x')}{G(x')} \frac{\partial \tilde{x}}{\partial w'} \]

for such \( w' \). But equation (14) implies

\[ 1 - v(x', .) \frac{h(x', G) G''(x')}{G'(x')} \frac{\partial \tilde{x}}{\partial w'} = 0 \]

at \( x' \) and combining yields

\[ \frac{\partial C}{\partial w'} = 1 - \frac{v(x, G)}{v(x', G)} < 0 \]

by lemma 3. Thus for \( w' < w(x, G) \), an increase in \( w' \) decreases cost \( C(.) \). The same argument establishes that increasing \( w' \) when \( x' \in (x, 1] \) is cost
increasing. Finally note for wages \( w' > w(1, G) \), the worker’s belief is fixed at \( \hat{x} = 1 \) and so higher wages are even further cost increasing, while wage \( w' < w(0, G) \) implies all workers quit and the firm makes zero profit. Hence given all other firms offer wages according to Proposition 1, the cost minimizing wage for any firm \( x \in [0, 1] \) is to offer \( w = w(x, G) \). This completes the proof of Proposition 1.

**Proof of lemma 3.**

As each type \( i \) firm \( x \in (x_{i-1}, x_i] \) enjoys the same value \( v_i \), optimal recruitment effort \( h^*(v_i) \) is the same for all such firms. Equation (11) then implies

\[
\lambda[1 - \tilde{F}(x_i, G)] \int_{x_i}^{1} \frac{h(z, G)dG(z)}{G(z)} = \sum_{j=i+1}^{I} \left[ h^*(v_j) \ln \frac{G_j}{G_{j-1}} \right].
\]  

(29)

Equal profit at all such firms implies the equilibrium wage equation

\[
w(x, G) + \lambda[1 - \tilde{F}(x, G)]v_i = w_{i-1} + v_i \sum_{j=i}^{I} \left[ h^*(v_j) \ln \frac{G_j}{G_{j-1}} \right]
\]

for all \( x \in (x_{i-1}, x_i] \). Putting \( x = x_i \) and using (29) one obtains

\[
w_i = w_{i-1} + v_i h^*(v_i) \ln \frac{G_i}{G_{i-1}}.
\]

As \( w_0 = b \) by Lemma 1, this recursion yields \( w_i(v, G) \) as stated in the lemma. The equation for \( q_i(v, G) \) follows from (29). The dynamics for \( G_j \) follow from standard turnover arguments. This completes the proof of Lemma 4.

**Proof of Theorem 2:**

The equilibrium wage function satisfies \( w'(p) = -v(p)q'(p) \) by equations (27). Given this fact, a differentiation of (25) implies

\[
v'(p) = \frac{1}{r + \delta + \mu + q(p) - h^*(v(p))}.
\]  

(30)

where \( h^*(v) = \arg\max_{h \geq 0} \{hv - c(h)\} \) by the envelope theorem. Further-
more, a differentiation of equations (27) and (26) yields

\[
q'(p) = -h^*(v(p)) \frac{G'(p)}{G(p)} \\
= -h^*(v(p)) \left( \frac{\mu_0 \Gamma_0'(p)}{\delta + \mu + \mu_0(\Gamma(p) - 1)} - \frac{q'(p)}{\delta + \mu + q(p)} \right) \\
= \frac{-h^*(v(p)) (\delta + \mu + q(p))}{\delta + \mu + q(p) - h^*(v(p))} \frac{\mu_0 \Gamma_0'(p)}{\delta + \mu + \mu_0(\Gamma(p) - 1)}.
\]

Any solution to these two differential equations consistent with the boundary conditions \(q(p) = 0\) and \(v(p) = v_0(q)\) where

\[
v_0(q) = \frac{p - b + \max_{h \geq 0} \{hv_0(q) - c(h)\}}{r + \delta + \mu + q}
\]

identifies a steady state. As this is a boundary value problem, existence and uniqueness of a solution are not guaranteed in general.

Figure 2 represents the phase diagram for the ODE system. As

\[
\frac{dv_0(q)}{dq} = \frac{v_0(q)}{r + \delta + \mu + q - h(v_0(q))}
\]

is the slope of the locus of point that satisfy equation (32), the curve labeled \(v_0(q)\) in the Figure converges asymptotically to zero as \(q\) becomes large. Although solutions trajectories shift with \(p\) (except in the case of a uniform distribution of productivity), they do not cross and their relative positions are as represented in the Figure where the arrows indicate the "direction of motion" as \(p\) increase from \(p\) to \(\overline{p}\).

Let the pair \(v(p, q)\) and \(q(p, q)\) represent the particular solution to the ODE system associated with a specific choice of \(q\). A steady state equilibrium, then, corresponds to a choice of \(q\) that satisfies

\[
q(\overline{p}, q) = q + \int_{p}^{\overline{p}} q'(p, q) dp = 0
\]

Obviously, \(q = 0\) is too small given \(\overline{p} > p\) and \(q = \infty\) is too large given \(\overline{p} < \infty\) as \(q'(p, q)\) is negative and finite. As \(q(p, q)\) is continuous in \(q\), at least one solution exists by the mean value theorem.
Figure 2: Phase Diagram for Equations (30) and (31)
For any choice of \( \theta \), suppose there exists some \( \epsilon \) such that \( \theta = v(p, \theta) = v(p, \theta + \epsilon) \) for any \( \epsilon > 0 \). As \( q < q + \epsilon \), the supposition implies that \( q(p, q) < q(p, q + \epsilon) \) and \( v(p, q) > v(p, q + \epsilon) \) for all \( \tilde{p} \) where \( \tilde{p} \) is the smallest solution to \( v(p, \tilde{q}) = v(p, \tilde{q} + \epsilon) \). These facts and that \( h^*(v) \) is increasing imply \( v'(p, \tilde{q}) > v'(p, \tilde{q} + \epsilon) \) by equation (30). Therefore,

\[
v(p, q) = v_0(q) + \int_0^p v'(p, q) \, dp > v_0(q + \epsilon) + \int_0^p v'(p, q + \epsilon) \, dp = v(p, q + \epsilon)
\]

which contradicts the supposition. Hence, an inspection of the phase diagram implies \( q(p, \tilde{q}) < q(p, \tilde{q} + \epsilon) \) for all \( p \in [p, \tilde{p}] \). As the same argument also yields \( q(p, \tilde{q}) < q(p, \tilde{q} - \epsilon) \), equation (34) has a unique solution.