The non-linear Calvo model at the zero bound: Some analytic solutions and numerical results

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very preliminary.
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Abstract

This paper shows closed form solutions for the non-linear Calvo model at the zero bond under the assumption that uncertainty is given by a two state Markov chain with an absorbing state. This allows us to explicitly compare the solution of the non-linear model to the better known log-linearized version. In line with the log-linear model, we confirm in the non-linear setting i) large drops in output as shocks become more persistent until bifurcation occurs, ii) large government spending multipliers that have to be above 1 and iii) the paradox of toil. These results are in contrast with some recent literature on non-linearities at the ZLB. Mostly this is because that literature assumes particular form of Rotemberg prices which leads to "implausible" large cost of price adjustment in a way we make precise. Overall the non-linear Calvo model behaves similarly as its linearized counterpart both qualitatively and quantitatively with some important caveats we document.

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1 Introduction

In this paper we derive closed form solutions for the non-linear Calvo model at the zero lower bound and verify that the results derived in the literature under the log-linearization go through qualitatively and to a large extent quantitatively in the non-linear model. This is important in light of recent findings in the literature which give rise to scepticism of the results derived in the class of New Keynesian models under log-linearization.

Using a linearized Calvo price adjustment based New Keynesian model, Christiano, Eichenbaum & Rebelo (2012), Eggertsson (2011), Woodford (2011) found that at the zero lower bound, the government spending policy is expansionary (multiplier is more than 1) while tax cuts are contractionary (also known as the 'paradox of toil'). Braun, Korber & Waki (2013) argue that the these results, derived under log-linearization, no longer apply when one considers a fully non-linear model with quadratic costs of price adjustment (or the Rotemberg model). They choose this particular form of price rigidity because 1) it is tractable due to its quadratic nature and 2) under loglinearization and specific parameter calibrations, the Rotemberg and the Calvo rigidity coincide.

The earlier literature carried out the analysis with log-linearized Calvo model because one had to keep track of the price dispersion in a non-linear model. First we show how a non-linear Calvo model can be made tractable using the specific labor makerts assumption of Woodford (2003), which gets rid of the price dispersion as a state variable[1]. Having, reduced the state space of our system, we impose the Eggertsson-Woodford (2003) assumption of a two state Markov chain on the evolution of the shock process to derive the main results: with probability μ, the economy stays in the recession/depression with nominal interest rate stuck at the zero lower bound and with reminaing probability $1-\mu$, the economy escapes the bad state. Doing this helps us explicitly plot the Aggregate Supply (AS or the New Keynesian Phillips Curve) and the Aggregate Demand (AD)curves which can then be compared with the analogous graphs in the linearized model.

We find that the non-linear model exhibits multiple equilibria when the zero lower bound on the nominal interest rate is binding. However, only one equilibrium survives the local determinacy criterion. The closed form solutions around that stable equilibrium show that the Calvo rigidity based non-linear and the linearized models are qualitatively very similar: The nonlinear AD curve is upward sloping and is steeper than the nonlinear AS curve as in the linearized model. A temporary increase in government spending (for the duration of binding zlb) is expansionary while a temporary tax cut is contractionary. As the probability of escape $1-\mu$ is reduced, the economy is expected to stay in the contractionary state for a longer duration and so the output and inflation drop is larger. The multipliers also become larger with the reduction in $1-\mu$ and they asymptote to a finite limit as $\mu \longrightarrow \bar{\mu}$. This is different in comparison with the linearized model where the multipliers explode to infinity at the limit.

We show these results for two scenarios - 'Great Depression' (GD) where output dropped by 30% and inflation by 10% and for 'Great Recession' (GR) scenario with output drop of 10% and inflation drop of 2%. In order to do this, we estimate the fully non-linear model under these scenarios using Bayesian methods. The table below previews our main results with regard to the government spending and tax multipliers.

[1] This tractable feature has been present in several past papers but was never used because one needs to keep track of price dispersion to undertake welfare analysis.

A temporary increase of 1% of government spending during the contractionary GD episode can lead to an increase in output by 1.71% in the nonlinear model compared to 2.29% in the linear model. Similarly a labor tax cut of 1% of steady state output leads to a reduction in output of 1.09% in the non-linear model. The size of multipliers is milder for the Great Recession episode.

Calibrations	GD in Non-Linear	GD in Linear	GR in Non Linear	GR in Linear
$\frac{\partial \hat{Y}}{\partial G}$	1.7141	2.2864	1.2874	1.1943
$\frac{\partial \hat{Y}}{\partial \tau^w}$	1.0863	1.1711	0.1805	0.1640

Further we show that because of the quadratic nature of its cost of adjustments, the Rotemberg model is more sensitive to non-linearities than a Calvo model. The resource costs very quickly become large (more than 100% for 8% drop in inflation) and hence the Rotemberg model may not be as handy for a robustness check.

The outline of the paper is as follows: In Section 2 we describe the basic benchmark Calvo model, show how tractability is achieved in the nonlinear model, derive analytical solutions for AS and AD curves. In section 3, we plot the analytical solutions for comparative statics with increase in government spending and tax cuts. In section 4, we undertake a Bayesian estimation of the model parameters and shocks and show how the linear and nonlienar models deliver similar results. In section 5, we present some evidence that there might be some problems when undertaking a non-linear analysis with the Rotemberg model and then we conclude.

2 Non-linear Calvo at the zero bound

2.1 Basic Calvo model in nonlinear form

In this section, we introduce the basic framework that we work with and clarify that we do not need to keep track of price dispersion as a state variable. We use a standard textbook model [9] with a representative household and industry specific labor markets. The household maximizes the discounted expected utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \xi_t \left[\frac{C_t^{1-\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}} - \lambda \frac{\int_0^1 n_t(i)^{1+\omega} di}{1+\omega} \right]$$

where C_t is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods

$$C_t = \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$

with an elasticity of substitution $\theta > 1$, $n_t(i)$ is the quantity of labor supplied of type i and $\omega > 0$ is the inverse of the Frisch elasticity of labor. Household's budget constraint:

$$\int_0^1 p_t(i) c_t(i) di + \mathbb{E}_t Q_{t,t+1} B_{t+1} \leq B_t + (1-\tau_t^w) \int_0^1 w_t(i) n_t(i) di + \int_0^1 \Pi_t(i) di,$$

where $p_t(i)$ is the price of good i in period t, $Q_{t,t+1}$ is the stochastic discount factor in period $t+1$ with respect to t, B_{t+1} denotes the state-contingent payoffs to the portfolio of financial assets purchased by the household in period t and sold in period $t+1$, $w_t(i)$ is the nominal wage rate in the ith industry in the economy, and $\Pi_t(i)$ is the nominal profits from the sale of good i.

The household demand for the differentiated good i is

$$c_t(i) = C_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta}$$

where P_t is the price level in period t, given by

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}$$

so that $P_t C_t = \int_0^1 p_t(i) c_t(i) di$.

Household's maximization problem yields the following first-order conditions:

$$Q_{t,T} = \beta^{T-t} \frac{\xi_T}{\xi_t} \left(\frac{C_T}{C_t}\right)^{-\tilde{\sigma}^{-1}} \frac{P_t}{P_T}$$

$$(1 - \tau_t^w) \frac{w_t(i)}{P_t} = \lambda n_t(i)^{\omega} C_t^{\tilde{\sigma}^{-1}}$$

and a transversality condition.

Every good i is produced by a monopolistically competitive firm. Production function of the firm is assumed to be linear in industry-specific-labor (and each firm takes factor prices given):

$$y_t(i) = n_t(i)$$

Furthermore the market clearing requires $y_t(i) = c_t(i)$ for every good i at all times t. This implies $Y_t = C_t$.

Using Calvo (1983) price-setting assumption, a fraction $0 < \alpha < 1$ of goods prices remain unchanged any period. With probability $1 - \alpha$, a firm is selected at random (independent of time of last adjustment) to adjust its good's price. Let p_t^* be the optimal reset price in period t. The Dixit-Stiglitz price index in perid t satisfies

$$\begin{aligned} P_t^{1-\theta} &\equiv \int_0^1 p_t(i)^{1-\theta} di \\ &= (1-\alpha) p_t^{*1-\theta} + \alpha \int_0^1 p_{t-1}(i)^{1-\theta} di \\ &= (1-\alpha) p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta} \end{aligned}$$

A supplier that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$\mathbb{E}_t \left\{ \sum_{j=t}^{\infty} \alpha^{j-t} Q_{t,j} \Pi\left(p_t(i), p_j^I, P_j, Y_j\right) \right\}$$

where α^{j-t} is the probability that a price last chosen in period t will not have been revised by period j, p_t^I is an index of prices charged in the industry I, P_t is the economy wide price index and the firm's profit function $\Pi(\cdot)$ is defined as

$$\Pi_t(i) = p_t(i)\left(\frac{p_t(i)}{P_t}\right)^{-\theta} Y_t - w_t^I \left(\frac{p_t(i)}{P_t}\right)^{-\theta} Y_t$$

The first order condition for optimal price setting is:

$$\mathbb{E}_t \left\{ \sum_{j=t}^{\infty} \alpha^{j-t} Q_{t,j} \Pi_1\left(p_t(i), p_j^I, P_j, Y_j\right) \right\} = 0$$

Since all firms in the Industry I reset the price in period t, this becomes:

$$\mathbb{E}_t \left\{ \sum_{j=t}^{\infty} \alpha^{j-t} Q_{t,j} \Pi_1\left(p_t(i), p_t(i), P_j, Y_j\right) \right\} = 0$$

where

$$\begin{aligned}\Pi_1\left(p_t(i), p_t(i), P_j, Y_j\right) &= (1-\theta)\left(\frac{p_t(i)}{P_j}\right)^{-\theta} Y_j + \theta \frac{w_t^I}{P_t}\left(\frac{p_t(i)}{P_t}\right)^{-\theta-1} Y_j \\ &= \left(\frac{p_t(i)}{P_j}\right)^{-\theta} Y_j \left[(1-\theta) + \theta \frac{\lambda}{(1-\tau_j^w)} C_j^{\tilde{\sigma}^{-1}} \left(\frac{p_t(i)}{P_j}\right)^{-\theta(1+\omega)-1} Y_j^{\omega}\right]\end{aligned}$$

Substituting the equilibrium value of the discount factor (and doing some algebra), we get:

$$\mathbb{E}_t \left\{ \sum_{j=t}^{\infty} (\alpha\beta)^{j-t} \xi_j C_j^{-\tilde{\sigma}^{-1}} \left(\frac{p_t^*}{P_j}\right)^{-\theta} Y_j \left[\frac{p_t^*}{P_j} - \frac{\theta}{\theta-1}\frac{\lambda}{(1-\tau_j^w)} C_j^{\tilde{\sigma}^{-1}} \left(\frac{p_t^*}{P_j}\right)^{-\theta\omega} Y_j^{\omega}\right]\right\} = 0$$

This gives the closed form solution :

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\theta}}$$

where F_t and K_t are aggregate variables given by:

$$F_t \equiv \mathbb{E}_t \sum_{j=t}^{\infty} (\alpha\beta)^{j-t} (1-\tau_j^w) \xi_j C_j^{-\tilde{\sigma}^{-1}} Y_j \left(\frac{P_j}{P_t}\right)^{\theta-1}$$

$$K_t \equiv \mathbb{E}_t \sum_{j=t}^{\infty} (\alpha\beta)^{j-t} \frac{\theta}{\theta-1} \lambda \xi_j Y_j^{1+\omega} \left(\frac{P_j}{P_t}\right)^{\theta(1+\omega)}$$

To summarize, the following equations define the equilibria

1. Euler Equation

$$1 = \beta(1+i_t)\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t}\right)^{-1/\tilde{\sigma}} \frac{\xi_{t+1}}{\xi_t} \frac{1}{\Pi_{t+1}}\right] \tag{1}$$

2. NK Phillips Curve

$$K_t = \frac{\theta}{\theta - 1}\lambda \xi_t Y_t^{1+\omega} + \alpha_c \beta \mathbb{E}_t \left[\Pi_{t+1}^{\theta(1+\omega)} K_{t+1} \right] \tag{2}$$

$$F_t = \xi_t C_t^{-\frac{1}{\sigma}} (1 - \tau_t^w) Y_t + \alpha \beta \mathbb{E}_t \left[\Pi_{t+1}^{(\theta-1)} F_{t+1} \right] \tag{3}$$

$$\frac{K_t}{F_t} = \left(\frac{1 - \alpha_c \Pi_t^{\theta-1}}{1 - \alpha_c} \right)^{\frac{1+\omega\theta}{1-\theta}} \tag{4}$$

3. Resource Constraint

$$Y_t = C_t + G_t \tag{5}$$

Price Dispersion at time t is defined as

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t} \right)^{-\theta(1+\omega)} di \tag{6}$$

Proposition 1 *In a representative agent New Keynesian economy with industry specific labor markets and Calvo price setting, an equilibrium is defined as a collection of stochastic processes for $\{C_t, Y_t, \Pi_t, K_t, F_t\}$ that solve eqns {1 - 5} given a path for $\{G_t, \tau_t^w\}$ determined by fiscal policy, a path for $\{i_t\}$ determined by monetary policy and a path for $\{\xi_t\}$. Price dispersion defined by eqn 6 does not feature as a state variable and is redundant.*

The reason the price dispersion does not enter the determination of the endogenous variables is pointed out in (Woodford) [9]. The reason is that the standard New Keynesian model assumes specific labor markets so that real marginal cost of supplying a given good depends only on $y_t(i)$, aggregate output Y_t, and the vector of aggregate shocks ξ_t. Alternatively in a model in which each firm hires from an economy-wide labor markets, then the real marginal cost instead depends also on an alternative measure of aggregate output $X_t = \int_0^1 y_t(i)di$ which in turn depends upon price dispersion. Using specific labor markets instead of economy wide factor market has important implications for real rigidities in the economy as stressed in [9], with the added benefit that one does not need to keep track of price dispersion as a state variable. Observe that if one wants to do welfare evalution, then price dispersion will enter social welfare, in which case one needs to keep track of this variable as in the opitmal polikcy exercise in Benigno and Woodford (2003).

2.2 Basic Calvo model in nonlinear form assuming a two state Markov chain with an absorbing state.

Having illustrated how the assumption of industry specific factor markets yields a perfectly forward looking system that does not feature a state variable, we can derive the closed form solution for the 'Aggregate Supply' and the 'Aggregate Demand' curves for the short run in the non-linear Calvo model. The key reason we can make this tractable is that we assume that the exogenous variable can only take on two value. In particular we assume ξ_t is a two-state Markov chain with an absorbing state.

More precisely we define the long run in the model, as in Eggertsson (2010), as a situation in which the shocks are at their steady state. The long run is an absorbing state. The short run, in contrast, is defined by period $t \geq 0$ at which time the shock ξ_S takes on a value below its steady state value $\xi_{L.}$ We assume that in each period $t < T^e$ when $\xi_t = \xi_S$ there is a constant probability $(1-\mu)$ that the shock reverts to its long run steady state value. T^e is the stochastic time period when the shock is back at the steady state level. So $t \geq T^e$ is defined as the long run.

We assume that the central bank sets and inflation target $\Pi_t = 1$ whenever it can. If this inflation target implies negative nominal interest rate, then we assume that instead it sets $i_t = 0$ in which case Π_t is endogenously determined. This is enough to pin down equilibrium and allows us to abstract from the issue of implementation of inflation policy outside of the zero bound, an important research topic that has been getting more and more attention.

As regards to the fiscal policy, we consider only temporary changes in government spending and/or labor taxes in the short run, i.e. variations in G_S and τ_S^w. They return back to steady state in the long run given by G_L and τ_L^w which we take to be some fixed numbers.

Under the assumption of a two state Markov process, the system of equations can be simplified into two equations that determined inflation and output in the short run. We choose units so that in steady state, i.e the long run, $Y_L = \xi_L = 1$, which impliles that $\lambda = (\frac{\theta-1}{\theta})(1-G_L)^{-\tilde{\sigma}^{-1}}(1-\tau_L^w)$. Furthermore, given equations (1)-(5) we have that in the long run $\Pi_L = Y_L = 1$, $1 + i_L = \beta^{-1}, C_L = 1 - G_L$ and

$$K_L = F_L = \frac{(1-\tau_L^w)(1-G_L)^{-\frac{1}{\tilde{\sigma}}}}{1-\alpha\beta} \tag{7}$$

Let us now turn to the determination of inflation and output in the short-run.

2.2.1 Deriving short run AS-AD curves

What makes the model particularly tractable is that it is purely forward looking, and our assumption of an absorbing steady state. This means that we can combine (1)-(5) into an AS and an AD curve which together determine Π_S and Y_S. In doing so we are assuming that the shock ξ_S is large enough so that the zero bound is binding, i.e., the solution $\Pi_S = 1$ would imply negative nominal interest rate. Accordingly, in writing aggregate demand we assume $i_S = 0$

1. **AD curve**: From the Euler equation (1) and the Resource Constraint (5), we obtain the AD curve relationship in the short run:

$$Y_S = G_S + (1-\bar{G})\left\{\left[1-\beta\mu\Pi_S^{-1}\right]\frac{\xi_S}{(1-\mu)\beta}\right\}^{\tilde{\sigma}} \tag{8}$$

2. **AS curve**: From the New-Keynesian Phillips curve (2)-(4) and the resource constraint (5), we can derive the AS curve in the short run:

$$\frac{\frac{\theta}{\theta-1}\lambda\xi_S Y_S^{1+\omega} + \alpha\beta(1-\mu)K_L}{\xi_S(Y_S-G_S)^{-\frac{1}{\tilde{\sigma}}}(1-\tau_S^w)Y_S + \alpha\beta\ (1-\mu)F_L}\frac{1-\alpha\beta\mu\Pi_S^{(\theta-1)}}{1-\alpha\beta\mu\Pi_S^{\theta(1+\omega)}} = \left(\frac{1-\alpha\Pi_S^{\theta-1}}{1-\alpha}\right)^{\frac{1+\omega\theta}{1-\theta}} \tag{9}$$

where K_L and F_L are given by (7)

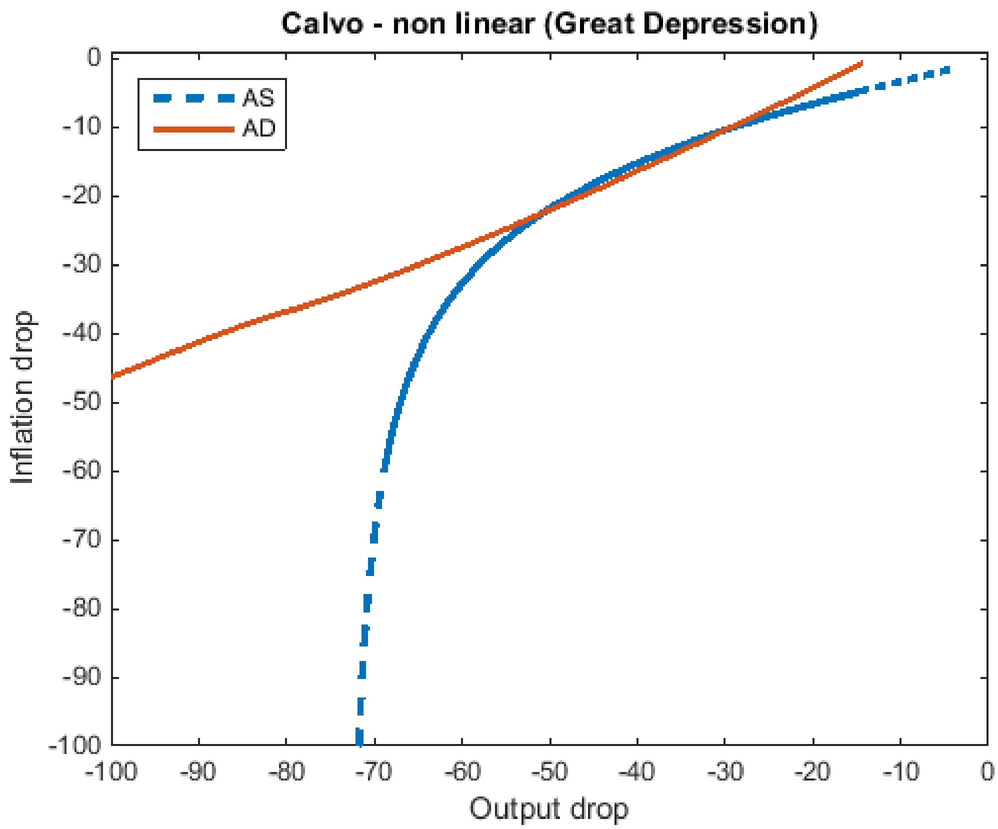


Figure 2.2.1 plots up these two curves. In plotting them up we have choosen parameters as in Eggertsson (2010), i.e. we want the model to deliver a Great Depression scenario of -30 percent drop in output and -10 percent drop in inflation where the two curves intersect. We will get into more details about these numbers shortly.

The red line is the Aggregate Demand curve (8) for the case when the zero lower bound is binding. It is an almost linear relationship between inflation and output. The main source of the non-linearity in the model is in dashed green line which is the Aggregate Supply curve (or the New Keynesian Phillips Curve) given by (9). As is obvious, there are two solutions to this system of equations (Moving from right to left, we call them the First and the Second Equilibrium respectively).

The first equilibria is the equilibria corresponding the a 30 percent output drop and 10 percent deflation. The second equilibria, however, is a hyperdeflation equilibria which is consistent with close to 30 percent drop in prices per annum and over 50 percent drop in output.

As we will see shortly, the first equilibria is very similar to, and has similar properties to the equilibria studied in the literature, such as Eggertsson and Woodford (2003). In particularly it is locally unique and determinate. The second equilibria, however, is indeterminate. This means that for small perturbation away from that steady state there are infinetely many solutions that locally satisfy the equilibrium conditions. Moreover, it can be show that the equilibria that is indeterminate is not learnable. Accordingly, in the next subsection, we will focus our analysis on the determinate equilibira.

We check for the determinacy of each solution as follows:

Under the two state assumption (with a binding zlb), the dynamic system can be written as:

1. Euler

$$0 = \beta\mu\mathbb{E}_t^{\mathbb{S}}\left[\left(\frac{Y_{t+1}^S - G^S}{Y_t^S - G^S}\right)^{-1/\tilde{\sigma}} \frac{1}{\Pi_{t+1}^S}\right] + \beta(1-\mu)\left[\left(\frac{1}{Y_t^S - G^S}\right)^{-1/\tilde{\sigma}} \frac{1}{\xi^S}\right] - 1 \equiv f^1(\mathbb{E}_t^{\mathbb{S}}Y_{t+1}^S, \mathbb{E}_t^{\mathbb{S}}\Pi_{t+1}^S, Y_t^S) \tag{10}$$

2. NK Phillips Curve

$$0 = \frac{\theta}{\theta-1}\lambda\xi^S(Y_t^S)^{1+\omega} + \alpha_c\beta(1-\mu)K^L + \alpha_c\beta\mu\mathbb{E}_t^{\mathbb{S}}\left[(\Pi_{t+1}^S)^{\theta(1+\omega)}K_{t+1}^S\right] - K_t^S \equiv f^2(\mathbb{E}_t^{\mathbb{S}}\Pi_{t+1}^S, \mathbb{E}_t^{\mathbb{S}}K_{t+1}^S, Y_t^S, K_t^S) \tag{11}$$

$$0 = \xi^S(Y_t^S - G^S)^{-\frac{1}{\sigma}}(1-\tau_t^w)Y_t^S + \alpha_c\beta(1-\mu)F^L + \alpha_c\beta\mu\mathbb{E}_t^{\mathbb{S}}\left[(\Pi_{t+1}^S)^{(\theta-1)}F_{t+1}^S\right] - F_t^S \equiv f^3(\mathbb{E}_t^{\mathbb{S}}\Pi_{t+1}^S, \mathbb{E}_t^{\mathbb{S}}F_{t+1}^S, Y_t^S, F_t^S) \tag{12}$$

$$0 = \frac{K_t^S}{F_t^S} - \left(\frac{1-\alpha_c(\Pi_t^S)^{\theta-1}}{1-\alpha_c}\right)^{\frac{1+\omega\theta}{1-\theta}} \equiv f^4(\Pi_t^S, K_t^S, F_t^S) \tag{13}$$

having substituted in the resource constraint and assuming that $G_{t+1}^S = G_t^S = G^S$. Observe that we denote by $\mathbb{E}^{\mathbb{S}}$ the expectation of each variable conditional on that the shock will remain in the low "short-run" state in the next period. Writing the system in this way, we have a regular rational expectation system which can be solved via standard methos, e.g. using Blanchard and Kahn.

Linearizing this system of equations yields:

$$\left[\frac{\partial f^1}{\partial \mathbb{E}_t^{\mathbb{S}}Y_{t+1}^S}\right]_{ss}\mathbb{E}^S\hat{Y}_{t+1}^S + \left[\frac{\partial f^1}{\partial Y_t^S}\right]_{ss}\hat{Y}_t^S + \left[\frac{\partial f^1}{\partial \mathbb{E}_t^{\mathbb{S}}\Pi_{t+1}^S}\right]_{ss}\mathbb{E}_t^S\pi_{t+1}^S = 0 \tag{14}$$

$$\left[\frac{\partial f^2}{\partial K_t^S}\right]_{ss}k_t^S + \left[\frac{\partial f^2}{\partial Y_t^S}\right]_{ss}\hat{Y}_t^S + \left[\frac{\partial f^2}{\partial \Pi_t^S}\right]_{ss}\pi_t^S + \left[\frac{\partial f^2}{\partial \mathbb{E}_t^{\mathbb{S}}K_{t+1}^S}\right]_{ss}\mathbb{E}^S\hat{K}_{t+2}^S = 0 \tag{15}$$

$$\left[\frac{\partial f^3}{\partial F_t^S}\right]\hat{F}_t^S + \left[\frac{\partial f^3}{\partial Y_t^S}\right]_{ss}\hat{Y}_t^S + \left[\frac{\partial f^3}{\partial \Pi_t^S}\right]_{ss}\pi_t^S + \left[\frac{\partial f^3}{\partial \mathbb{E}_t^{\mathbb{S}}F_{t+1}^S}\right]_{ss}\mathbb{E}_t^S\hat{F}_{t+1}^S = 0 \tag{16}$$

$$\left[\frac{\partial f^4}{\partial K_t^S}\right]_{ss}\hat{K}_t^S + \left[\frac{\partial f^4}{\partial F_t^S}\right]_{ss}\hat{F}_t^S + \left[\frac{\partial f^4}{\partial \Pi_t^S}\right]_{ss}\pi_t^S = 0 \tag{17}$$

where small letters $\{\hat{Y}_t^S, \pi_t^S, \hat{K}_t^S, \hat{F}_t^S\}$ indicate deviations of the variables from the solution around which the equations are linearized. This system of equations is determinate if there are at least 3 unstable eigenvalues outsite of the unit circle. Numerically (using reds and solds), we found that the second equilibrium fails to meet the determinacy conditions and hence is locally unstable. Henceforth, we only investigate the properties of the stable equilibrium (in this case, it is the one that produces 30% drop in Output and 10% drop in inflation). Note that as shown in Figure 1b, the AS and AD curves are almost linear up to the stable equilibrium[2]. Next, we compare this system to the solution obtained from a linearized system of equations.

[2] We are not suggesting there exists any deep scientific insight for this feature of the Calvo model. It just so happens that the non-linearities do not play a quantitatively big role in the non-linear Calvo model as compared to the Rotemberg model. Recent literature has spurred use of non-linear techniques. Our results are in no way meant to discredit these approaches. Rather we are very much interested in their development and merely point out in this paper that while those techniques are still under development, economic analysis with the linearized Calvo model in a New Keynesian setting is not off the mark.

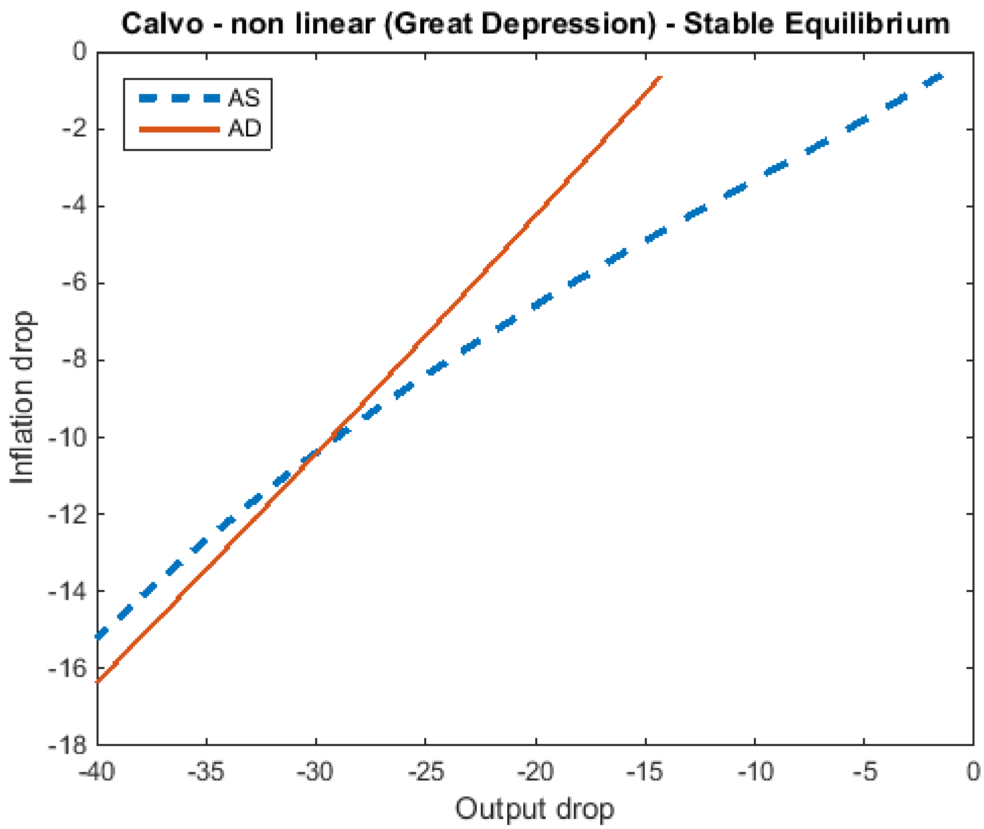


Figure 1b: Stable Equilibrium

2.3 Bifurcation and output collapses

A key prediction of the linearized New Keynesian model is that for small shocks, there can be a large drop in output. This occurs due to a bifurcation of the linearized model. A natural question is if this is an artifact of the linearization. Here we confirm this phenomenom in the nonlinear model. As was shown in Eggertsson (2011), the log linearized model under the assumption of a two state Markov chain with an absorbing state can be expressed as a system of AS-AD curves:

1. Aggregate Demand

$$\hat{Y}_S = \hat{G}_S + \frac{\sigma}{(1-\mu)}(\mu\pi_S - r^e_S)$$

   where $\hat{Y}_S \doteq \frac{Y_S - \bar{Y}}{\bar{Y}}$; $\hat{G}_S \doteq \frac{G_S - \bar{G}}{\bar{Y}}$; $\pi_S \doteq \Pi_S - 1$; $r^e_S \doteq \log\beta^{-1} + (1-\mu)\log\xi_S$ and $\tilde{\sigma} = \sigma(1-\bar{G})$

2. Aggregate Supply

$$(1-\beta\mu)\pi_S = \kappa\hat{Y}_S + \kappa\psi\left(\chi^w \hat{\tau^w_S} - \sigma^{-1}\hat{G}_S\right)$$

   where$\chi^w \equiv \frac{1}{1-\bar{\tau}^w}$; $\hat{\tau}^w_S \equiv \tau^w_S - \bar{\tau}_w$; $\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)(\omega+\sigma^{-1})}{\alpha(1+\omega\theta)}$ and $\psi \equiv \frac{1}{\sigma^{-1}+\omega}$

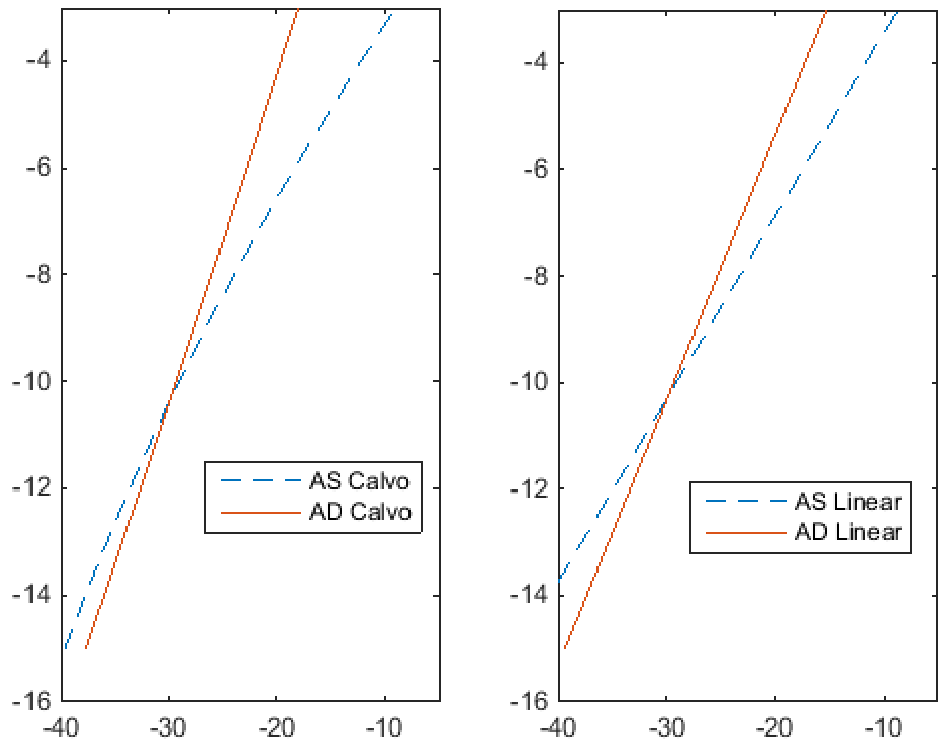


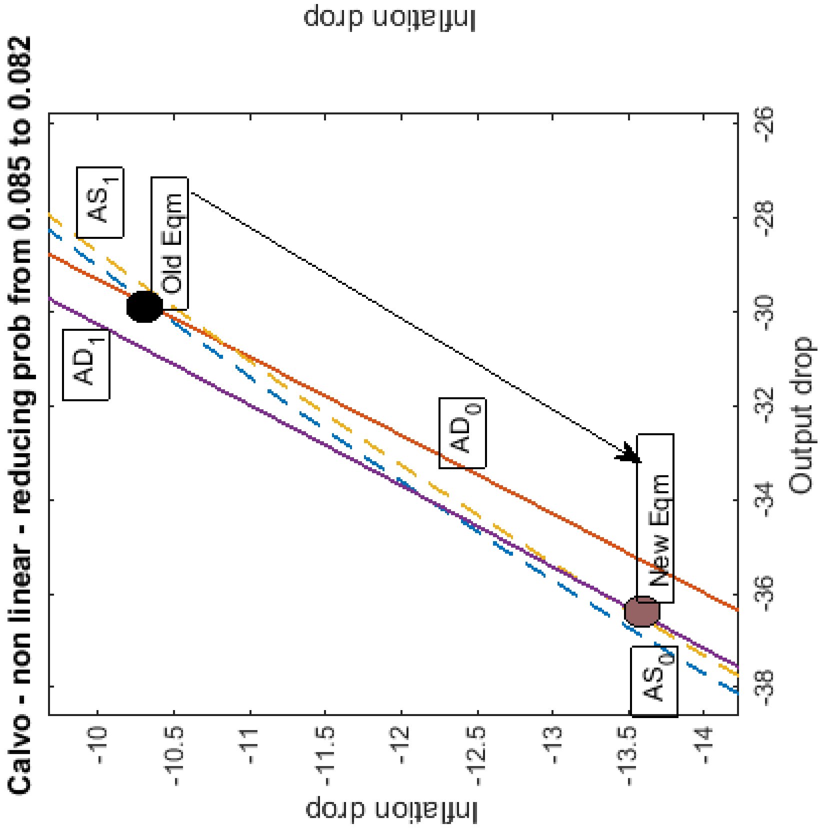
Figure 2 plots the AS and AD curves for the non-linear Calvo and the linear model. The non-linear and the linear models are calibrated[3] such that each generates a 30% drop in output and 10% drop in inflation at the intersection of the curves. Proposition 3 in Eggertsson 2011 establishes that the conditions for local determinacy in this economy at the zero lower bound is

$$L(\mu) \equiv (1-\mu)(1-\mu\beta) - \kappa\mu\sigma > 0 \tag{C2}$$

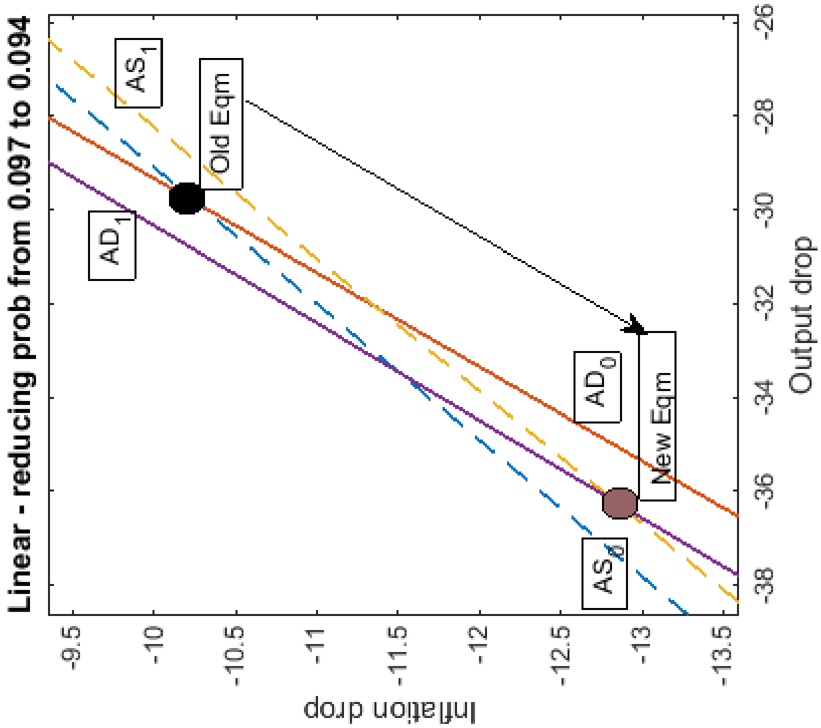
which is the as stating that the AD curve needs to be steeper than the AS curve. This will always be the case for a small enough μ, i.e. if the probability of going back to "long run" is large enough. As the probability to staying in the depresses state increases, the two curves become closer to paralell and the drop in output and inflation goes to -∞ and explodes until at a critical value $\bar{\mu}$ i), no solution exists (the lines are paralell). We call this the bifurcation point. When $\mu > \bar{\mu}$ there is interminacy in the linearized model. This explosive behavior is at the heart of the New Keynesian model, because it explains how "small" shocks can trigger large drops in output. In figure X we show an example of a simple comparativ static s as one increases μ,, i.e. it leads to bigger output drop and bigger deflation, while figure X shows the exact bifurcation point. Figure X shows how output and inflation behaves as a function of μ. As the figure illustrates output and prices drop without a bound converging to $-\infty$ at the bifurcation point.

[3]The calibration strategy involved a Bayesian estimation to match 30% output drop and 10% drop in inflation in the Great Depression scenario. It is elaborated in Denes and Eggertsson (2009). The calibrated parameters for the linear model are the same as reported in Eggertsson (2011). The parameters for the non-linear model are reported in Section 4.

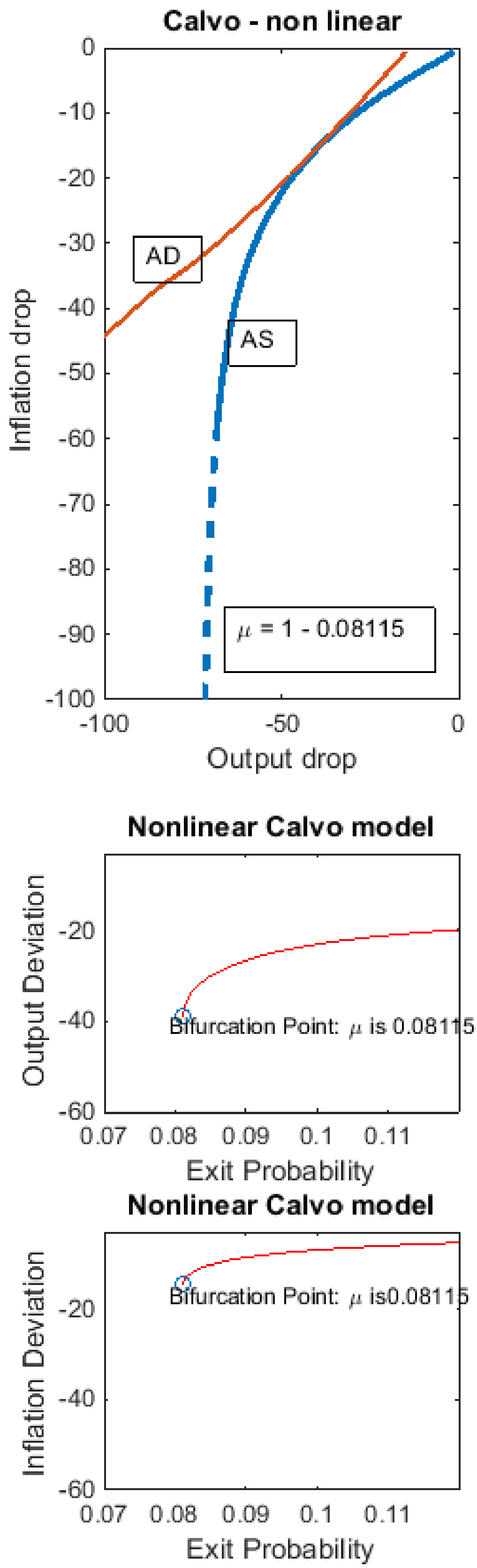
How does this explosive behavior show up in the non-linear model? Is it only an artifact of the linearization method as suggested by some authors? In the non-linear model the relative slope of the two curves in the normal equilibrium is always of the form shown in the figure in the normal equilibrium. As shown in figure X, then an incrase in μ again leads to the two curves becoming more and more paralell, and thus output and inflation drop more. In the non-linear model, however, output and inflation do not explode at the bifurcation point. Instead, as seen in figure X, bifurcation occur at μ=X where deflation is X and the output drop is X. As can be seen in figure X this happens as the two curves no longer interesect, i.e. the AS curve is convex and is tangent to the AD curve, at finite values of output and inflation. An interesting implication is that as one increases μ further then the non-linear model has no solution. Hence the indeterminacy region in the linearized model does not seem of much practical interest; in the non-linear counterpart this region of the papermeter space corresponds to non-existence of equilibria. The bound C2 us thus not only important for determinacy, it also is important for existence. We will now focus on the case in which the model does not bifurcate, and focus on the determinate equilibria in the non-linear model. To what extent do the qualitative and quantiative prediction of the linear and non-linear models co-incide?



Calvo - non linear - reducing prob from 0.085 to 0.082
Inflation drop
Output drop
AS1
AS0
AD1
AD0
Old Eqm
New Eqm
-10
-10.5
-11
-11.5
-12
-12.5
-13
-13.5
-14
-38
-36
-34
-32
-30
-28
-26



Linear - reducing prob from 0.097 to 0.094
Inflation drop
Output drop
AS1
AS0
AD1
AD0
Old Eqm
New Eqm
-9.5
-10
-10.5
-11
-11.5
-12
-12.5
-13
-13.5
-38
-36
-34
-32
-30
-28
-26



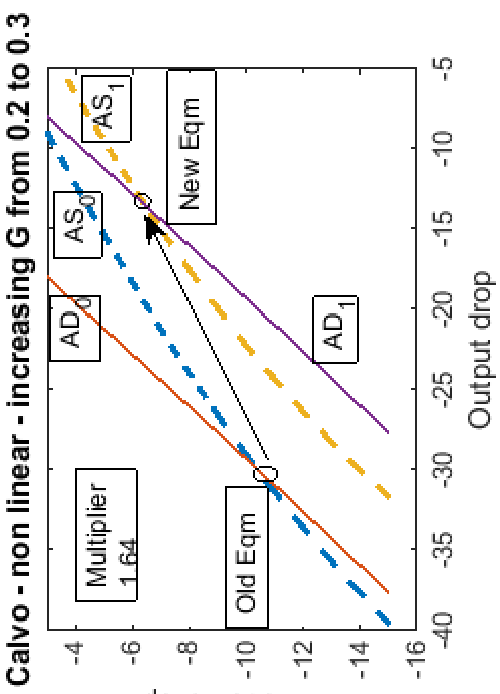
Calvo - non linear
Inflation drop
Output drop
AD
AS
μ = 1 - 0.08115
Linear
Inflation drop
Output drop
AD
AS
μ = 1 - 0.0812
Nonlinear Calvo model
Output Deviation
Exit Probability
Bifurcation Point: μ is 0.08115
Linearized model
Output Deviation
Exit Probability
μ is 1- 0.0812
Nonlinear Calvo model
Inflation Deviation
Exit Probability
Bifurcation Point: μ is0.08115
Linearized model
Inflation Deviation
Exit Probability
μ is 1- 0.0812

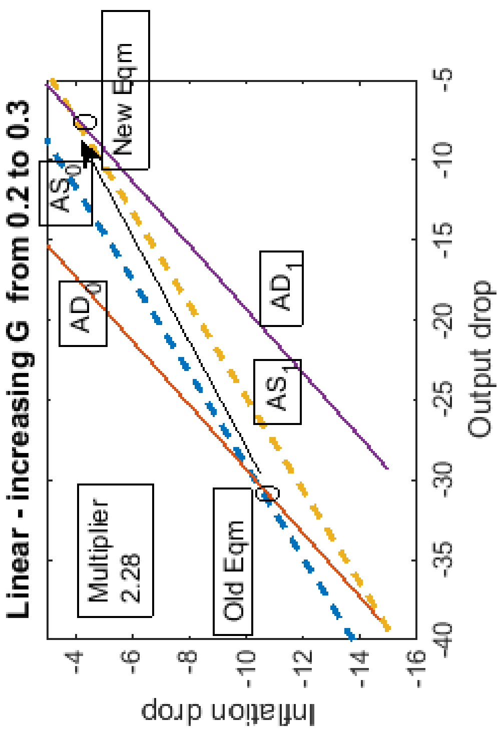
3 Multipliers in the linear and nonlinear model

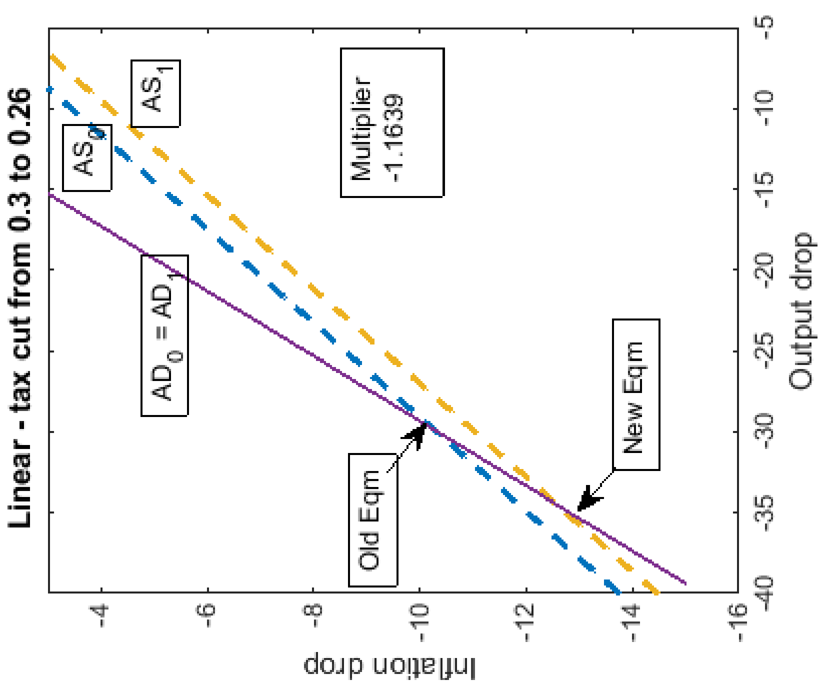
Table 1

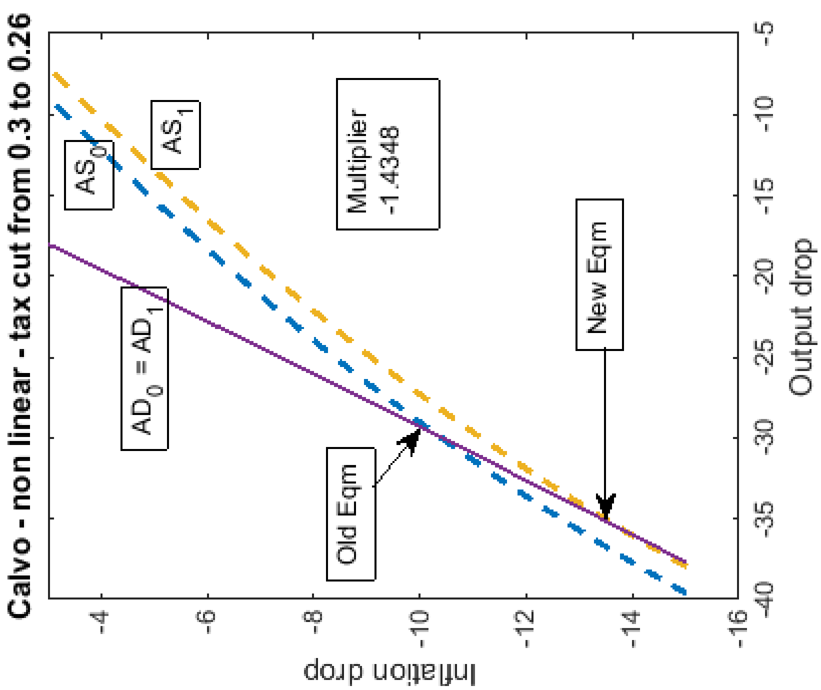
Calibrations	GD in Non-Linear	GD in Linear	GR in Non Linear	GR in Linear
$\frac{\partial \hat{Y}}{\partial G}$	1.7141	2.2864	1.2874	1.1943
$\frac{\partial \hat{Y}}{\partial \tau^w}$	1.0863	1.1711	0.1805	0.1640

Table 1 summarizes the multiplier of government spending and labor tax cuts in the model at zero interest rate in two numerical examples, to be discussed in better detail below. In one experiment Great Depression (GD) the parameters and shocks are chosen so to match 30 percent drop in output and 10 percent in inflationl this is the numbers we have already been using in our figures. In the second, Great Recession (GR), they are choosen to match 10 percent drop in output and 2 percent deflation. For the Great Depression scenario the multiplier is 1.7 in the non-linear model, and 2.3 in the nonlinear one. For the Great Recession scenario, however, the non-liner multiplier is larger but that difference is not large. Similar comments applies to the wage cut multiplier. Overall the message is that the multipliers are of similar order in the two models. Figure X shows the comparative static of increasing government spending in the linear and non-linear model. Qualitatively the effects are the same, since the relative slopes of the two curves are the same, albeit the exact number differ. We also show comparative static for labor tax cut in figure X.

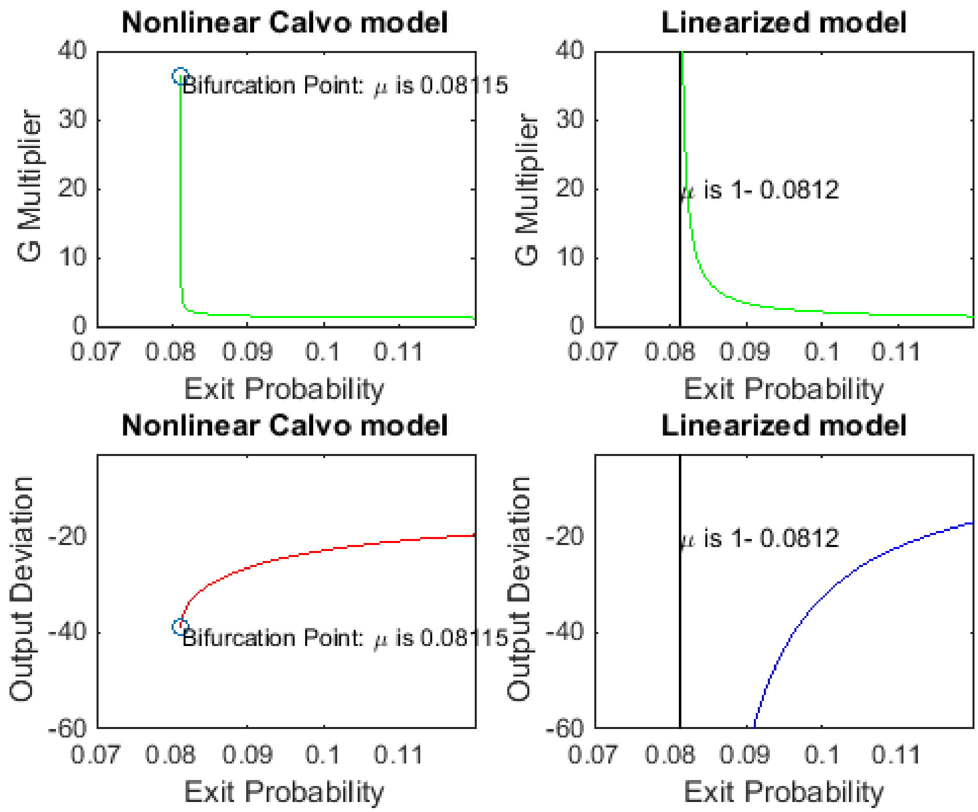

Calvo - non linear - increasing G from 0.2 to 0.3
Multiplier
1.64
Old Eqm
New Eqm
AD_0
AS_0
AS_1
AD_1
Inflation drop
Output drop
-4
-6
-8
-10
-12
-14
-16
-40
-35
-30
-25
-20
-15
-10
-5


Linear - increasing G from 0.2 to 0.3
Multiplier
2.28
Old Eqm
New Eqm
AD_0
AS_0
AS_1
AD_1
Inflation drop
Output drop
-4
-6
-8
-10
-12
-14
-16
-40
-35
-30
-25
-20
-15
-10
-5


Linear - tax cut from 0.3 to 0.26
Multiplier
-1.1639
AS0
AS1
AD0 = AD1
Old Eqm
New Eqm
Output drop
Inflation drop


Calvo - non linear - tax cut from 0.3 to 0.26
Multiplier
-1.4348
AS0
AS1
AD0 = AD1
Old Eqm
New Eqm
Output drop
Inflation drop

Perhaps more instructive is to look at the basic behavior of the multiplier accross the two models. The key economic insight of the literature that relied on linearization methods was that as the output drop become bigger, the government spending multiplier becomes larger. We see this phenomenom in figure X. In both models. As the probability of the staying at the zero bound increases, then the drop in output intensifies. In the linearized model this means that the multiplier of government spending goes to ∞ as the drop in output goes to -∞. In the non-linear model, however, bifurcation occurs at a finite level of output drop at about -40 percent. At that point the multiplier of government spending is about 35, hence the multiplier is increasing at a faster rate than the drop in output.



4 Quantitative comparison to the linearized model

In this section, we describe how we calibrated our model and then show the quantitative results. As was found in the earlier literature (Eggertsson 2011, Christiano, Eichenbaum & Rebelo 2012, Woodford 2011 among others), at the zero lower bound the government spending multiplier is larger than 1 and the tax cut multiplier is negative both for the 'Great Depression' and the 'Great Recession' scenarios.

We define the Great Depression scenario to be an event in which output dropped by 30% and inflation (annual) by 10%. On the other hand, the Great Recession is defined as a less extreme event with 10% drop in output and 2% drop in inflation.[4] Consequently we separately estimate the structural parameters and shocks in our model for each scenario by maximizing the posterior distribution of the model to match these

[4] Actual decline in inflation was much less than this. Del Negro, Giannoni & Schorfheide (2015) show how standard DSGE models with financial frictions can match the actual drop in inflation. We here abstract from financial frcitions primarily to keep it similar to previous literature where results shown in this paper were derived.

data points exactly while matching as closely as possible the priors that we choose for these parameters and shocks . The priors used are the same as used for estimating the linearized model in Denes, Gilbukh & Eggertsson (2013) reproduced below. The estimation method is described in more detail in Denes & Eggertsson (2009).

Priors for the structural parameters and the shocks				
	distribution	Prior 5%	Prior 50%	Prior 95%
α	beta	0.5757	0.6612	0.7402
β	beta	0.9949	0.9968	0.9981
$1-\mu$	beta	0.0198	0.074	0.1788
σ^{-1}	gamma	1.2545	1.9585	2.8871
ω	gamma	0.1519	0.82	2.4631
θ	gamma	3.7817	7.6283	13.4871
r_S^e	gamma	0.0036	0.0094	0.0196

Posteriors for Great Depression Calibration in Nonlinear Model				
	Posterior 5%	Posterior 50%	Posterior 95%	Mode
α	0.697	0.7632	0.8238	0.7704
β	0.9949	0.9968	0.9982	0.9971
$1-\mu$	0.0657	0.0951	0.1346	0.0852
σ^{-1}	0.725	1.1251	1.6853	1.111
ω	0.75	1.4957	2.4359	1.4012
θ	7.667	12.4474	19.1998	12.9272
r_S^e	-0.037	-0.0237	-0.0134	-0.0171

The posterior distribution of the fiscal multipliers (Great Depression) in nonlinear model				
	Posterior 5%	Posterior 50%	Posterior 95%	Mode
tax cut multiplier i=0:	-1.8744	-0.4998	0.1967	-1.0863
govt spending multiplier i=0:	1.0859	1.368	1.9714	1.7141

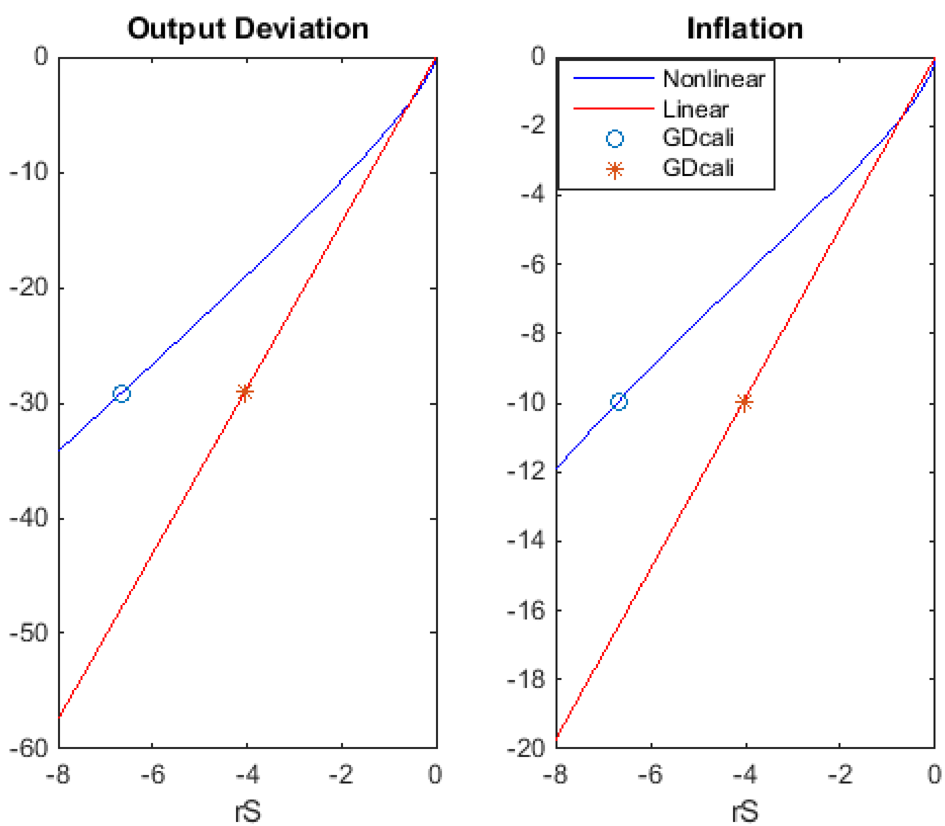
Posteriors for Great Recession Calibration in Nonlinear Model				
	Posterior 5%	Posterior 50%	Posterior 95%	Mode
α	0.6785	0.7456	0.806	0.7534
β	0.9949	0.9968	0.9981	0.997
$1-\mu$	0.0687	0.1071	0.1733	0.0892
σ^{-1}	0.933	1.4571	2.1747	1.4242
ω	1.2617	2.5929	4.7351	2.3944
θ	8.6289	13.5677	20.1804	13.4151
r_S^e	-0.0244	-0.014	-0.0079	-0.0099

The posterior distribution of the fiscal multipliers (Great Recession) in nonlinear model				
	Posterior 5%	Posterior 50%	Posterior 95%	Mode
tax cut multiplier i=0:	-0.2791	-0.1119	-0.0485	-0.1806
govt spending multiplier i=0:	1.0878	1.1878	1.3628	1.2874

4.1 Comparison with the Linear Model

Calibrations	GD in Non-Linear	GD in Linear	GR in Non Linear	GR in Linear
$\hat{Y}$	-29.91%	-29.93%	-9.8295	-9.80%
π	-10.23 %	-10.30 %	-2.9049	-2.02%
$\frac{\partial \hat{Y}}{\partial G}$	1.7141	2.2738	1.2874	1.1943
$\frac{\partial \hat{Y}}{\partial \tau^w}$	1.0863	1.1602	0.1806	0.1640
α	0.7704	0.7731	0.7534	0.7839
β	0.9971	0.9970	0.9970	0.9970
σ^{-1}	1.1110	1.1422	1.4242	1.2248
ω	1.4012	1.5685	2.3944	1.6930
θ	12.9272	12.6780	13.4151	13.2320
$1-\mu$	0.0852	0.0988	0.0892	0.1434
r^e_S	-0.0171	-0.0105	-0.0099	-0.0129

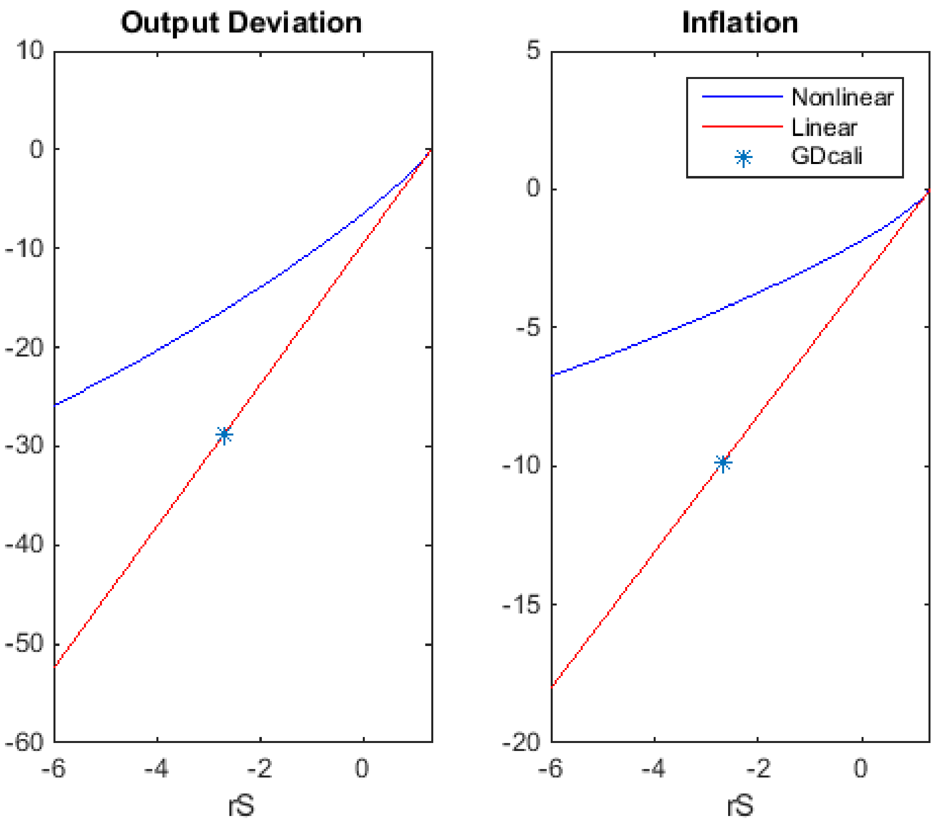
1. Sensitivity to shock



Further, when the zero lower bound is enforced at $\log \beta^{-1}$, the two models (linear and the nonlinear) intersect at the zero bound. The following graph plots[5] the values of the stable equilibrium under the

[5] We use the parameter calibrations from Eggertsson (2011) to plot both the curves. Later we also show how these two curves look when plotted under differnet calibrations, i.e. the non-linear model and the linear model separately generate 30% drop in output and 10% drop in inflation

non-linear model along with the unique values of output drop and inflation drop in the linear model as the preference shock increases (from right to left). It shows how the non-linearities come into play as we move away from the zero lower bound.



5 Comparison to the Rotemberg model.

Now write the Rotemberg model directly up in the AS-AD form for the two state Markov process (we can just cite somebody for the details).

1. (a) Euler

$$1 = \beta(1-\mu)\left(\frac{1-\bar{G}}{C_S}\right)^{-1/\sigma}\frac{1}{\xi_S} + \beta\mu\frac{1}{\Pi_S}$$

   (b) NK Phillips Curve

$$[1-\beta\mu](\Pi_S - 1)\Pi_S = \frac{\theta-1}{\alpha_r}\left(\frac{\theta}{\theta-1}\frac{\lambda C_S^{\sigma^{-1}} Y_S^{\omega}}{(1-\tau_S^w)} - 1\right)$$

   where $\lambda \equiv \frac{\theta-1}{\theta}(1-\tau_w)(1-\bar{G})^{\sigma^{-1}}$.

   (c) Resource Constraint

$$Y_S\left[1 - \frac{\alpha_r}{2}(\Pi_S - 1)^2\right] = C_S + G_S,$$

   where $\alpha_r \equiv \frac{\alpha_c(\theta-1)(1+\omega\theta)}{(1-\alpha_c)(1-\alpha_c\beta)}$.

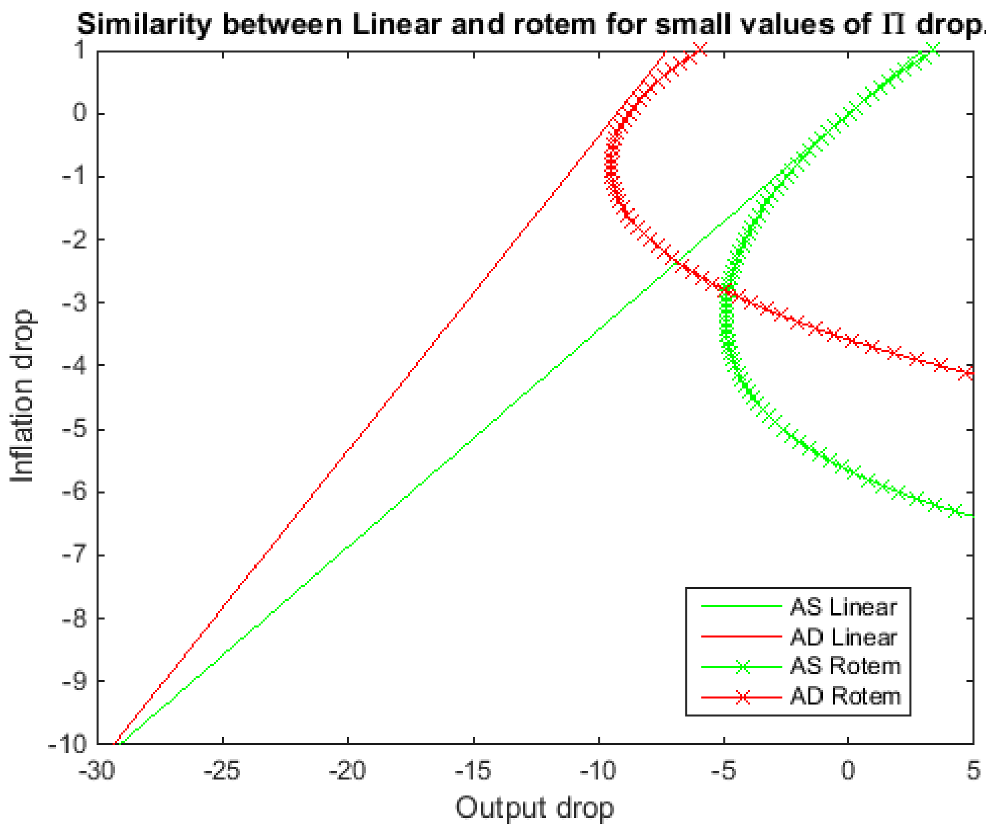
5.0.1 Deriving the AS-AD curves

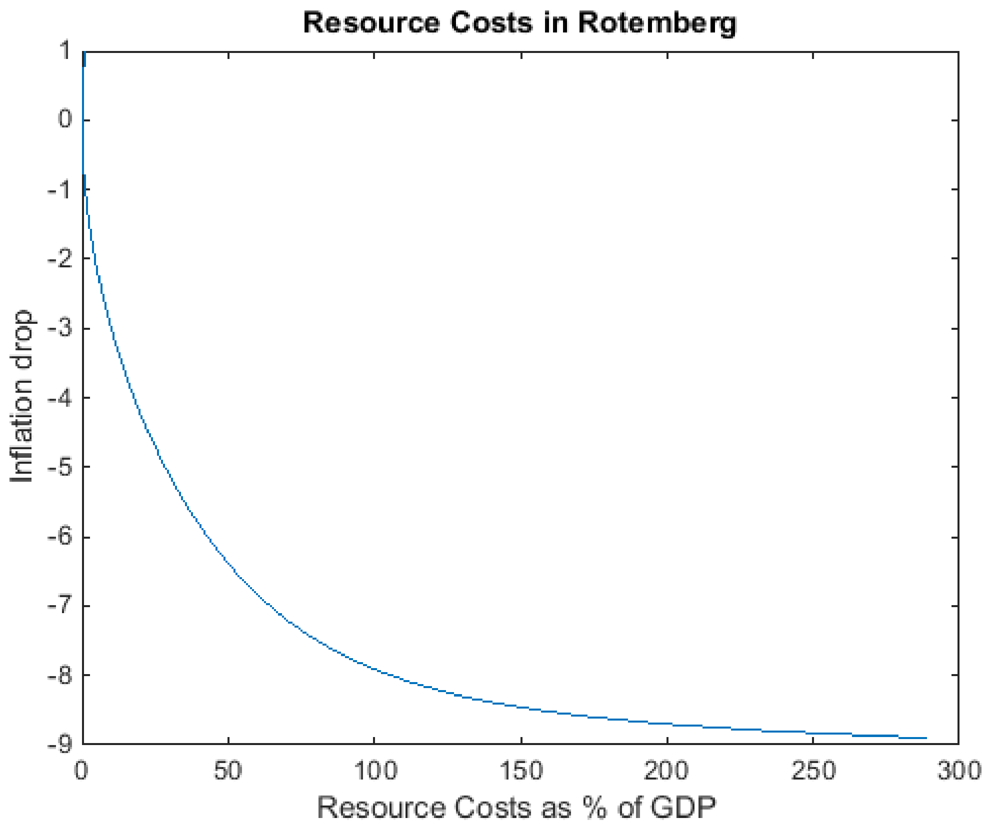
1. (a) **AD curve**: From the Euler equation and the Resource Constraint, we obtain the AD curve (relationship between Inflation and Output) in the low state:

$$Y_S = \left[1 - \frac{\alpha_r}{2}(\Pi_S - 1)^2\right]^{-1} \left\{ G_S + (1-\bar{G}) \left\{ \left[1 - \beta\mu\Pi_S^{-1}\right] \frac{\xi_S}{(1-\mu)\beta} \right\}^{\sigma} \right\}$$

(b) **AS curve**: From the New-Keynesian Phillips curve and the recourse constraint, we can derive the AS curve for the low state:

$$[1-\beta\mu](\Pi_S - 1)\Pi_S = \frac{\theta - 1}{\alpha_r} \left(\frac{\theta}{\theta - 1} \frac{\lambda\{Y_S\left[1 - \frac{\alpha_r}{2}(\Pi_S - 1)^2\right] - G_S\}^{\sigma^{-1}} Y_S^{\omega}}{(1-\tau_S^w)} - 1 \right)$$





How large is the multiplier at this stage? Differences?

Baseline Model→ Shock Calibration↓	Rotemberg
Non-Linear($r^e_S = -0.0098, p = 0.070$)	0.2680
Linear($r^e_S = -0.0104, p = 0.097$)	0.4957
Great Recession in Non-Linear	0.7042
Great Recession in Linear (EJ)	0.7583

Paradox of toil?

Baseline Model→ Shock Calibration↓	Rotemberg
Non-Linear($r^e_S = -0.0066, 1 - \mu = 0.0721$)	0.7179
Linear($r_S = -0.0104, 1 - \mu = 0.097$)	0.4942
Great Recession in Non-Linear	-0.2665
Great Recession in Linear (EJ)	-0.2183

6 Conclusion

to be written

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A Appendix

A.1 System of equations for the Non-linear Calvo model under two-state Markov process assumption

1. Euler

$$1 = \beta(1-\mu)\left(\frac{C_H}{C_S}\right)^{-1/\sigma}\frac{1}{\xi_S}\frac{1}{\Pi_L} + \beta\mu\frac{1}{\Pi_S}$$

where $C_H = 1 - \bar{G}$, and $\Pi_L = 1$.

2. NK Phillips Curve

$$\begin{aligned} K_S &= \frac{\theta}{\theta-1}\lambda\xi_S Y_S^{1+\omega} + \alpha_c\beta(1-\mu)K_L + \alpha_c\beta\mu\Pi_S^{\theta(1+\omega)}K_S \\ F_S &= \xi_L C_S^{-\frac{1}{\sigma}}(1-\tau_S^w)Y_S + \alpha_c\beta\ (1-\mu)F_L + \alpha_c\beta\mu\Pi_S^{(\theta-1)}F_S \\ \frac{K_S}{FS} &= \left(\frac{1-\alpha_c\Pi_S^{\theta-1}}{1-\alpha_c}\right)^{\frac{1+\omega\theta}{1-\theta}} \end{aligned}$$

where $\lambda \equiv \frac{\theta-1}{\theta}(1-\tau_w)(1-\bar{G})^{\sigma^{-1}}$.

3. Monetary Policy Rule:

$$i_t = i_{zlb} \text{ ,in the short run}$$

4. Resource Constraint

$$Y_t = C_t + G_t$$

5. Fiscal Policy rule: Normal Government Spending $\bar{G}$ is calibrated at 0.2 (20% of GDP). Labor Taxes $\bar{\tau}^w$. are calibrated at 0.3. To calculate fiscal multipliers: when ZLB is binding, $G_S > \bar{G}$. For Paradox of toil, $\tau^w_S < \bar{\tau}^w$.

6. Preference Shock

$$\xi_S < \xi_L \equiv 1$$

A.2 *Mapping the shock in the linear Calvo with the non-linear Calvo*

$$\hat{r}^e_S = \log \beta^{-1} + (1-\mu) \log \xi_S$$
$$\implies \log \xi_S = \frac{(r^e_S - \log \beta^{-1})}{(1-\mu)}$$
$$\implies \xi_S = \exp(\frac{(r^e_S - \log \beta^{-1})}{(1-\mu)})$$

Defines a correspondance between ξ_S *and* $\hat{r}^e_S a$

A.3 Determinacy in the linear model - Proposition 3 from Eggertsson 2011

Lemma 2 *An approximate equilibrium defined by the collection of stochastic processes* $\{\hat{y}_S, \pi_S\}$ *at zero interest rates is a locally unique bounded equilibrium, for a given path of these variables after the ZLB stops binding and a given value of* r^e_S. *if C1 and C2 hold, where*

$$r^e_S < -\Gamma_{\tau^w} \hat{\tau^w_S} - \Gamma_G \hat{g}_S \tag{C1}$$

$$L(\mu) \equiv (1-\mu)(1-\mu\beta) - \kappa\mu\sigma > 0 \tag{C2}$$