The Political Economy of Underfunded Municipal Pension Plans*

Jeffrey Brinkman
Federal Reserve Bank of Philadelphia

Daniele Coen-Pirani
University of Pittsburgh

Holger Sieg
University of Pennsylvania and NBER

February 9, 2015

Abstract

The purpose of this paper is to provide an explanation of the political and economic determinants of underfunding of municipal pension funds. We develop a new dynamic politico-economic model within an overlapping generations framework. The key insight of the model is that underfunding can result

---

*The views expressed here are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System.
in equilibrium even if individuals are fully informed, perfectly rational, and forward looking, and policies are capitalized in housing or land prices. Funding policies matter if housing also serves as collateral for households that are potentially credit constrained. The model suggests that differences in funding levels are systematically related to differences in economic fundamentals such as wage levels, the size of the public sector in a city, and the compensation of public sector workers measured by the current wage and retirement benefits. Finally, our analysis has some important policy implications. A policy intervention that mandates higher funding rates for municipalities than those adopted in equilibrium improves household welfare.

KEYWORDS: Unfunded liabilities, Political Economy, Land Prices, Capitalization.
1 Introduction

A large number of municipalities in the U.S. have taken on a significant amount of debt, primarily by underfunding their pension plans. Underfunding implies that a municipality incurs debt which breaks the link between current taxation and expenditure policies, allowing the municipal government to potentially shift the tax burden among cohorts. Given that cities face stringent requirements to balance their budgets each year, underfunding of pension plans is one of the few viable options for cities to effectively take on debt that is not linked to capital expenditures. What is commonly overlooked in this debate is that there large differences in the funding levels of local pensions plans, even among the largest U.S. cities. Despite the potential importance of underfunding, we do not fully understand the political and economic determinants that generate these differences.

To address this research question we consider an overlapping generations model of an economy characterized by a unit measure of municipalities and a unit measure of population. Municipalities are all identical and individuals differ only along the age dimension. Each municipality has a fixed supply of land. Individuals are free to move across municipalities when they are young but are immobile when old. As young, individuals work in a municipality, purchase land there, and consume land’s services and goods. When old, individuals sell their land and consume its proceedings. Individuals can borrow and save at the same exogenous rate as the local government. Individual borrowing is, however, constrained, and households can only borrow up to a constant fraction of their next period value of housing wealth.

In addition to private goods, individuals in each municipality consume an exogenous amount of public goods per capita supplied by municipal workers. While we abstract from modeling the latter explicitly, we assume that public sector workers are compensated for current services through a combination of wages and promised
future pension benefits. Public expenditures are financed through a tax on land values. A local government in each municipality has to pay for current wages by raising taxes, but can choose the extent to which it sets aside funds to meet future pensions. Our model thus allows us to study the interaction between land prices, population inflows, the determination of pension funding by local governments in the presence of capitalization effects.

Voters determine tax and funding policies in equilibrium at each point of time. We assume that the older cohort of households is decisive at the ballot box, which implies that the local government wants to maximize current period prices of housing values. A key assumption of our model is that politicians cannot commit to policies in the future. Policies are, therefore, chosen sequentially. Each period the government can re-optimize and deviate from previous period policies. As a consequence, the optimal government policy must be self-enforcing or time consistent. The characterization of a Markov Perfect Equilibrium in our model follows the pioneering work by Krusell, Quadrini and Rios-Rull (1997) and Krusell and Rios-Rull (1999).

Solving these models is difficult, since individual and policy makers need to consistently forecast how government policies and potential deviations from optimal policies affect the behavior of individuals and governments in the future. It is well understood that it is difficult to obtain analytical solutions to these style of infinite horizon dynamic models.¹ One contribution of this paper is that we provide sufficient conditions that allow us to provide analytical characterization of equilibrium and its properties. In particular, we show that, depending on the values of key parameters, there exist two types of equilibria.

¹To the best of our knowledge, the only a few papers that work out an analytical solution to Markov Perfect Equilibria of these type of dynamic political economy models. One prominent example is the work by Hassler, Mora, Storesletten and Ziliboti (2003). Other papers that analytically characterize the equilibrium of infinite horizon political economy models are Grossman and Helpman (1998), Duggan (2000), and Battaglini and Coate (2008).
The first type of equilibrium arises when borrowing constraints are not binding. Our analysis reinforces the notion that asset markets for land or housing severely limit the opportunity for local governments to manipulate the distribution of welfare across cohorts. Current funding policies affect current and future tax rates. An increase in future tax rates is fully capitalized in the future housing prices. The current generation of households benefits from lower current taxes. Since they also own housing in the community, they are hurt by the reduction in the asset value of their house. If credit constraints are not binding in equilibrium, households can smoothly transfer wealth across both periods off-setting any potential effects on the distribution of income across periods. The two pricing effects perfectly off-set each other. Since the effects of any debt policy implemented by a local government can be fully off-set by actions of individuals, funding levels of municipal pensions plans are not uniquely determined in equilibrium. There exists a variety of policies that are all consistent with optimal individual behavior in dynamic equilibrium. All policies give rise to the same level of welfare.

The second case arises when borrowing constraints are binding in equilibrium. A decrease in future housing prices also reduces the collateral value of housing. As a consequence young households respond by increasing their demand for housing (collateral). The decrease in current taxes is not fully offset by a decrease in future housing prices. Housing demand by young household increases which then leads to an increase in current housing prices. As a consequence there is some scope for affecting current housing prices and thus the distribution of welfare by underfunding public pension funds. When households cannot fully undo the consequences of funding or debt policies of municipal governments, these policies affect the distribution of welfare, which then potentially gives rise to an intergenerational conflict. The outcome of this conflict will largely be determined by the political decision making process. We assume that old households are decisive at the ballot box, which implies current
funding choices are chosen to maximize current land prices. We can also show that the equilibrium is not efficient. Household welfare depends on the optimal funding policy implemented in equilibrium.

We derive a closed form solution for the optimal level of funding if borrowing constraints are binding in equilibrium. We show that funding levels are systematically related to economic fundamentals such as wage levels, the size of the public sector in the city, and the compensation for public sector workers measured by the current wage and retirement benefits. Moreover, funding levels are systematically linked to parameters of the model such as the degree of myopia of households, and the prevailing interest rate and the severity of the borrowing constraints. As such our model accounts for numerous factors that may explain the heterogeneity in observed funding policies within cities.

Finally, our analysis has some important policy implications. In particular, we show that there is some scope for an intervention by the state or federal government. Any policy that mandates higher funding rates for municipalities than those adopted in equilibrium improve household welfare. Our empirical analysis suggests that a number of cities are likely to be close to the minimum acceptable funding policies.

Our paper is also related to the existing literature in public economics that considers the impact of public pension funding policies on the distribution of economic welfare. The key insight of the analysis is then that underfunding can result in a dynamic model in which individuals are fully informed, perfectly rational and forward looking. Other potential explanations for underfunding rely on informational asymmetries and behavioral biases (Glaeser and Ponzetto, 2014) or myopic behavior (Bagchi, 2013). Epple and Schipper (1981) explain underfunding as a result of population or income growth. Mummy (1978) develops a model in which a municipality has a natural advantage in credit markets over households and can borrow at lower interest rates (Mummy, 1978). Other paper rely on restrictions in mobility to gen-
erate underfunding (Inman, 1982, Schulz and Sjostrom, 2001). As we argued above, the key challenge of each model is not only to explain the existence of underfunding, but to explain the large degree in heterogeneity in observed funding policies.

2 A Model of Underfunding and Capitalization

2.1 Framework

2.1.1 Individual Preferences and Budget Constraint

Homogenous agents have preferences described by the following utility function:

\[ U (c_{yt}, l_t, c_{ot+1}) \]  

where \( c_{yt} \) denotes consumption of the numeraire good when young, \( l_t \) denotes the services of the land purchased by the agent, and \( c_{ot+1} \) denotes consumption when old. The utility function \( U (.,.,.) \) is well-behaved and strictly concave in its arguments.

Each agent is endowed with \( w \) units of the consumption good when young. Its budget constraint is:

\[ w = c_{yt} + (1 + \tau_t) q_t l_t + \frac{b_{t+1}}{R} \]  

\[ c_{ot+1} = q_{t+1} l_t + b_{t+1} \]

where \( q_t \) denotes the price of land in a municipality in period \( t \), \( b_{t+1} \) is the quantity of bonds purchased (or issued) by the agent and \( R > 1 \) is the exogenous gross interest rate. Notice that there are two assets in this economy, land and a risk-less asset (the bond \( b_{t+1} \)). We assume that, while borrowing is allowed, it is constrained to a fraction
of the collateral value of land:

$$b_{t+1} \geq -\kappa q_{t+1} l_t$$

where $0 < \kappa \leq 1$ is a parameter that indexes the extent of the borrowing constraint. When $\kappa = 1$ agents can borrow freely against the future value of land. Since all municipalities are homogeneous, we omit a location subscript from the variables.

### 2.1.2 Municipal Government’s Budget

The government of a municipality finances the provision of a local public good (police, education, etc.). By assumption it takes $g$ public sector workers for each young person living in the municipality to provide public services. A public sector employee works for an exogenously set current wage $w^g$ and pension benefit $b^g$. The government of the municipality collects revenue $\tau_t q_t$ by taxing property values and uses it to pay the wage $w^g$ of current public sector workers, to fund their promised retirement benefits $b^g$, and to pay for the unfunded portion of the pension benefits of last period’s public sector workers. Thus, in period $t$ the municipality’s budget constraint is:

$$\tau_t q_t = gn_{t+1} \left[ w^g + \frac{f_t b^g}{R} \right] + gn_t b^g (1 - f_t)$$

where $f_t$ is fraction of pensions due in period $t$ that is funded, $n_{t+1}$ and $n_t$ are the measures of respectively young and old agents residing in the municipality in period $t$. We assume that $f_t$ is constrained to be between $f_{\text{min}} \geq 0$ (if $f_t = 0$ all future pension benefits are unfunded) and one (the municipality fully funds the future pensions of its employees). We interpret the lower bound $f_{\text{min}}$ as a policy parameter that might be manipulated by a higher level of government such as a state government.
2.1.3 Equilibrium in a Municipality

The supply of land in each municipality is normalized to one. Since only young workers demand land, the total demand for land in a municipality is given by $n_{t+1}l_t$ and land market equilibrium requires that

$$n_{t+1}l_t = 1.$$ (5)

In order to attract workers a municipality has to offer a utility level $\bar{V}_t$ to a young agent. Migration occurs in each period after municipal policies have been set.

2.1.4 Policy Making

The only policy decision in this economy in each period $t$ is the mix $(\tau_t, f_{t+1})$ of current taxes and funding of future public sector pensions. We assume that $(\tau_t, f_{t+1})$ is chosen in each period by the current old generation. Since consumption when old only depends on land values, this is equivalent to assume that policy is set to maximize current land values. When choosing $(\tau_t, f_{t+1})$ to maximize $q_t$ the policy-maker takes as given the utility level $\bar{V}_t$, but understands the effect of its policies on population flows from other localities and on future policies chosen by future policy-makers.

2.1.5 General Equilibrium

We require that each young agent in the economy is located in a municipality in each period $t$. Formally, since there is a measure one of localities in the economy, this is equivalent to imposing that in equilibrium:

$$n_{t+1} = 1.$$ (6)
2.2 Recursive Formulation and Definition of Equilibrium

It is convenient to cast the model in recursive form and then focus on a recursive equilibrium without commitment, following Krusell, Quadrini and Rios-Rull (1997), Krusell and Rios-Rull (2000), and Persson and Tabellini (2002). The state variables for each municipality are the measure of old agents and the fraction of pensions that are funded, or \((n,f)\). The measure of old agents is relevant because it determines the total pensions due to retiring public sector workers, while the fraction of pensions that is funded determines the need for current taxes to pay for the promises made in the previous period.

Let \(f' = F(n,f)\) denote a funding policy of the municipal government that begins a period with state \((n,f)\), conditional on a utility level \(\bar{V}\) that the municipality has to offer to attract residents. In what follows we leave the dependence of the function \(F\) and the other equilibrium functions on \(\bar{V}\) implicit in order not to overly complicate the notation. Similarly, let \(\tau = T(n,f;F)\) denote the current period property tax rate in a municipality that follows the funding rule \(F\). Let \(q(n,f;F)\) denote the price of land in a municipality that begins a period with state \((n,f)\) and whose government follows a policy rule \(F\). Let \(n' = N(n,f;F)\) denote the young population in a municipality that begins a period with state \((n,f)\). Last, let \(l(n,f;F)\) and \(b(n,f;F)\) denote the quantity of land and bonds that maximize the young agent’s utility subject to the
budget and borrowing constraints:

\[
[l(n,f;F), b(n,f;F)] = \arg \max_{l,b} U(c_y, l, c_o)
\]  

s.t.

\[
c_y = w - (1 + T(n,f;F))q(n,f;F)l - \frac{b'}{R}
\]

\[
c_o = q(N(n,f;F), F(n,f); F)l + b'
\]

\[
b' \geq -\kappa q(N(n,f;F), F(n,f); F)l,
\]

and use the notation \(V(n,f;F)\) to denote the maximum utility achieved when demand for land and bonds is \(l(n,f;F)\) and \(b(n,f;F)\).

We can now define an economic equilibrium given a policy rule.

**Definition 1** An economic equilibrium under a policy rule \(F\)

Fix the funding rule \(F(n,f)\) and a utility level \(\nabla\). An equilibrium given this policy rule is given by the functions \(l(n,f;F)\), \(b(n,f;F)\), \(V(n,f;F)\), \(q(n,f;F)\), \(T(n,f;F)\), \(N(n,f;F)\) such that:

1. The quantities of land and bonds \(l(n,f;F)\) and \(b(n,f;F)\) solve problem (7).

2. The market for land clears in each municipality:

\[
N(n,f;F)l(n,f;F) = 1.
\]

3. Young individuals are indifferent between living in the municipality or elsewhere:

\[
V(n,f;F) = \nabla.
\]
4. The local government’s budget constraint holds in each municipality:

\[
T(n, f; F) q(n, f; F) = gN(n, f; F) \left[ w^g + \frac{F(n, f) b^g}{R} \right] + gnb^g (1 - f).
\]

In order to endogenize the policy rule \( F \), it is necessary to define an equilibrium after a one-period deviation from the rule. This is because the current policymaker only controls current period funding and taxes and takes as given the behavior of future policymakers. Restricting attention to one-period deviations implies that all future policy makers will adhere to the policy rule given by \( F \).

Let \( \bar{f} \) denote the funding level chosen in the current period that deviates from the policy rule \( f' = F(n, f) \). A current period deviation will be associated with different current taxes, population flows, and land prices, and thus different choices of land and potentially bonds on the part of young agents. Let these be respectively denoted by \( \bar{T}(n, f, \bar{f}; F) \), \( \bar{N}(n, f, \bar{f}; F) \), \( \bar{q}(n, f, \bar{f}; F) \) and new land choices of a young agent be such that:

\[
\begin{align*}
&\left[ \bar{t}(n, f, \bar{f}; F), \bar{b}(n, f, \bar{f}; F) \right] = \arg \max_{l,b'} U(c_y, l, c_o) \\
&\text{s.t.} \\
&c_y = w - \left( 1 + \bar{T}(n, f, \bar{f}; F) \right) \bar{q}(n, f, \bar{f}; F) l - b' \\
&c_o = q\left( \bar{N}(n, f, \bar{f}; F), \bar{f}; F \right) l + b' \\
&b' \geq -\kappa q\left( \bar{N}(n, f, \bar{f}; F), \bar{f}; F \right) l.
\end{align*}
\]

Let \( V(n, f, \bar{f}; F) \) denote the associated maximum level of utility. Notice that the young agent faces prices \( \bar{q} \) in the current period, but (correctly) anticipates that the pricing function will revert back to \( q \) in the following period. Hence, the agent’s consumption when old is computed using the equilibrium pricing function. Notice that the policy-maker in each municipality takes the utility level \( V \) as given when
choosing \( \tilde{f}' \).

With this notation in hand, we can define an economic equilibrium after a one-period deviation \( \tilde{f}' \) from the policy rule \( F \):

**Definition 2** An equilibrium after a one-period deviation \( \tilde{f}' \) from the policy rule \( F \)

Set the utility level \( \overline{V} \). Then, an equilibrium after a one-period deviation \( \tilde{f}' \) is given by the functions \( \tilde{l}(n, f, \tilde{f}'; F) \), \( \tilde{b}(n, f, \tilde{f}'; F) \), \( \tilde{V}(n, f, \tilde{f}'; F) \), \( \tilde{q}(n, f, \tilde{f}'; F) \), \( \tilde{T}(n, f, \tilde{f}'; F) \), \( \tilde{N}(n, f, \tilde{f}'; F) \) such that:

1. The functions \( \tilde{l}(n, f, \tilde{f}'; F) \) and bonds \( \tilde{b}(n, f, \tilde{f}'; F) \) solve problem (8).

2. The market for land clears in each municipality:

\[
\tilde{N}(n, f, \tilde{f}'; F) \tilde{l}(n, f, \tilde{f}'; F) = 1.
\]

3. Young individuals are indifferent between living in the municipality or elsewhere:

\[
\tilde{V}(n, f, \tilde{f}'; F) = \overline{V}.
\]

4. The local government’s budget constraint holds in each municipality:

\[
\tilde{T}(n, f, \tilde{f}'; F) \tilde{q}(n, f, \tilde{f}'; F) = g \tilde{N}(n, f, \tilde{f}'; F) \left[ w^g + \frac{\tilde{f}' b^g}{R} \right] + g n b^g (1 - f) .
\]

We can finally define an equilibrium without commitment for this economy.

**Definition 3** An equilibrium without commitment

Fix the utility level \( \overline{V} \). An equilibrium without commitment is given by a policy rule \( F \), and set of functions \( l(n, f; F) \), \( b(n, f; F) \), \( V(n, f; F) \), \( q(n, f; F) \), \( T(n, f; F) \),
\[ N(n, f; F) \text{ and } \tilde{l}(n, f, \tilde{f}; F), \; \tilde{b}(n, f, \tilde{f}; F), \; \tilde{V}(n, f, \tilde{f}; F), \; \tilde{q}(n, f, \tilde{f}; F), \; \tilde{T}(n, f, \tilde{f}; F), \; \tilde{N}(n, f, \tilde{f}; F) \] such that:

1. The functions \( l(n, f; F), \; b(n, f; F), \; V(n, f; F), \; q(n, f; F), \; T(n, f; F), \; N(n, f; F) \) constitute an economic equilibrium under \( F \) according to Definition 1.

2. The functions \( \tilde{l}(n, f, \tilde{f}; F), \; \tilde{b}(n, f, \tilde{f}; F), \; \tilde{V}(n, f, \tilde{f}; F), \; \tilde{q}(n, f, \tilde{f}; F), \; q(n, f; F), \; \tilde{T}(n, f, \tilde{f}; F), \; \tilde{N}(n, f, \tilde{f}; F) \) constitute an economic equilibrium after a one-period deviation from \( F \) according to Definition 2.

3. The policymaker has no incentive to deviate from \( F \) in any period and for any state, taking into account the economic equilibrium after a one-period deviation:

\[
F(n, f) = \arg \max_{\tilde{f}} \tilde{q}(n, f, \tilde{f}; F)
\]

for all \((n, f)\).

Finally, we can endogenize the reservation utility by requiring that all agents reside in one of the communities at each point in time. A general equilibrium without commitment an endogenous reservation utility is given by a utility level \( \tilde{V} \) and the set of functions that are part of an equilibrium without commitment (Definition 3) such that:

\[
N(n, f; F) = 1 \quad (9)
\]

for all \((n, f)\).
3 Properties of Equilibria

In this section we show how the model outlined in the previous section admits a closed-form solution when the utility function takes the following logarithmic form:

\[ U(c_y, l, c_o) = \ln c_y + \psi \ln l + \beta \ln c_o, \]  

where \( \psi > 0 \) and \( 0 < \beta < 1 \). We organize our results in a series of propositions whose proofs are contained in the paper’s appendix. Two cases are possible in equilibrium.

In the benchmark case, the borrowing constraint does not bind in equilibrium. We show that the pension funding rule is irrelevant for all the equilibrium variables. In such equilibrium agents land prices adjust to offset any attempt by the old generation to underfund the pension system. Lower funding leads to lower current land prices gross of taxes and lower future prices. These forces cancel each other so the demand for land stays constant. Young agents save more in anticipation of lower land prices when old.

In the second case, the borrowing constraint binds in equilibrium. As a consequence, markets are not complete and there is some scope for old households to manipulate current period housing prices via funding decisions. When the borrowing constraint is binding, the same two effects arise as in the benchmark case. The reduction in \( \tilde{f} \) lowers the after-tax price of land today and leads to an expansion in the demand for land. Similarly, there is a countervailing effect will lower the price of land tomorrow, potentially reducing the demand today as the young today have to sell the land tomorrow. When the borrowing constraint is binding, the second effect is present but does not fully offset the first one. As a consequence, current period prices are a monotonically decreasing function in \( \tilde{f} \).

It turns out that a simple condition on the model’s parameters separates these
two cases. The condition involves the land share parameter \( \psi \), the borrowing limit parameter \( \kappa \), the net interest rate \( R - 1 \), and the discount factor \( \beta \):

\[
\psi \leq \frac{\beta (R - 1)}{1 - \kappa}.
\] (11)

### 3.1 Equilibrium with Non-binding Borrowing Constraints

We start by considering the case in which the borrowing constraint is not binding in equilibrium. The following results then hold.

**Proposition 1 Policy Independence:**

*If \( \psi \) is smaller than the right-hand side of equation (11), then there exists a continuum of equilibria without commitment in which the funding rule is a constant \( f^* \in [f_{\min}, 1] \). These equilibria differ only in terms of \( f^* \) and constant land prices \( q^* \):

\[
q^* = \alpha_q^* - gb^\beta N^* (1 - f^*),
\] (12)

where \( \alpha_q^* \) is a function of the parameters defined in the Appendix. All equilibria are characterized by the same population \( N^* \) and the same consumption of goods and land.*

The intuition for this result is as follows. When the borrowing constraint is not binding, agents can transfer consumption across periods at a relative price \( R^{-1} \). Attempts by the old generation to underfund the pension system have no effect on land prices because the price increase generated by lower taxes is fully offset by the expectation of lower future prices that drive current prices down. In other words, the function \( \tilde{q}(n, f, \tilde{f}^0; F) \) is independent of the one-period deviation \( \tilde{f}^0 \) in this case.

The key to this result is the ability of agents to freely borrow and lend at the rate \( R^{-1} \). When a young agent looks ahead at the price of land in the next period
(when she is old), she anticipates that land prices will be lower due to underfunding and discounts each future dollar using the market interest rate. Thus, while reducing future pension funding by a dollar reduces current taxes by $R^{-1}$ dollars, it also reduces future land prices by a dollar by $R^{-1}$ dollars. Thus, these two effects exactly offset each other and pension funding becomes irrelevant for all economic variables (except land prices). The condition that the interest rate and/or the discount factor are high enough relative to $\psi$ in equation (11) is sufficient to guarantee that the borrowing constraint is never binding either in equilibrium or following a deviation.

3.2 Equilibrium with Binding Borrowing Constraints

Now consider the case in which the borrowing constraint is binding. As we discussed above, this case arises if the land share parameter $\psi$ is larger than the right-hand side of equation (11), or:

$$\psi > \frac{\beta (R - 1)}{1 - \kappa}. \quad (13)$$

Then, another type of equilibrium emerges characterized by a binding borrowing constraint. In this case, the choice of the funding policy has an impact on the current period price.

**Proposition 2 Policy Relevance:**
If equation (13) holds and the relative size $g$ of local government is sufficiently small, then there exists a equilibrium without commitment for the economy described above in which the funding rule is a constant given by the maximum between $f_{min}$ and the following expression:

$$f^* = \frac{R}{\kappa \psi + (\beta + \psi) (R - 1)} \left\{ \left( \frac{\beta (R - 1)}{1 - \kappa} - \psi \right) \left( \frac{\psi + \beta}{1 + \psi + \beta g b^\omega g} - \frac{w g^\omega}{b^\omega} - \frac{\kappa}{R} \right) + \psi \right\}, \quad (14)$$

with $f^* < 1$. 

17
The other functions that comprise the equilibrium are, for convenience, reported in the Appendix.

The intuition behind Proposition 2 can be best grasped by considering the land price function that the old generation seeks to maximize (see Section B.2 in the Appendix for the exact expression and its derivation). The relevant terms of this function that depend on \( f' \) are reported below:

\[
\tilde{q}(n, f, \tilde{f}; F) \propto \left[ \frac{\psi + \beta}{1 + \psi + \beta} w - q \left( w^g + \frac{\kappa}{R} b^g + \frac{\tilde{f}' b^g (1 - \kappa)}{R} \right) \right] q \left( \tilde{N} \left( \tilde{f}'; F \right), \tilde{f}'; F \right)^{\frac{\beta}{\psi + \beta}}
\]

where \( \tilde{f}' \) is the level of funding controlled by the old and the equilibrium land pricing function is:

\[
q \left( \tilde{N} \left( \tilde{f}'; F \right), \tilde{f}'; F \right) = \alpha_q^{**} - gb^g \tilde{N} \left( \tilde{f}'; F \right) \left( 1 - \tilde{f}' \right),
\]

where \( \alpha_q^{**} \) is a constant and \( \tilde{N} \left( \tilde{f}'; F \right) \) is the young population that settles in the locality when funding is \( \tilde{f}' \). It is defined in the Appendix and can be shown to increase monotonically in \( \tilde{f}' \). The pricing function \( q \) is the equilibrium pricing function that prevails in the period after the deviation while the function \( \tilde{q} \) represents the pricing function that applies after a deviation of policy in the current period.

Notice, first of all, that the current state of the economy does not interact with \( \tilde{f}' \), establishing the result that the optimal funding level is not state-dependent.

The young generation faces the following trade-off. One the one hand, a lower level of funding today (i.e. a lower \( \tilde{f}' \)), will reduce current taxes and future land prices, just as in the economy in which the borrowing constraint is not binding. However, in this case since borrowing is constrained, a one dollar decline in future land prices tightens the agent’s borrowing constraint by \( \kappa < 1 \) dollars, and produces a reduction in the current demand for land by the young that is increasing in \( \kappa \). In the limit, if \( \kappa = 0 \), the current demand for land is unaffected by future land prices. Given this
limited dependence, this first effect implies that the old cohort has an incentive to set \( \tilde{f}' \) as low as possible. Formally, this effect can be seen inside the square brackets in equation (15); as long as \( \kappa < 1 \), the term in square brackets declines in \( \tilde{f}' \).

However, there is a second effect that acts to partially offset the first one. When the borrowing constraint binds, the expected future land price has a second independent effect on current land prices. Recall that the first effect of a lower future land price is to affect current demand for land with its importance being determined by \( \kappa \). The second effect is to affect directly the lifetime utility offered by the location. A lower future land price decreases lifetime utility from the perspective of a young agent. Since the level of utility is fixed at \( \overline{V} \), the agent must be indifferent between this location and others only if the current land price falls, i.e., the city becomes cheaper today. This second effect is captured by the term \( q \) at the end of equation (15). Notice that its importance depends positively on the discount factor parameter \( \beta \).

The old generation sets \( \tilde{f}' \) to balance these two effects at the margin. This optimization yields the solution in equation (14), taking into account the possibility that the optimal level of funding violates the lower bound \( f_{\text{min}} \).

4 Policy Implications

We can summarize the policy implications of our model in the following proposition which applies for the version of our model with endogenous reservation utility:

**Proposition 3**

a) If individuals are not borrowing constrained, then there is no scope for any policy intervention by a higher level of government.

b) If individuals are borrowing constrained, individual welfare is monotonically in-
creasing the minimum funding level \( f_{\text{min}} \) all the way to \( f_{\text{min}} = 1 \).

The intuition behind Proposition 3 is the following. Consider the Ricardian case first. Pension funding does not matter for any of the variables that enter into an agent’s utility and so it does not matter for the equilibrium utility \( \bar{V}^* \). The situation is different when individuals are borrowing constrained. Provided the other assumptions of Proposition 2 are also satisfied, underfunding occurs in equilibrium. Welfare might be improved by increasing the minimum funding level \( f_{\text{min}} \) all the way to \( f_{\text{min}} = 1 \). The reason for this is that underfunding increases demand for land in a location by reducing current gross land prices. As a result a location’s population is smaller than it would be under full-funding. In general equilibrium, the entire population has to be distributed across localities, so each locality has to absorb the same unit measure of young agents. This is achieved by reducing the overall utility \( \bar{V}^{**} \) of living in such an economy. Intuitively, by underfunding, each location creates a negative externality on other locations as it forces other locations to absorb the young agents who do not locate there, resulting into lower utility for everybody. The policy intervention can correct this negative externality.

5 Conclusions

We have explored in this paper the political and economic determinants that explain underfunding of municipal pension plans using a new dynamic politico-economic model. The key insight of our the model is that underfunding can result in equilibrium even if individuals are fully informed, perfectly rational, and forward looking, and policies are capitalized in housing or land prices. Funding policies matter if housing also serves as collateral for households that are potentially credit constrained. We have shown that our analysis has some important policy implications. A policy in-

20
vention that mandates higher funding rates for municipalities than those adopted in equilibrium improves household welfare.

We view this paper as providing ample scope for future research. Proposition 2 suggests that funding levels are systematically related to economic fundamentals such as wage levels, the size of the public sector in the city, and the compensation for public sector workers measured by the current wage and retirement benefits. Moreover, funding levels should be systematically linked to parameters of the model such as the degree of myopia of households, and the prevailing interest rate and the severity of the borrowing constraints. As such our model could potentially accounts for numerous channels that may explain the heterogeneity in observed funding policies within cites. More empirical research is clearly needed to isolate the key factors that predict funding levels.
References


A Proof of Proposition 1

We first consider the equilibrium for given policy and the in the next section we consider an equilibrium after a deviation.

A.1 Equilibrium Under a Constant Policy

Denote by $f^*$ the constant funding policy we guess:

$$f^* = F(n, f),$$

for all $(n, f)$. In what follows I replace $F$ by $f^*$ as an argument of the equilibrium functions.

Consider the problem (7) faced by a young agent and impose that the borrowing constraint is not binding:

$$b' > -\kappa q(N(n, f; f^*), f^*; f^*)l.$$  \hspace{1cm} (17)

At the end of this section we verify that this is always the case in equilibrium. The budget constraints are:

$$c_y = w - (1 + T(n, f; f^*))q(n, f; f^*)l - b'/R$$  \hspace{1cm} (18)

$$c_o = q(N(n, f; f^*), f^*; f^*)l + b'.$$

Collapse into one budget constraint and collect:

$$c_y = w - [(1 + T(n, f; f^*))q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*)/R]l - c_o/R.$$
The optimization problem is now:
\[
\max_{l,c_o} \left\{ \ln \left[ w - \left( (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right) l - c_o/R \right] + \psi \ln l + \beta \ln c_o \right\}.
\]

The interior first-order conditions for consumption when old and land are:
\[
\frac{1}{w - \left( (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right) l - c_o/R} = \frac{R\beta}{c_o},
\]
\[
\frac{(1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R}{w - \left( (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right) l - c_o/R} = \frac{\psi}{l}.
\]

Solve for \(c_o\) from the first equation:
\[
c_o = \frac{R\beta}{1 + \beta} \left[ w - \left( (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right) l \right].
\]

Replace in the first-order condition for land:
\[
(1 + \beta) \frac{(1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R}{w - \left( (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right) l} = \frac{\psi}{l}.
\]

Solve for \(l\):
\[
l(n, f; f^*) = \frac{\psi}{1 + \beta + \psi} \left[ (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right] w.
\]

This implies that consumption when old is given by:
\[
c_o = \frac{R\beta}{1 + \beta + \psi} w,
\]
and consumption when young by:
\[
c_y = \frac{w}{1 + \beta + \psi}.
\]
Impose land market equilibrium (equation 5) and solve for \( N(n, f; f^*) \):

\[
N(n, f; f^*) = \frac{1 + \psi + \beta}{\psi} \frac{1}{w} \left[ (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right].
\]

(19)

Notice that the indirect utility function of a young agent is:

\[
V(n, f; f^*) = \ln \frac{w}{1 + \psi + \beta} - \psi \ln N(n, f; f^*) + \beta \ln \frac{R\beta}{1 + \beta + \psi} w.
\]

(20)

Impose that \( V(n, f; f^*) = \nabla \)

\[
\nabla = \ln \frac{w}{1 + \psi + \beta} - \psi \ln N(n, f; f^*) + \beta \ln \frac{R\beta}{1 + \beta + \psi} w,
\]

and solve for \( N(n, f; f^*) = N^* \):

\[
N^* = \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1 + \beta}{\psi}} (R\beta)^{\frac{\beta}{\psi}} \exp \left( -\frac{\nabla}{\psi} \right).
\]

(21)

Now replace it into the definition of \( N(n, f; f^*) \) in equation (19):

\[
\psi \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1 + \beta + \psi}{\psi}} (R\beta)^{\frac{\beta}{\psi}} \exp \left( -\frac{\nabla}{\psi} \right)
= \left[ (1 + T(n, f; f^*)) q(n, f; f^*) - q(N(n, f; f^*), f^*; f^*) / R \right].
\]

Solve for the price of land gross of taxes:

\[
(1 + T(n, f; f^*)) q(n, f; f^*) = \psi \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1 + \beta + \psi}{\psi}} (R\beta)^{\frac{\beta}{\psi}} \exp \left( -\frac{\nabla}{\psi} \right) + q(N^*, f^*; f^*) / R.
\]

(22)
From the government’s budget (4) the tax rate is such that:

\[
T(n, f; f^*) q(n, f; f^*) = gN^* \left[ w^g + \frac{f^* b^g}{R} \right] + \frac{gb^g}{1-f^*} (1-f) .
\] (23)

Use this equation to write the price of land as:

\[
q(n, f; f^*) = \psi \left( \frac{w}{1+\psi+\beta} \right)^{\frac{1+\beta+\psi}{\psi}} (R\beta)^{\frac{1}{\psi}} \exp \left( -\frac{V}{\psi} \right) + q(N^*, f^*; f^*) / R - gN^* \left( w^g + \frac{f^* b^g}{R} \right) - \frac{gb^g}{1-f} (1-f) .
\]

This can be written more compactly as:

\[
q(n, f; f^*) = \alpha_q^* - gb^g n (1-f) ,
\] (24)

where

\[
\alpha_q^* = \psi \left( \frac{w}{1+\psi+\beta} \right)^{\frac{1+\beta+\psi}{\psi}} (R\beta)^{\frac{1}{\psi}} \exp \left( -\frac{V}{\psi} \right) + q(N^*, f^*; f^*) / R - gN^* \left( w^g + \frac{f^* b^g}{R} \right) .
\] (25)

The tax rate \( T(n, f; f^*) \) must then satisfy (23). Now, solve for \( q(N^*, f^*; f^*) \):

\[
q(N^*, f^*; f^*) = \alpha_q^* - gb^g N^* (1-f^*)
\]

Replace \( \alpha_q^* \):

\[
q(N^*, f^*; f^*) = \psi \left( \frac{w}{1+\psi+\beta} \right)^{\frac{1+\beta+\psi}{\psi}} (R\beta)^{\frac{1}{\psi}} \exp \left( -\frac{V}{\psi} \right) + q(N^*, f^*; f^*) / R - gN^* \left( w^g + \frac{f^* b^g}{R} \right) - gb^g N^* (1-f^*) .
\]
Solve for \(q(N^*, f^*; f^*)\):

\[
q(N^*, f^*; f^*) = (1 - 1/R)^{-1} \psi \left( \frac{w}{1 + \psi + \beta} \right)^{1+\beta+\psi} (R\beta)^{\beta} \exp \left( -\frac{V}{\psi} \right) - N^* g \left[ \frac{R}{R-1} (w^g + b^g) - f^* b^g \right],
\]

and replace \(N^*\) to obtain:

\[
q(N^*, f^*; f^*) = (R\beta)^{\beta} \exp \left( -\frac{V}{\psi} \right) \left( \frac{w}{1 + \psi + \beta} \right)^{1+\beta} \times \left[ \frac{R}{R-1} \left( \frac{\psi w}{1 + \psi + \beta} \right) + g \left( f^* b^g - \frac{R}{R-1} (w^g + b^g) \right) \right].
\]

Finally, check that agents are not constrained if

\[
\frac{\beta (R-1)}{\psi (1-\kappa)} > 1.
\]

From equation (18), the demand for bonds is:

\[
b' = (w - (1 + T(n, f; f^*)) q(n, f; f^*) l(n, f; f^*) - c_y) R.
\]

Imposing that the borrowing constraint does not bind (equation 17) and replacing \(l(n, f; f^*)\) and simplify:

\[
\frac{\beta + \psi}{\psi} R > \frac{(1 + T(n, f; f^*)) q(n, f; f^*) R - \kappa q(N^*, f^*; f^*)}{(1 + T(n, f; f^*)) q(n, f; f^*) - q(N^*, f^*; f^*) / R}.
\]

Recall that

\[
(1 + T(n, f; f^*)) q(n, f; f^*) - q(N^*, f^*; f^*) / R = \psi \left( \frac{w}{1 + \psi + \beta} \right)^{1+\beta+\psi} (R\beta)^{\beta} \exp \left( -\frac{V}{\psi} \right),
\]
and use it to get:

\[
\frac{(1 + T(n; f; f^*) - q(n; f; f^*) )}{(1 + T(n; f; f^*))} \frac{q(n; f; f^*) - q(N^*; f^*; f^*) / R}{R + \frac{q(N^*; f^*; f^*)(1 - \kappa)}{\psi(\frac{w}{1 + \psi + \beta})^{1+\beta+\psi} (R \beta)^\frac{\beta}{\psi} \exp \left(-\frac{\Sigma}{\psi} \right)}}.
\]

Replace the expression for \( q(N^*; f^*; f^*) \) and collect to get:

\[
\frac{(1 + T(n; f; f^*) - q(n; f; f^*) )}{(1 + T(n; f; f^*))} \frac{q(n; f; f^*) - q(N^*; f^*; f^*) / R}{R + (1 - \kappa) \times \left[ (1 - 1/R)^{-1} + g \frac{1 + \psi + \beta}{\psi w} \left( f^* b^g - \frac{R}{R - 1} (w^g + b^g) \right) \right]}.\]

Impose the inequality:

\[
\frac{\beta + \psi}{\psi} R > \frac{(1 + T(n; f; f^*) - q(n; f; f^*) )}{(1 + T(n; f; f^*))} \frac{q(n; f; f^*) - q(N^*; f^*; f^*) / R}{R + \frac{q(N^*; f^*; f^*)(1 - \kappa)}{\psi(\frac{w}{1 + \psi + \beta})^{1+\beta+\psi} (R \beta)^\frac{\beta}{\psi} \exp \left(-\frac{\Sigma}{\psi} \right)}}.
\]

and simplify to obtain:

\[
\frac{(\beta(R - 1) / \psi(1 - \kappa) - 1)}{R / (R - 1)} > g \frac{1 + \psi + \beta}{\psi w} \left( f^* b^g - \frac{R}{R - 1} (w^g + b^g) \right).
\]

Under the assumption that:

\[
\frac{\beta(R - 1)}{\psi(1 - \kappa)} > 1,
\]

the left-hand side is positive while the restriction that \( f^* \leq 1 \) guarantees that the right-hand side is negative. Thus, the inequality is always satisfied and the borrowing constraint never binds.
A.2 Equilibrium Under a Deviation

Consider a deviation $\tilde{f}'$ from the policy $f^*$. There are two cases according to whether the deviation induces young agents to borrow or not to borrow up to the limit.

A.2.1 Interior solution

If, given $\tilde{f}'$, young agents do not borrow to the limit, then

$$\tilde{l}(n, f, \tilde{f}'; f^*) = \frac{\psi}{1 + \beta + \psi} \left[ \left( 1 + \tilde{T}(n, f, \tilde{f}'; f^*) \right) \tilde{q}(n, f, \tilde{f}'; f^*) - q \left( \tilde{N}(n, f, \tilde{f}'; f^*), \tilde{f}'; f^* \right) / R \right]$$

$$c_o = \frac{R \beta}{1 + \beta + \psi} w,$$

$$c_y = \frac{w}{1 + \beta + \psi}.$$

The market clearing condition for land yields:

$$\tilde{N}(n, f, \tilde{f}'; f^*) = \frac{1 + \psi + \beta}{\psi} \frac{1}{w} \left[ \left( 1 + \tilde{T}(n, f, \tilde{f}'; f^*) \right) \tilde{q}(n, f, \tilde{f}'; f^*) - q \left( \tilde{N}(n, f, \tilde{f}'; f^*), \tilde{f}'; f^* \right) / R \right]$$

Replacing this into the utility function yields:

$$\tilde{V}(n, f, \tilde{f}'; f^*) = \ln \frac{w}{1 + \psi + \beta} - \psi \ln \tilde{N}(n, f, \tilde{f}'; f^*) + \beta \ln \frac{R \beta}{1 + \beta + \psi} w.$$

Impose that $\tilde{V}(n, f, \tilde{f}'; f^*) = \overline{V}$ and solve for $N^*$:

$$N^* = \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1 + \beta}{\psi}} (R \beta)^{\frac{\beta}{\psi}} \exp \left( -\frac{\overline{V}}{\psi} \right).$$
This is the same population as in equilibrium. Replace it in equation (26) to get:

\[
\left(1 + \tilde{T} \left( n, f, \tilde{f}^t; f^* \right) \right) \tilde{q} \left( n, f, \tilde{f}^t; f^* \right) = \frac{\psi w}{1 + \psi + \beta} N^* + q \left( \tilde{N} \left( n, f, \tilde{f}^t; f^* \right), \tilde{f}^t; f^* \right) / R.
\] (27)

From the expression (24) for equilibrium land prices:

\[
q \left( \tilde{N} \left( n, f, \tilde{f}^t; f^* \right), \tilde{f}^t; f^* \right) = \alpha_q^* - gb^\theta N^* \left( 1 - \tilde{f}^t \right),
\]

where \( \alpha_q^* \) is defined in equation (25).

From the government’s budget:

\[
\tilde{T} \left( n, f, \tilde{f}^t; f^* \right) \tilde{q} \left( n, f, \tilde{f}^t; f^* \right) = gN^* \left[ w^g + \frac{\tilde{f}^t b^\theta}{R} \right] + gn b^\theta \left( 1 - f \right).
\]

Replacing these two expressions into (27) we obtain:

\[
\tilde{q} \left( n, f, \tilde{f}^t; f^* \right) = \frac{\psi w}{1 + \psi + \beta} N^* + \frac{\alpha_q^*}{R} - gb^\theta N^* \frac{1 - \tilde{f}^t}{R} - gN^* \left[ w^g + \frac{\tilde{f}^t b^\theta}{R} \right] - gn b^\theta \left( 1 - f \right)
\]

or

\[
\tilde{q} \left( n, f, \tilde{f}^t; f^* \right) = \frac{\psi w}{1 + \psi + \beta} N^* + \frac{\alpha_q^*}{R} - gb^\theta N^* \frac{1 - \tilde{f}^t}{R} - gN^* w^g - gn b^\theta \left( 1 - f \right).
\]

This expression does not depend on \( \tilde{f}^t \). Thus, any \( f^* \) can be supported as the old agent is indifferent with regard to the most profitable deviation (there is none).
A.2.2 Borrowing to the limit

There might be deviations $\tilde{f}'$ such that the agent wants to borrow up to the limit.

The demand for bonds by the young is:

$$b_0 = R = w \frac{1 + \tilde{T} \left(n, f, \tilde{f}' ; f^* \right)}{1 + \tilde{T} \left(n, f, \tilde{f}' ; f^* \right)} \tilde{q} \left(n, f, \tilde{f}' ; f^* \right) \tilde{l} \left(n, f, \tilde{f}' ; f^* \right), \quad (28)$$

where:

$$\tilde{l} \left(n, f, \tilde{f}' ; f^* \right) = \frac{\psi}{1 + \beta + \psi} \frac{w}{\left[ \tilde{T} \left(n, f, \tilde{f}' ; f^* \right) \tilde{q} \left(n, f, \tilde{f}' ; f^* \right) - q \left(\tilde{N} \left(n, f, \tilde{f}' ; f^* \right) \tilde{f}' ; f^* \right) / R \right].}$$

The borrowing constraint is not binding when:

$$b' \geq -\kappa q \left(\tilde{N} \left(n, f, \tilde{f}' ; f^* \right) ; \tilde{f}' ; f^* \right) \tilde{l} \left(n, f, \tilde{f}' ; f^* \right). \quad (29)$$

Putting (28) and (29) together we obtain:

$$R w - R c_y - R \left[1 + \tilde{T} \left(n, f, \tilde{f}' ; f^* \right) \right] \tilde{q} \left(n, f, \tilde{f}' ; f^* \right) \tilde{l} \left(n, f, \tilde{f}' ; f^* \right) \geq -\kappa q \left(\tilde{N} \left(n, f, \tilde{f}' ; f^* \right) ; \tilde{f}' ; f^* \right) \tilde{l} \left(n, f, \tilde{f}' ; f^* \right).$$

Rearrange to obtain:

$$R w - R c_y \geq \left[ R \left(1 + \tilde{T} \left(n, f, \tilde{f}' ; f^* \right) \right) \tilde{q} \left(n, f, \tilde{f}' ; f^* \right) - \kappa q \left(N^* , \tilde{f}' ; f^* \right) \right] \tilde{l} \left(n, f, \tilde{f}' ; f^* \right).$$

Replace the demand for labor:
$$R^{\beta + \psi} \geq \frac{\left[ R \left( 1 + \tilde{T} \left( n, f, \tilde{f}^*; f^* \right) \right) \tilde{q} \left( n, f, \tilde{f}^*; f^* \right) - \kappa q \left( N^*, \tilde{f}^*; f^* \right) \right]}{\left( 1 + \tilde{T} \left( n, f, \tilde{f}^*; f^* \right) \right) \tilde{q} \left( n, f, \tilde{f}^*; f^* \right) - q \left( \tilde{N} \left( n, f, \tilde{f}^*; f^* \right), \tilde{f}^*; f^* \right) / R}.$$  

Simplify to obtain:

$$R^{\beta} \left( 1 + \tilde{T} \left( n, f, \tilde{f}^*; f^* \right) \right) \tilde{q} \left( n, f, \tilde{f}^*; f^* \right) \geq (\beta + \psi (1 - \kappa)) q \left( N^*, \tilde{f}^*; f^* \right).$$  

Replace the expression on the left-hand side using (27) and simplify:

$$R^{\beta} \frac{w}{1 + \psi + \beta} N^* \geq (1 - \kappa) q \left( N^*, \tilde{f}^*; f^* \right).$$  

Replace $q \left( N^*, \tilde{f}^*; f^* \right)$:

$$R^{\beta} \frac{w}{1 + \psi + \beta} N^* \geq (1 - \kappa) \alpha_q - (1 - \kappa) gb^\theta N^* \left( 1 - \tilde{f}^* \right).$$  

Solve for $\tilde{f}^*$:

$$R^{\beta} \frac{w}{1 + \psi + \beta} N^* \geq (1 - \kappa) \alpha_q - (1 - \kappa) gb^\theta N^* + (1 - \kappa) gb^\theta N^* \tilde{f}^*.$$  

Thus:

$$\tilde{f}^* \leq \frac{R^{\beta} \frac{w}{1 + \psi + \beta} N^* - \alpha_q}{gb^\theta N^*} + 1. \quad (30)$$  

Now, replace $\alpha^*_q$ from equation (25), keeping in mind the definition of $q \left( n, f; f^* \right)$:

$$\alpha^*_q = \psi \left( \frac{w}{1 + \psi + \beta} \right)^{1 + \phi + \psi} (R^{\beta})^{\phi} \exp \left( -\frac{\psi}{w} \right) + \frac{\alpha_q - gb^\theta N^*(1 - f^*)}{R} - gN^* \left( w^q + \frac{f^*b^q}{R} \right).$$
This can be solved for $\alpha_q^*$:

$$
\alpha_q^* \equiv (1 - 1/R)^{-1} \psi \left( \frac{w}{1 + \psi + \beta} \right) N^* - (1 - 1/R)^{-1} \frac{g b^\theta N^* (1 - f^*)}{R} - (1 - 1/R)^{-1} g N^* \left( w^g + \frac{f^* b^g}{R} \right).
$$

It follows that:

$$
\frac{\alpha_q^*}{g b^\theta N^*} = \frac{1}{R - 1} \left[ R \left( \frac{w \psi}{1 + \psi + \beta} \right) \frac{1}{g b^\theta} - 1 - R \frac{w^g}{b^g} \right].
$$

Replace it in the inequality (30) and simplify:

$$
\tilde{f}' \leq \frac{R}{R - 1} \left\{ \left( \frac{\beta (R - 1)}{1 - \kappa} - \psi \right) \frac{w}{g b^\theta} \frac{1}{1 + \psi + \beta} + 1 + \frac{w^g}{b^g} \right\}. \quad (31)
$$

This is the cut-off such that for less funding $\tilde{f}'$ than the expression on the right-hand side the borrowing constraint does not bind. Notice that the expression on the right-hand side is larger than one if we assume that:

$$
\frac{\beta (R - 1)}{1 - \kappa} - \psi \geq 0.
$$

This is a sufficient condition only. Since $\tilde{f}'$ is always less than one, the inequality (31) is always satisfied. Thus, in every deviation between zero and one, the borrowing constraint is never binding.

Finally impose

$$
N^* = 1
$$

from equation (21) and solve for $V^*$ in equation (60).
B Proof of Proposition 2

We divide the proof in different parts. First, in Section B.1 we guess that the equilibrium policy is a constant and solve for the equilibrium given this policy. Second, in Section B.2 we consider a one-period deviation from this policy. Third, in Section B.3 we impose consistency between the optimal deviation and the equilibrium policy and solve for the latter.

B.1 Equilibrium Under a Constant Policy

Denote by $f^+$ the constant funding policy we guess:

$$f^+ = F(n, f),$$

for all $(n, f)$. In what follows I replace $F$ with $f^+$ in the equilibrium functions.

Consider the problem (7) faced by a young agent and impose that the borrowing constraint is binding:

$$b' = -\kappa q \left( N(n, f; f^+) , f^+; f^+ \right) l.$$

At the end of this section we impose the condition on $f^+$ that guarantees that this is the case. Use the borrowing constraint to replace $b'$ in the budget constraints, which become:

$$c_y = w + \left( \kappa q \left( N(n, f; f^+) , f^+; f^+ \right) / R - \left( 1 + T(n, f; f^+) \right) q(n, f; f^+) \right) l$$

$$c_o = (1 - \kappa) q \left( N(n, f; f^+) , f^+; f^+ \right) l.$$
The optimization problem is now:

$$\max_{l} \left\{ \ln \left[ w + (\kappa q (N (n, f; f^+), f^+; f^+)) / R - (1 + T (n, f; f^+)) q (n, f; f^+) \right] l + \psi \ln l + \beta \ln (1 - \kappa) q (N (n, f; f^+), f^+; f^+) l \right\}.$$ 

The first-order condition for land is:

$$\frac{(\kappa q (N (n, f; f^+), f^+; f^+) / R - (1 + T (n, f; f^+)) q (n, f; f^+))}{w + (\kappa q (N (n, f; f^+), f^+; f^+) / R - (1 + T (n, f; f^+)) q (n, f; f^+) / R} + \frac{\psi + \beta}{l} = 0.$$

Solve for the demand for land:

$$l (n, f; f^+) = \frac{\psi + \beta}{1 + \psi + \beta (1 + T (n, f; f^+)) q (n, f; f^+) - \kappa q (N (n, f; f^+), f^+; f^+) / R} \frac{w}{1 + \psi + \beta}.$$

Impose land market equilibrium (equation 5) and solve for $N (n, f; f^+)$:

$$N (n, f; f^+) = \frac{1 + \psi + \beta}{\psi + \beta} \frac{1}{w} \left[ (1 + T (n, f; f^+)) q (n, f; f^+) - \kappa q (N (n, f; f^+), f^+; f^+) / R \right].$$

Notice that the indirect utility function of a young agent is:

$$V (n, f; f^+) = \ln \frac{w}{1 + \psi + \beta} + \psi \ln (1 - \kappa) - \psi \ln N (n, f; f^+) + \beta q (N (n, f; f^+), f^+; f^+).$$

(33)
Replace $N(n, f; f^+)$ into it and impose that $V(n, f; f^+) = \overline{V}$:

\[
\ln \frac{w}{1 + \psi + \beta} + \beta \ln (1 - \kappa) - (\psi + \beta) \ln \frac{1 + \psi + \beta}{\psi + \beta} \frac{1}{w} [(1 + T(n, f; f^+)) q(n, f; f^+) - \kappa q(N(n, f; f^+), f^+; f^+) / R] + \beta \ln q(N(n, f; f^+), f^+; f^+) = \overline{V}.
\]

Solve for the price of land gross of taxes:

\[
q(n, f; f^+) (1 + T(n, f; f^+)) = \alpha_N q(N(n, f; f^+), f^+; f^+) \frac{\beta}{\psi + \beta} + \kappa q(N(n, f; f^+), f^+; f^+) / R,
\]

where

\[
\alpha_N = \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1 + \psi + \beta}{\psi + \beta}} (1 - \kappa) \frac{\beta}{\psi + \beta} (\psi + \beta)^{\psi + \beta} (\psi + \beta)^{\psi + \beta} (1 - \kappa) \exp \left( -\frac{\overline{V}}{\psi + \beta} \right).
\]

Replace this expression in equation (32) to get:

\[
N(n, f; f^+) = \frac{1 + \psi + \beta}{\psi + \beta} \frac{1}{w} \alpha_N q(N(n, f; f^+), f^+; f^+) \frac{\beta}{\psi + \beta}.
\]

This shows that $N(n, f; f^+)$ is a constant that is independently of the state $(n, f)$.

Denote it by $N^{**} = N(n, f; f^+)$ for all $(n, f)$.

From the government’s budget (4) the tax rate is such that:

\[
T(n, f; f^+) q(n, f; f^+) = gN^{**} \left[ \frac{w \alpha + \frac{f^+ b^g}{R}}{g N^{**}} \right] + gnb^g (1 - f).
\]

Use equation (34) and (35), to write the price of land as:

\[
q(n, f; f^+) = \alpha_N q(N^{**}, f^+; f^+) \frac{\beta}{\psi + \beta} + \kappa q(N^{**}, f^+; f^+) / R - gN^{**} \left( \frac{w \alpha + \frac{f^+ b^g}{R}}{g N^{**}} \right) - gnb^g (1 - f).
\]
This can be written more compactly as:

\[
q(n, f; f^+) = \alpha_q^{**} - gb^n (1 - f),
\]

(36)

where

\[
\alpha_q^{**} \equiv \alpha_N q(N^{**}, f^+; f^+) \frac{\beta}{\psi + \beta} + \kappa q(N^{**}, f^+; f^+) / R - g N^{**} \left( w^g + \frac{f^g + (1 - f^+)}{R} \right). \tag{37}
\]

Notice that \( \alpha_q^{**} \) is a function of \( N^{**} \). The tax rate \( T(n, f; f^+) \) must then satisfy (35). Now, solve for \([N^{**}, q(N^{**}, f^+; f^+)]\) using these two equations:

\[
N^{**} = \frac{1 + \psi + \beta}{\psi + \beta} \frac{1}{w} \alpha_N q(N^{**}, f^+; f^+) \frac{\beta}{\psi + \beta}, \tag{38}
\]

\[
q(N^{**}, f^+; f^+) = \alpha_q^{**} - gb^g N^{**} (1 - f^+). \tag{39}
\]

Replace \( \alpha_q^{**} \) in the second:

\[
q(N^{**}, f^+; f^+) = \alpha_N q(N^{**}, f^+; f^+) \frac{\beta}{\psi + \beta} + \kappa q(N^{**}, f^+; f^+) / R - g N^* \left( w^g + b^g \frac{f^+ + (1 - f^+)}{R} \right). \tag{40}
\]

Solve for \( q(N^{**}, f^+; f^+) \):

\[
q(N^{**}, f^+; f^+) = (1 - \kappa / R)^{-1} \left[ \alpha_N q(N^{**}, f^+; f^+) \frac{\beta}{\psi + \beta} - g N^* \left( w^g + b^g \frac{f^+ + (1 - f^+)}{R} \right) \right]. \tag{41}
\]

Use equation (38) to solve for

\[
\alpha_N q(N^{**}, f^+; f^+) \frac{\beta}{\psi + \beta} = \frac{\psi + \beta}{1 + \psi + \beta} w N^*. \tag{41}
\]
and replace it in equation (40):

\[ q(N^*, f^+; f^+) = (1 - \kappa / R)^{-1} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + b^g \frac{1 + (1 - f^+) (R - 1)}{R} \right) \right\} N^*. \]  

(42)

Replace the land price back into (41):

\[ \alpha_N (1 - \kappa / R)^{-\frac{\alpha}{\psi + \beta}} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + b^g \frac{1 + (1 - f^+) (R - 1)}{R} \right) \right\} \frac{\psi}{\psi + \beta} N^* N^* \]

\[ = \frac{\psi + \beta}{1 + \psi + \beta} w N^* \]

and solve for \( N^* \):

\[ (N^*)^{\frac{\psi}{\psi + \beta}} = (1 - \kappa / R)^{-\frac{\alpha}{\psi + \beta}} \alpha_N \frac{1 + \psi + \beta}{w} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + b^g \frac{1 + (1 - f^+) (R - 1)}{R} \right) \right\} \frac{\psi}{\psi + \beta}. \]

Simplify this expression to obtain:

\[ N^* = (1 - \kappa / R)^{-\frac{\alpha}{\psi + \beta}} \left( \frac{\alpha_N}{w} \frac{1 + \psi + \beta}{\psi + \beta} \right)^{\frac{\psi + \beta}{\psi}} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + b^g \left( 1 - \frac{(R - 1) f^+}{R} \right) \right) \right\}^{1 + \frac{\psi}{\psi + \beta}}, \]

or equivalently the expression in the text of the proposition.

Notice that a higher \( f^+ \) leads to a larger population in steady state. Given this we can compute \( q(N^*, f^+; f^+) \) from (42):

\[ q(N^*, f^+; f^+) = (1 - \kappa / R)^{-\left(1 + \frac{\alpha}{\psi} \right)} \left( \frac{\alpha_N}{w} \frac{1 + \psi + \beta}{\psi + \beta} \right)^{\frac{\psi + \beta}{\psi}} \times \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + b^g \left( 1 - \frac{(R - 1) f^+}{R} \right) \right) \right\}^{-1 + \frac{\psi}{\psi + \beta}}. \]

(43)

Land prices increase in \( f^+ \) in the steady state. Finally, given \( N^* \) and \( q(N^*, f^+; f^+) \) we can solve for the coefficient \( \alpha^*_q \) in equation (37).
The last step of the proof is to impose the condition that the agent chooses to borrow an amount equal to the borrowing limit or equivalently that the Euler equation for bonds holds as a strict inequality:

\[ \frac{1}{c_y} > \frac{\beta R}{c_o} \]  

(44)

where

\[ c_y = \frac{w}{1 + \psi + \beta} \]
\[ c_o = (1 - \kappa) q (N (n, f; f^+), f^+; f^+) l (n, f; f^+) \]

where recall that \( q (N (n, f; f^+), f^+; f^+) = q (N^{**}, f^+; f^+) \). Replacing these and \( l (n, f; f^+) \) into equation (44) and simplifying we obtain the inequality:

\[ 1 > \frac{\beta [R (1 + T (n, f; f^+)) q (n, f; f^+) - \kappa q (N^{**}, f^+; f^+)]}{(1 - \kappa) q (N^{**}, f^+; f^+) (\psi + \beta)}. \]

Notice that

\[ R (1 + T (n, f; f^+)) q (n, f; f^+) - \kappa q (N^{**}, f^+; f^+) = \alpha_N q (N^{**}, f^+; f^+) \psi^{\beta \psi + \beta}. \]

Thus, also replacing \( \alpha_N \), the inequality becomes:

\[ 1 > \frac{\beta}{[(1 - \kappa) q (N^{**}, f^+; f^+)]^{\psi + \beta}} \left( \frac{w}{1 + \psi + \beta} \right)^{1+\psi+\beta} \exp \left( - \frac{\psi}{\psi + \beta} \right). \]

After appropriate substitutions this inequality becomes the following constraint on \( f^+ \):

\[ f^+ > \frac{R}{R - 1} \left[ \frac{w^g}{b^{\psi + \beta}} + 1 - \frac{w}{g b^{\psi + \beta}} \frac{1}{1 + \psi + \beta} \left( \psi - \beta \frac{R - 1}{1 - \kappa} \right) \right]. \]  

(45)
Notice that $f^+$ is by definition bounded from below by $f_{\text{min}} \geq 0$. Thus, if $g$ is sufficiently small, as assumed in Proposition 2, point 2., then the right-hand side of this inequality is always negative and the inequality is always satisfied.

### B.2 Equilibrium After a One Period Deviation

Consider now the equilibrium after a one-period deviation in the funding level to $	ilde{f}'$.

I consider deviation in which the borrowing constraint is binding so that:

$$b' = -\kappa q \left( \tilde{N} \left( n, f, \tilde{f}'; f^+ \right) \right) l.$$  

In this case the demand for land is:

$$\tilde{l} \left( n, f, \tilde{f}'; f^+ \right) = \frac{\psi + \beta}{1 + \psi + \beta} \left( 1 + \tilde{N} \left( n, f, \tilde{f}'; f^+ \right) \right) \frac{w}{\tilde{q} \left( n, f, \tilde{f}'; f^+ \right) - \kappa q \left( \tilde{N} \left( n, f, \tilde{f}'; f^+ \right) \right)}.$$  

The market clearing condition for land implies that population is given by:

$$\tilde{N} \left( n, f, \tilde{f}'; f^+ \right) = \frac{1 + \psi + \beta}{\psi + \beta} \frac{1}{w} \times$$

$$\left[ \left( 1 + \tilde{T} \left( n, f, \tilde{f}'; f^+ \right) \right) \tilde{q} \left( n, f, \tilde{f}'; f^+ \right) - \kappa q \left( \tilde{N} \left( n, f, \tilde{f}'; f^+ \right) \right) / R \right].$$

Notice that the indirect utility of an agent who is young at the time of the deviation is:

$$\tilde{V} \left( n, f, \tilde{f}'; f^+ \right) = \ln \frac{w}{1 + \psi + \beta} + \beta \ln (1 - \kappa) - (\psi + \beta) \ln \tilde{N} \left( n, f, \tilde{f}'; f^+ \right)$$

$$+ \beta \ln q \left( \tilde{N} \left( n, f, \tilde{f}'; f^+ \right) \right),$$

where the land price on the right-hand side of this equation is determined by the
equilibrium land pricing function.

Impose that $\bar{V} \left( n, f, \bar{f}^i; f^+ \right) = V$ and replace into it the expression for population:

$$\ln \frac{w}{1 + \psi + \beta} + \beta \ln (1 - \kappa) - (\psi + \beta) \ln \frac{1 + \psi + \beta}{\psi + \beta} \frac{1}{w} \left[ (1 + \bar{T} \left( n, f, \bar{f}^i; f^+ \right)) \bar{q} \left( n, f, \bar{f}^i; f^+ \right) - \kappa q \left( \bar{N} \left( n, f, \bar{f}^i; f^+ \right), \bar{f}^i; f^+ \right) / R \right] + \beta \ln q \left( \bar{N} \left( n, f, \bar{f}^i; f^+ \right), \bar{f}^i; f^+ \right) = V.$$ 

Solve it for the recursive equation for land prices:

$$\left( 1 + \bar{T} \left( n, f, \bar{f}^i; f^+ \right) \right) \bar{q} \left( n, f, \bar{f}^i; f^+ \right) = \alpha_N q \left( \bar{N} \left( n, f, \bar{f}^i; f^+ \right), \bar{f}^i; f^+ \right) \left( \frac{\beta}{1 + \psi + \beta} \right)^{\frac{\beta}{\psi + \beta}} (49)$$

Notice that from equation (36):

$$q \left( \bar{N} \left( n, f, \bar{f}^i; f^+ \right), \bar{f}^i; f^+ \right) = \alpha_q^{**} - g b^\theta \bar{N} \left( n, f, \bar{f}^i; f^+ \right) \left( 1 - \bar{f}^i \right). (50)$$

Replace this into the right-hand side of equation (49)

$$\left( 1 + \bar{T} \left( n, f, \bar{f}^i; f^+ \right) \right) \bar{q} \left( n, f, \bar{f}^i; f^+ \right) = \alpha_N q \left( \bar{N} \left( n, f, \bar{f}^i; f^+ \right), \bar{f}^i; f^+ \right) \left( 1 - \bar{f}^i \right) \left( \frac{\beta}{1 + \psi + \beta} \right)^{\frac{\beta}{\psi + \beta}} + \kappa \left[ \alpha_q^{**} - g b^\theta \bar{N} \left( n, f, \bar{f}^i; f^+ \right) \left( 1 - \bar{f}^i \right) \right] / R,$$

and the latter in (47) to obtain the implicit expression for $\bar{N} \left( n, f, \bar{f}^i; f^+ \right)$:

$$\bar{N} \left( n, f, \bar{f}^i; f^+ \right) = \frac{1 + \psi + \beta}{\psi + \beta} \frac{\alpha_N}{w} \left[ \alpha_q^{**} - g b^\theta \bar{N} \left( n, f, \bar{f}^i; f^+ \right) \left( 1 - \bar{f}^i \right) \right] \left( \frac{\beta}{\psi + \beta} \right)^{\frac{\beta}{\psi + \beta}}. (51)$$

This equation shows that $\bar{N} \left( n, f, \bar{f}^i; f^+ \right)$ does not depend on $(n, f)$ but only on
\[ \tilde{f}^t \text{. Thus, we can write:} \]
\[ \tilde{N} \left( n, f, \tilde{f}^t; f^+ \right) = \tilde{N}(\tilde{f}^t). \]

where \( \tilde{N}(\tilde{f}^t) \) is implicitly defined by (51).

The budget constraint of the government is:
\[ \tilde{T} \left( n, f, \tilde{f}^t; f^+ \right) \tilde{q} \left( n, f, \tilde{f}^t; f^+ \right) = g\tilde{N}(\tilde{f}^t) \left[ w^g + \frac{\tilde{f}^t b^g}{R} \right] + gnb^g (1 - f). \quad (52) \]

Then, the land price equation (49) implies that
\[ \tilde{q} \left( n, f, \tilde{f}^t; f^+ \right) = \alpha_N q \left( \tilde{N}(\tilde{f}^t), \tilde{f}^t; f^+ \right) \frac{\psi}{\psi + \gamma} + \kappa q \left( \tilde{N}(\tilde{f}^t), \tilde{f}^t; f^+ \right) / R - g\tilde{N}(\tilde{f}^t) \left[ w^g + \frac{\tilde{f}^t b^g}{R} \right] - gnb^g (1 - f). \]

This can be simplified by using equations (50) and (51):
\[ \tilde{q} \left( n, f, \tilde{f}^t; f^+ \right) = \frac{\kappa \alpha_q^*}{R} \left[ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + \kappa \frac{b^g}{R} + \frac{\tilde{f}^t b^g (1 - \kappa)}{R} \right) \right] \tilde{N}(\tilde{f}^t) - gnb^g (1 - f). \quad (53) \]

The last step is to impose the condition that the agent chooses to borrow the maximum amount when facing the prices derived above. This requires that equation (44) holds when
\[ c_y = \frac{w}{1 + \psi + \beta} \]
\[ c_o = (1 - \kappa) q \left( \tilde{N}(\tilde{f}^t), \tilde{f}^t; f^+ \right) \tilde{I} \left( n, f, \tilde{f}^t; f^+ \right). \]

Imposing the inequality in (44) and simplifying we obtain the following restriction:
\[ \tilde{N} \left( \tilde{f}^t \right) > \frac{1 + \psi + \beta}{\psi + \beta} \frac{\alpha_N}{w} \left( \frac{\beta R}{(1 - \kappa) (\psi + \beta \alpha_N)} \right)^{\frac{\psi}{\psi + \beta}}. \quad (54) \]
Notice that the function \( \tilde{N}(\tilde{f}') \) is strictly increasing in \( \tilde{f}' \), so the condition above can equivalently be written in terms of \( \tilde{f}' \):

\[
\tilde{f}' > \tilde{N}^{-1}\left[ \frac{1 + \psi + \beta}{\psi + \beta} \frac{\alpha_N}{w} \left( \frac{\beta R}{(1 - \kappa)(\psi + \beta)\alpha_N} \right) \frac{\beta}{\psi} \right]
\]

where \( \tilde{N}^{-1} \) is the inverse function of \( \tilde{N}(\tilde{f}') \) in (51). The condition above becomes:

\[
\tilde{f}' > -\frac{\alpha_q^{**}}{gb^0} \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1+\beta}{\psi}} \exp \left( \frac{\bar{V}}{\psi} \right) (\beta R)^{-\frac{\beta}{\psi}} + \frac{w}{gb^0} \frac{\beta R}{1 - \kappa} \frac{1}{1 + \psi + \beta} + 1.
\]

It is possible to show that under Assumption 13 the term on the right-hand side of this inequality can be made arbitrarily negative as \( g \to 0 \). Thus, in this case the borrowing constraint will always be binding. The equilibrium after a deviation is therefore comprised by the land price equation after a deviation (53), the population equation after a deviation (51), the tax function \( \tilde{T} \left( n, f, \tilde{f}'; f^+ \right) \) from (52), the land demand function (46), and the indirect utility function (48).

### B.3 Equilibrium Without Commitment

The policy-maker chooses \( \tilde{f}' \) to maximize \( \tilde{q} \left( n, f, \tilde{f}'; f^+ \right) \), taking as given \( f^+ \). In what follows, we first characterize the first-order condition for its problem and then impose consistency, i.e., that the optimal deviation is consistent with the policy rule postulated in part B.1. We operate under the assumption that the optimal deviation, i.e. \( f^+ \), is such that the borrowing constraint binds. The condition under which this is the case is in equation (45), which is always satisfied if \( g \) is small enough.

Notice from (53) that the optimal deviation is independent of \( (n, f) \), which is consistent with our guess that the equilibrium policy is a constant. The function \( \tilde{q} \left( n, f, \tilde{f}'; f^+ \right) \) is concave in \( \tilde{f}' \). There are two cases to consider, depending on...
whether the equilibrium solution is interior or not.

For an interior solution we set the first derivative of the land price function after a deviation to zero:

$$\frac{\partial \bar{q}(n, f, \bar{f}; f^+)}{\partial \bar{f}} = 0.$$ 

The full solution to the problem is then obtained by imposing that the optimal deviation $\bar{f}^*$ is equal to $f^*$, when the function $f^+$ is uniformly equal to $f^*$. Formally, if the solution is interior it satisfies:

$$\frac{\partial \bar{q}(n, f, f^*; f^*)}{\partial \bar{f}} = 0.$$

Alternatively, we could have a corner solution at $f_{\text{min}}$. Imposing consistency, the following condition has to be verified in equilibrium:

$$\frac{\partial \bar{q}(n, f, f_{\text{min}}; f_{\text{min}})}{\partial \bar{f}} < 0.$$

Given that the pricing function $\bar{q}(n, f, \bar{f}; f^+)$ is concave in $\bar{f}$, we can then solve for the interior solution first, and then verify whether it is larger than $f_{\text{min}}$. If not, then the equilibrium is simply $f_{\text{min}} \geq 0$.

The first order condition is:

$$\bar{N}'(\bar{f}) \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + \frac{\kappa}{R} b^g + \frac{\bar{f} b^g (1 - \kappa)}{R} \right) \right\} = \bar{N}(\bar{f}) \frac{gb^g (1 - \kappa)}{R}. \quad (55)$$

Notice that applying the implicit function theorem to equation (51):

$$\bar{N}(\bar{f}) = \left[ \frac{1 + \psi + \beta}{w} \frac{\alpha_N}{\psi + \beta} \left[ \alpha_{q^*} - gb^g N(\bar{f}) (1 - \bar{f}) \right] \right]^{\frac{\beta}{\psi + \beta}} = 0,$$
we obtain:
\[
\frac{\partial \tilde{N}}{\partial \tilde{f}'} = \frac{1+\psi+\beta}{\psi+\beta} \frac{\alpha_q^{**} - gb^g \tilde{N}(\tilde{f}')(1-\tilde{f}')}{w} \left[ \frac{\beta}{\psi+\beta} - 1 \right] \frac{\beta}{\psi+\beta} gb^g \tilde{N}(\tilde{f}') \left( 1 - \tilde{f}' \right).
\]

Use the definition of \( \tilde{N}(\tilde{f}') \) this derivative can be written as:
\[
\frac{\partial \tilde{N}}{\partial \tilde{f}'} = \frac{1+\psi+\beta}{\psi+\beta} \frac{\alpha_q^{**} - gb^g \tilde{N}(\tilde{f}')(1-\tilde{f}')}{w} \left[ \frac{\beta}{\psi+\beta} - 1 \right] \frac{\beta}{\psi+\beta} gb^g \tilde{N}(\tilde{f}') \left( 1 - \tilde{f}' \right)
= \frac{\tilde{N}(\tilde{f}')^2 \beta gb^g}{\alpha_q^{**} (\psi + \beta) - gb^g \tilde{N}(\tilde{f}')(1-\tilde{f}')}
\]

Use it in the foc (55):
\[
\frac{\tilde{N}(\tilde{f}')^2 \beta gb^g}{\alpha_q^{**} (\psi + \beta) - gb^g \tilde{N}(\tilde{f}')(1-\tilde{f}')} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( \frac{w^g + \kappa b^g}{R} + \frac{\tilde{f}'b^g (1 - \kappa)}{R} \right) \right\} = \frac{\tilde{N}(\tilde{f}') gb^g (1 - \kappa)}{R}.
\]

Simplify:
\[
\frac{\tilde{N}(\tilde{f}')^2 \beta}{\alpha_q^{**} (\psi + \beta) - gb^g \tilde{N}(\tilde{f}')(1-\tilde{f}')} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( \frac{w^g + \kappa b^g}{R} + \frac{\tilde{f}'b^g (1 - \kappa)}{R} \right) \right\} = 1 - \frac{\kappa}{R}.
\]

This equation, together with the definition of the function \( \tilde{N}(\tilde{f}') \) determines the optimal deviation \( \tilde{f}^+ \). Consistency then requires
\[
\tilde{f}^+ = f^+.
\]
The only point where $f^+$ enters equations (51) and (56) is through the coefficient $\alpha_{q}^{**}$ defined in (37). So the optimal policy is $f^+$ such that:

\[
\frac{\tilde{N}(f^+)^\beta}{\alpha_{q}^{**} (\psi + \beta) - \psi gb^{q} \tilde{N}(f^+) (1 - f^+)} \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( \frac{w^g + \frac{\kappa}{R} b^g + \frac{f^+ b^g (1 - \kappa)}{R}}{1 - \kappa} \right) \right\} = \frac{1 - \kappa}{R}. \tag{57}
\]

Notice that

\[
\tilde{N}(f^+) = N^{**}
\]

because the population when policy is optimal must equal to equilibrium population. Moreover, by definition (37):

\[
\alpha_{q}^{**} = \left( 1 - \frac{\kappa}{R} \right)^{-1} \left[ \frac{\psi + \beta}{1 + \psi + \beta} w - \frac{\kappa}{R} gb^{q} (1 - f^+) - g \left( \frac{w^g + \frac{f^+ b^g}{R}}{1 - \kappa} \right) \right] N^{**}. \tag{58}
\]

Replace this into (57) to get:

\[
\frac{N^{**} \beta}{(1 - \frac{\kappa}{R})^{-1} \left[ \frac{\psi + \beta}{1 + \psi + \beta} w - \frac{\kappa}{R} gb^{q} (1 - f^+) - g \left( \frac{w^g + \frac{f^+ b^g}{R}}{1 - \kappa} \right) \right] N^{**} (\psi + \beta) - \psi gb^{q} N^{**} (1 - f^+)} \times \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( \frac{w^g + \frac{\kappa}{R} b^g + \frac{f^+ b^g (1 - \kappa)}{R}}{1 - \kappa} \right) \right\} = \frac{1 - \kappa}{R}
\]

Simplify $N^{**}$:

\[
\frac{\beta}{(1 - \frac{\kappa}{R})^{-1} \left[ \frac{\psi + \beta}{1 + \psi + \beta} w - \frac{\kappa}{R} gb^{q} (1 + f^+) - g \left( \frac{w^g + \frac{f^+ b^g}{R}}{1 - \kappa} \right) \right] (\psi + \beta) - \psi gb^{q} (1 - f^+)} \times \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( \frac{w^g + \frac{\kappa}{R} b^g + \frac{f^+ b^g (1 - \kappa)}{R}}{1 - \kappa} \right) \right\} = \frac{1 - \kappa}{R}
\]
Keep simplifying:

\[ \beta (R - \kappa) \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + \frac{\kappa}{R} b^g + \frac{f^g b^g (1 - \kappa)}{R} \right) \right\} \]

\[ = (1 - \kappa) \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + \frac{\kappa}{R} b^g + \frac{f^g b^g (1 - \kappa)}{R} \right) \right\} (\psi + \beta) - (1 - \kappa) \psi g^b g^b (1 - f^+). \]

Simplify more:

\[ [(1 - \kappa) \psi - \beta (R - 1)] \left\{ \frac{\psi + \beta}{1 + \psi + \beta} w - g \left( w^g + \frac{\kappa}{R} b^g + \frac{f^g b^g (1 - \kappa)}{R} \right) \right\} = (1 - \kappa) \psi g^b g^b (1 - f^+). \]

Solve for the interior solution \( f^+ \), denoted by \( f^{**} \):

\[
 f^{**} = \frac{R}{\kappa \psi + (\beta + \psi) (R - 1)} \left\{ \left( \frac{\beta (R - 1)}{1 - \kappa} - \psi \right) \left( \frac{\psi + \beta}{1 + \psi + \beta} g^b g^b - \left( \frac{w^g}{b^g} + \frac{\kappa}{R} \right) \right) + \psi \right\}.
\]

Alternatively, if

\[ f^{**} < f_{\min} \]

then the equilibrium solution is simply \( f_{\min} \).

Impose the general equilibrium condition that

\[ N(n, f; f^+) = 1 \]

or replacing the solution for \( N(n, f; f^+) \):

\[ 1 = \left( \frac{w}{1 + \psi + \beta} \right)^{\frac{1}{\psi}} \exp \left( -\frac{V}{\psi} \right) \times \left\{ \left( \frac{1 - \kappa}{1 - \kappa/R} \right) gb^g \left( \frac{\psi + \beta}{1 + \psi + \beta} gb^b - \frac{w^g}{b^g} - 1 + \frac{R - 1}{R} \max (f_{\min}, f^{**}) \right) \right\}^{\frac{\psi}{\beta}}. \]
Solve for \( V^{**} \):

\[
V^{**} = \ln \frac{w}{1 + \psi + \beta} + \beta \ln \left\{ \left( \frac{1 - \kappa}{1 - \kappa/R} \right)^{gb^\theta} \left( \frac{\psi + \beta}{1 + \psi + \beta} \frac{w}{gb^\theta} - \frac{w^g}{b^\theta} - 1 + \frac{R - 1}{R} \max(f_{\min}, f^{**}) \right) \right\}.
\]

C Proof of Proposition 3

a) Consider the case studied in Section 3.1. We need to show that the equilibrium level of utility \( V^* \) achieved in this economy is also independent of \( f^* \). This is obtained by imposing that the supply of population in the economy is equal to the demand by all cities or:

\[
N(n, f; F) = 1
\]

for all \((n, f)\). In any equilibrium of the economy described in Section 3.1 the utility level \( V^* \) is the same and equal to:

\[
V^* = \ln \frac{w}{1 + \psi + \beta} + \beta \ln \left( \frac{R\beta w}{1 + \psi + \beta} \right).
\]  \hspace{1cm} (60)

Notice that the expression for \( V^* \) is independent of \( f^* \).

b) Consider the case studied in Section 3.2. Following the same argument as above, the utility level \( V^{**} \) in equilibrium is given by:

\[
V^{**} = \ln \frac{w}{1 + \psi + \beta} + \beta \ln \left\{ \left( \frac{1 - \kappa}{1 - \kappa/R} \right)^{gb^\theta} \left( \frac{\psi + \beta}{1 + \psi + \beta} \frac{w}{gb^\theta} - \frac{w^g}{b^\theta} - 1 + \frac{R - 1}{R} \max(f_{\min}, f^{**}) \right) \right\}.
\]

Notice that the general equilibrium level of utility \( V^{**} \) is an increasing function of the pension funding policy adopted by the city. Q.E.D.