Advertising, Innovation and Economic Growth*

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Abstract

We develop a model of firm dynamics through product innovation that explicitly incorporates advertising decisions by firms. We model advertising by constructing a framework that unifies a number of facts identified by the empirical marketing literature. The model is then used to explain several empirical regularities across firm sizes using U.S. data. Through a novel interaction between R&D and advertising, we are able to explain empirically observed deviations from Gibrat's law, as well as the behavior of advertising expenditures across firms, the degree of substitution between R&D and advertising expenditures as firms grow large, and broadly the effects of advertising on both firm and economic growth. We find that smaller firms can be both more innovation- and advertising-intensive as in the data even when there exist increasing returns to scale in research.

JEL codes: E2; L1; M3; O31, O32, O33, and O41.

Keywords: Endogenous growth; Advertising; Innovation; Research and Development; Firm size distribution.

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1 Introduction

Since the seminal contributions of [Romer \(1990\)](#), [Grossman and Helpman \(1991\)](#) and [Aghion and Howitt \(1992\)](#), the economic literature has widely emphasized the role of innovation in the process of economic growth. Research and Development (R&D) introduces goods of higher quality and enhanced production technologies which increase living standards. Most advanced economies spend a non-negligible share of their income on R&D. In the United States, the share of R&D expenditures over GDP fluctuated between 2.27 and 2.82% over the period 1981 – 2013 (Figure 1).

Figure 1: R&D and advertising expenditures as a share of GDP (U.S.). Source: OECD databases (for R&D) and Coen Structured Advertising Expenditure Dataset (for advertising).

From a firm’s perspective, innovation is strategically used to increase profits. By performing R&D, firms can increase their market share by selling higher quality goods and diverting demand from lower quality products. Innovation can take two different forms with slightly different effects. By increasing the quality of the goods that they already produce (incremental/internal innovation), incumbent firms can increase their market share. On the other hand, external/radical innovation (i.e. the process through which a firm enters a new market) increases firms’ profit through creative destruction displacing the former incumbent by a new higher quality good. In both cases, nevertheless, increasing product quality through innovation acts as a consumer demand shifter.

While these processes and their implications for economic growth are relatively well acknowledged in the literature, there exist other tools that firms can use to shift demand towards their products. One obvious and non-negligible example is advertising. By advertising their goods, firms can alter consumers’ perceptions and ultimately increase demand for their products. This suggests that innovation and advertising may be substitutable tools in the quest of firms for higher profits. Furthermore, advertising decisions by firms are potentially not neutral in terms of global economic growth in the economy. While firms in the U.S. spend around 2.2% of GDP on advertising each year (Figure 1), the

growth literature is silent on the potential interaction between R&D and advertising expenditures and economic growth.

In this paper, we propose a model of endogenous growth with explicit R&D and advertising choices by firms. Firms are heterogeneous and own different portfolios of intermediate goods that they monopolistically supply. Firms are characterized by the number of goods in their portfolio and their associated quality which grows on a ladder through innovation. R&D in the model takes two forms. Through internal R&D, firms can increase the quality of the goods that they own. The resulting incremental improvement in the quality of their products allows firms to increase their market shares and their profits. On the other hand, external R&D allows firms to improve on the quality of a good that they do not own and displace the former producer through creative destruction. External R&D is also available to potential entrants (firms which do not own any good in their portfolio). Besides R&D, firms can use an alternative tool to expand their market shares and profit. Even though the actual (intrinsic) quality of goods in the economy moves along a quality ladder only through R&D and innovation, firms can use advertising to influence the perception that consumers have of these goods. In our model, advertising acts as a demand shifter. By spending more resources on advertising, firms can increase the quality perceived by consumers and hence increase the demand for this good. This increases market shares and profit. Even though R&D and advertising in our model can both increase sales and profits at any time, long term growth comes only from R&D and innovation. Nevertheless, the short-run substitutability between R&D and advertising as tools to increase sales and profits can have an impact on the amount of research performed by firms and hence on innovation and long term economic growth.

In addition to short-term substitutability between R&D and advertising, our model allows for heterogeneity in terms of R&D and advertising costs across firm size. In particular, we show that if larger firms (firms with larger number of products) have a cost advantage in terms of advertising, our model can explain several facts related to firm dynamics that have been documented in the literature. Figure 2 shows the average growth rate of firms by size quintiles.¹ Growth is measured as the rate of increase in sales (Panel A.) and employment (Panel B.). Growth rates are higher for smaller firms and are consistently decreasing in firm size. Panel C. and D. of Figure 2 shows that small firms also tend to be relatively more R&D intensive when R&D intensity is measured as R&D expenditures per employee and as a percentage of sales. This suggests that smaller firms experience higher growth through higher investments in R&D.

Consistent with these facts, our model obtains a deviation from Gibrat's law coming from a higher R&D intensity (i.e., investment in R&D normalized by firm size) of smaller firms. From that perspective, the contribution of this paper is to show that these deviations endogenously arise from firms'

¹Quintiles are based on the number of employees. Data relates to all U.S. listed companies from 1981 and comes from Compustat database.

optimal allocation of resources to R&D and advertising. We document two facts related to advertising using U.S. firm-level data. First, we find a negative relationship between advertising intensity (i.e., advertising expenditures normalized by size) and firm size which can be observed in Panel E. and F. of Figure 2. Second, we find that larger firms rely relatively more on advertising compared to R&D as the ratio of R&D to advertising expenditures decreases with firm size (Panels G and H in Figure 2). This suggests that firms optimally choose to change the mix of R&D and advertising and switch to relatively more advertising as they grow larger. These two facts are consistent with the implications of our model and can help explain the observed patterns in terms of firm R&D and growth across firm size.

Figure 2: Firm growth, R&D intensity and advertising intensity by firm size.

Note: Quintiles are based on size measured by employment. R&D and advertising intensity (Panel C - F) are measured as the ratio of total R&D and advertising expenditures to total sales or employment within the decile. Panel G (respectively H) report the ratio by decile of Panel C (D) and E (F).

The main mechanisms in our model are based on microfounded evidence from the marketing literature. There is ample evidence that larger firms have a cost advantage in terms of advertising compared to firms with fewer products. One reason for this size advantage comes from a spillover effect of advertising through different goods under the same brand name. By advertising one product under a given brand name, a firm can influence not only the perception of the quality of the advertised good but also of other goods sharing the same brand name. The marketing literature on umbrella branding and brand equity finds empirical evidence for the existence of such a spillover effect. The existence of this spillover effect affects firms' incentives to engage in R&D. As smaller firms gain relatively more in terms of advertising spillover from acquiring an additional product, they optimally choose to perform relatively more R&D and hence grow relatively faster than large firms. This mechanism also leads to decreasing advertising intensity across firm size even though larger firms rely relatively more on

advertising than R&D to increase sales and profit.

The rest of the paper is organized as follows. Section 2 presents a comprehensive review of the literature, with special focus on the marketing literatures that we use to motivate our advertising modeling approach. Section 3 discusses empirical facts related to Gibrat’s law, R&D and advertising intensity, as well as the relative use of R&D and advertising across firm size. We then present the model and describe its comparative statics in Section 4. In particular, we show how advertising affects the firm size distribution, firm entry rates and economic growth. In Section 5, we calibrate the model and show that it is able to replicate the main empirical regularities described in Section 3. Section 6 concludes. Proofs are included in the Appendix.

2 Related literature

Our work is related to several different literatures. First and foremost, we build upon models of endogenous firm growth through product innovation. This area was pioneered by Klette and Kortum (2004), who built a stylized version of the neo-Schumpeterian creative-destruction models of Grossman and Helpman (1991) and Aghion and Howitt (1992) into a model of multi-product firm dynamics. Their work is able to exhibit several behaviors that are consistent with empirical finding coming from micro-level data, especially regarding the right-skewness in the firm size distribution, the persistence in firms’ R&D investments, and the volatility of innovation in the cross-section of firms.²

However, by assuming that firm productivity scales perfectly with size, this framework fails to explain why firms of different sizes may grow at different rates. In fact, their model delivers a theoretical version of Gibrat’s law (Gibrat (1931)), in that firm growth is independent of size. While this observation was fairly accepted among applied theorists at the time, empirical studies have since argued that there may exist deviations especially among small continuing establishments. Conditional on survival, small establishments grow faster than large ones, and net exit rates are a decreasing function of size.³

One contributor to such phenomenon is that smaller firms tend to be more innovation-intensive.⁴ Accordingly, a wave of second-generation models of innovation-driven firm growth has emerged from subsequent applied work by Acemoglu, Akcigit, Bloom, and Kerr (2013), Acemoglu and Cao (2015) and Akcigit and Kerr (2015), among others. These papers have proposed extending the Klette and

²The applied work by Lentz and Mortensen (2008) further unveiled the predictive power of the Klette and Kortum (2004) model by putting it to a rigorous empirical test.

³See Hall (1987), Sutton (1997), Caves (1998), and Geroski (1998), all of which argue that growth rates of surviving firms tend to systematically decrease with increasing firm size. Audretsch, Klomp, Santarelli, and Thurik (2004) argue that Gibrat’s law deviations are especially prevalent within the manufacturing industry, and Rossi-Hansberg and Wright (2007) find similarly that U.S. sectors with larger physical capital shares exhibit significantly stronger scale dependence in establishment dynamics and distributions. For a survey of the empirical literature, see Santarelli, Klomp, and Thurnik (2006).

⁴Cohen and Klepper (1996) show that small innovating firms generate more innovations per dollar spent in R&D, and Akcigit and Kerr (2015) show that small firms spend more in R&D per dollar of sales. Using Compustat data, we confirm some of these results in Section 3.

Kortum (2004) framework to include heterogeneity in innovation technologies in order to incorporate a more meaningful interaction between different types of research (e.g. product versus process, radical versus incremental), as well as between entrants and incumbents. Most of the theoretical approaches to date impose parametric restrictions on the innovation sector in order to limit growth and feature size-dependence in equilibrium.⁵ In contrast, we propose a theory that, while building upon the aforementioned work, does not hinge on specific technological size-dependences in R&D. Yet, our theory is able to deliver that smaller firms spend in equilibrium a higher fraction of sales to R&D expenditures, and that those firms grow faster as a result. In particular, we are able to obtain these results even when there exist constant or increasing returns to scale in research, and no type of innovation is more radical than any other.

Our novel trade-off between innovation and advertising decisions of firms introduces the required non-homogeneities in equilibrium. Specifically, we rely on two main strands of the marketing literature in order to build a motive for advertising into an endogenous growth framework.

First, in our model firms benefit from the value of their brand name: there exist spillovers between goods within the firm, in the sense that increasing advertising expenditures on one good not only increases sales for that good but also indirectly for all other goods under the brand.⁶ A long tradition in the marketing literature identifies this phenomenon as the equity value of so-called “umbrella-branding”. Lane and Jacobson (1995) and Tauber (1981, 1988) show that brand developments can decrease marketing costs, and Rangaswamy, Burke, and Oliver (1993) show that they can enhance marketing productivity. Moreover, Smith and Park (1992) show that they can help capture greater market share. These effects are present in our model. Using household scanner panel data for the U.S., Balachander and Ghose (2003) find a positive and significant reciprocal spillover effect (i.e. an effect from advertising of a child brand on the choice of a parent brand by households) for multiple product categories and geographic markets. Similarly, in our formulation, the spillover effect works both from the newly acquired products to old products, and vice versa. Dacin and Smith (1994) show, by means of controlled experiments, a positive relationship between the number of products affiliated with a brand and consumers’ confidence in the quality extension of the brand. Erdem (1998) argues empirically that the quality perceptions of a brand in a product category are positively affected by the consumer’s experience with the same brand in a different category, because branding allows consumers to learn faster about quality through use experience. More particularly for our purposes, Erdem and Sun (2002) show that the effects identified in Erdem (1998) further translate to positive spillovers between advertising and sales for different goods within the same umbrella brand. Specifically, such effects are

⁵For instance, Akcigit and Kerr (2015) assume weak scalability in the innovation technology that is used most intensively by small firms; Acemoglu and Cao (2015) assume that innovations by small firms lead to higher step sizes of technical advance; and Acemoglu, Akcigit, Bloom, and Kerr (2013), who assume an exogenous stochastic process for firms’ productive capacity, impose that lower capacity states are highly absorbing so that most large surviving firms belong to the high state.

⁶In the model, we take as given that each firm has a unique brand, and we abstract from brand choice.

present in innovation-driven industries, e.g. the pharmaceutical industry (see [Suppliet \(2015\)](#)). More generally, [Ailawadi, Harlam, César, and Trounce \(2006\)](#), [Leeflang, Parreno-Selva, Dijk, and Wittink \(2008\)](#), [Bezawada, Balachander, Kannan, and Shankar \(2009\)](#), and [Leeflang and Parreno-Selva \(2012\)](#) all show significant cross-category sales and advertising dependencies for a variety of retail markets.

The second observation that we draw from the marketing literature is that higher-quality goods within the firm benefit more from advertising: everything else equal, a good that compares relatively better to other goods within the brand in terms of quality experiences higher increases in sales per dollar spent into advertising. This issue dates back to at least [Nelson \(1974\)](#), who shows that more heavily advertised products are more likely to be of high quality.⁷ In subsequent empirical work, [Archibald, Haulman, and Moody \(1983\)](#) and [Caves and Greene \(1996\)](#) find a positive correlation between advertising and quality for innovative goods as well as for experience goods, i.e. goods for which the experience of buyers is useful in making brand choices. [Tellis and Fornell \(1988\)](#) examine the relationship over the product life cycle, and find that advertising and profitability are both positively correlated to quality. Further, [Homer \(1995\)](#), [Kirmani and Wright \(1989\)](#) and [Kirmani \(1990, 1997\)](#) offer experimental evidence suggesting that the positive relationship exists because consumers perceive higher advertising expenditures indicating that the good is of high quality. These dimensions can be accommodated into our theory, because in our model advertising is persuasive rather than merely informative: not only advertising raises product awareness, but firms try to directly increase their stock value by marketing their higher quality products.⁸

More broadly, our theoretical approach is also consistent with standard modeling in both Marketing and Economics. On the one hand, the marketing literature typically argues that advertising firms value “goodwill”, namely the stock of advertising accumulated over time. The most extended approach consists of posing a logit discrete-choice model of demand, coupled with an explicit law of motion for goodwill.⁹ In the empirical application, most studies find that the lag coefficient on goodwill is less than unity on a weekly frequency, meaning that, over a one-year period (our frequency in the estimation), the lag coefficient is nearly zero. This is in line with our assumption that advertising has an instantaneous effect on firms’ operating profits, while all dynamic effects operate indirectly through innovation.

Regarding modeling advertising in Economics, our paper relates to a long tradition of studying advertising as an explicit factor affecting consumer tastes, as in [Dorfman and Steiner \(1954\)](#), [Dixit and Norman \(1978\)](#), [Becker and Murphy \(1993\)](#) and [Benhabib and Bisin \(2002\)](#). Our approach is iso-morphic to those and can be thought of a microfoundation, because firms’ advertising effectively

⁷In this empirical literature, quality is usually proxied by consumer reports and rankings of consumer satisfaction.

⁸[Marquardt and McGann \(1975\)](#) and [Rotfeld and Rotzoll \(1976\)](#) offer further empirical evidence on the advertising-quality correlation, and [Kirmani and Rao \(2000\)](#) and [Bagwell \(2007\)](#) survey the literature.

⁹See [Dubé, Hitsch, and Manchanda \(2005\)](#) and [Doganoglu and Klapper \(2006\)](#). For direct applications of this approach in Macroeconomics, see [Nervole and Arrow \(1962\)](#) and [Molinari and Turino \(2009\)](#).

acts as a perfect demand shifter in equilibrium. A parallel body of literature introduces a role for valuable customer capital into macro models, viewing marketing as a tool to build continuing buyer-seller relationships for firms due to either the existence of frictions in product markets (e.g. [Gourio and Rudanko \(2014\)](#)) or because of costs to market penetration (e.g. [Arkolakis \(Forthcoming\)](#) and [Eaton, Eslava, Jinkins, Krizan, and Tybout \(2014\)](#)). Like us, [Perla \(2015\)](#) views advertising as a way of raising product awareness among customers, but his paper focuses on the implications of product sorting by consumers on the industry life cycle and the degree of market competition, while we focus on the effects of advertising on growth. To our knowledge, we are the first to consider this important interaction.

3 Empirical Findings

In this section, we present the results of a few regressions about firm growth, R&D and advertising expenditures. We use data on U.S. listed companies from the Compustat database over the period 1981-2015 (35 years). For all regressions, we use firms with strictly positive investment in R&D and advertising expenditures. Firms reporting negative sales or employment are excluded from the sample. We also exclude firm experiencing year-on-year growth of more than 1000%.¹⁰ Age is measured as the number of years since the firm first appeared in the database.¹¹

In Table 1, we regress one measure of sale revenue and two measures of profit (earning before interest and tax and operating income) on R&D and advertising expenditures. Controlling for firm size (measured by employment), these results confirm that firms spending more on R&D and advertising have higher sales and profits, which suggests that R&D and advertising are two alternative tools that firms can use to increase both sales and profits. This is in line with the assumptions that we make in our model which aim at describing the optimal choice of firms regarding R&D and advertising and their potential interaction.

¹⁰Using firms with growth rates above 1000% does not change the sign nor the significance of the coefficients of interest.

¹¹This way of measuring age in the Compustat database is standard in the literature and has been used among others by [Shumway \(2001\)](#), [Lubos and Veronesi \(2003\)](#), [Fama and French \(2004\)](#) and [Chun, Kim, Morck, and Yeung \(2008\)](#).

	(1) Sales	(2) Sales	(3) Sales	(4) EBIT	(5) EBIT	(6) EBIT	(7) Operating Income	(8) Operating Income	(9) Operating Income
<i>R&D</i>	8.234*** (0.0730)	8.198*** (0.0734)	8.018*** (0.0756)	1.210*** (0.0138)	1.192*** (0.0139)	1.229*** (0.0141)	1.769*** (0.0180)	1.755*** (0.0181)	1.769*** (0.0186)
<i>Advertising</i>	5.988*** (0.102)	5.971*** (0.102)	5.663*** (0.105)	0.815*** (0.0193)	0.806*** (0.0193)	0.859*** (0.0195)	1.174*** (0.0255)	1.167*** (0.0254)	1.188*** (0.0261)
<i>Emp.</i>	101.1*** (1.128)	100.4*** (1.138)	105.2*** (1.220)	3.869*** (0.214)	3.510*** (0.215)	2.971*** (0.227)	10.92*** (0.278)	10.64*** (0.280)	10.69*** (0.299)
<i>Age</i>		13.32*** (2.852)	10.22*** (2.995)		6.709*** (0.539)	7.169*** (0.557)		5.167*** (0.701)	5.344*** (0.735)
<i>Asset Equity</i>			1.152 (1.227)			-0.108 (0.228)			-0.0352 (0.301)
<i>Constant</i>	-718.3 (630.5)	-934.4 (632.0)	-950.9 (656.1)	-54.09 (119.5)	-163.0 (119.5)	-169.1 (122.0)	-101.4 (155.7)	-184.1 (155.9)	-195.6 (161.6)
Year dummies	Y	Y	Y	Y	Y	Y	Y	Y	Y
Industry FE	Y	Y	Y	Y	Y	Y	Y	Y	Y
Observations	30310	30310	28423	30309	30309	28423	30106	30106	28245
<i>R</i> ²	0.81	0.81	0.81	0.56	0.57	0.59	0.67	0.67	0.68

Standard errors in parentheses. Sales, EBIT, Operating Income, R&D and advertising expenditures in thousands U.S.D. Employment in thousands.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 1: Sale and profit regressions

The next two tables confirm two facts which have already been highlighted in the literature. First we show that there exists a significant deviation from Gibrat’s law. Table 2 shows that larger firms tend to experience lower rates of growth both in terms of sales and employment. This phenomenon has been studied in the literature within different theoretical frameworks. Models of firm dynamics following [Hopenhayn \(1992\)](#) based on stochastic (Markov) productivity shocks can replicate the negative correlation between firm size and firm growth. If the set of productivity shocks is finite and mean-reverting, larger firms (i.e., firms with higher productivity) eventually face a limit to growth and experience lower average growth rates than small firms. [Jovanovic \(1982\)](#) proposes a model of learning in which firms receive noisy signals about their productivity (or cost function) and update their beliefs accordingly. Firms with beliefs below a certain threshold optimally choose to exit. Younger firms are more uncertain about their productivity and hence learn more than older firms. This leads to stronger revision of their beliefs and hence higher growth rates (among survivors).¹² On average, younger firms grow faster and are larger than older ones. The correlation between firm size and firm growth comes through an age effect and the positive correlation between age and size. In this type of models, the observed association between firm size and firm growth should vanish as one controls for firm age. [Cooley and Quadrini \(2001\)](#) link the dependence of firm growth on size and age to financial market frictions. In a model of firm dynamics, they show that the interaction of persistent productivity shocks with costly equity issuance and default generates a higher financial leverage for small and young firms and qualitatively replicates the decreasing relationship between firm size, firm age and expected growth. In Table 2, we find that the deviation from Gibrat’s law still holds even after controlling for firm age and leverage.

Recently, the economic growth literature has started investigating this departure from constant growth rate across firm size and has linked it to another empirical fact: the lower R&D intensity of larger firms ([Acemoglu and Cao \(2015\)](#) and [Akcigit and Kerr \(2015\)](#) among others). As larger firms invest relatively less in innovation, they experience relatively lower rates of growth. [Acemoglu, Akcigit, Bloom, and Kerr \(2013\)](#) and [Acemoglu, Akcigit, and Celik \(2014\)](#) also propose models in which older firms invest relatively less in R&D and grow more slowly as a result. In these two models, firm growth and firm size are indirectly related through the positive correlation between firm age and size.

Table 3 shows that this pattern holds in our database as well. Larger firms are less R&D intensive when R&D intensity is measured by both R&D expenditure per employee or as a share of sales. In addition, we control for firm age and leverage and find that the negative relationship between R&D intensity and firm size still holds. We can also notice that the effect of age on R&D intensity is ambiguous. We find that R&D intensity significantly decreases with age when size is measured by sales whereas the relationship between R&D intensity and firm age is not significant when size is measured by the number of employees. This suggests that the significant negative relationship between firm

¹²In the long run, surviving firms learn their true productivity and hence do not grow any more.

	(1) $\frac{\Delta sales}{sales}$	(2) $\frac{\Delta sales}{sales}$	(3) $\frac{\Delta sales}{sales}$	(4) $\frac{\Delta emp.}{emp.}$	(5) $\frac{\Delta emp.}{emp.}$	(6) $\frac{\Delta emp.}{emp.}$
$\log(sales)$	-0.0665*** (0.000693)	-0.0607*** (0.000743)	-0.0599*** (0.000775)			
$\log(emp.)$				-0.0275*** (0.000562)	-0.0218*** (0.000605)	-0.0201*** (0.000630)
Age		-0.00322*** (0.000149)	-0.00314*** (0.000147)		-0.00275*** (0.000110)	-0.00275*** (0.000110)
$\log\left(\frac{Assets}{Equity}\right)$			-0.000111 (0.00247)			-0.0354*** (0.00188)
$Constant$	0.431*** (0.0181)	0.447*** (0.0181)	0.449*** (0.0177)	0.0574*** (0.0129)	0.0922*** (0.0129)	0.118*** (0.0128)
Year dummies	Y	Y	Y	Y	Y	Y
Industry FE	Y	Y	Y	Y	Y	Y
Observations	200503	200503	186709	180790	180790	169323
R^2	0.07	0.08	0.08	0.03	0.03	0.04

t-Standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 2: Firm growth regressions

	(1) $\log\left(\frac{R\&D}{sales}\right)$	(2) $\log\left(\frac{R\&D}{sales}\right)$	(3) $\log\left(\frac{R\&D}{sales}\right)$	(4) $\log\left(\frac{R\&D}{emp.}\right)$	(5) $\log\left(\frac{R\&D}{emp.}\right)$	(6) $\log\left(\frac{R\&D}{emp.}\right)$
$\log(sales)$	-0.284*** (0.00185)	-0.291*** (0.00202)	-0.2671*** (0.00217)			
$\log(emp.)$				-0.152*** (0.00210)	-0.151*** (0.00231)	-0.135*** (0.00246)
Age		0.00358*** (0.000414)	0.00316*** (0.000421)		-0.000345 (0.000410)	-0.000472 (0.000419)
$\log\left(\frac{Assets}{Equity}\right)$			-0.142*** (0.00750)			-0.163*** (0.00768)
$Constant$	-2.701*** (0.0571)	-2.724*** (0.0571)	-2.751*** (0.0575)	0.188*** (0.0547)	0.193*** (0.0550)	0.296*** (0.0555)
Year dummies	Y	Y	Y	Y	Y	Y
Industry FE	Y	Y	Y	Y	Y	Y
Observations	84499	84499	77979	78705	78705	73227
R^2	0.55	0.55	0.55	0.51	0.51	0.52

t-Standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 3: R&D intensity regressions

growth and age does not come through the effect of age on R&D.

The fact that size remains significant after controlling for age and leverage both in growth and R&D intensity regressions suggests that stories about firm and product life-cycle and financial constraints cannot entirely explain the deviation from Gibrat's law observed in the data. We propose an alternative explanation for this deviation based on the substitutability between R&D and advertising in firm's optimal decisions.

Before describing the model and the mechanisms leading to the observed deviation from Gibrat's law and decreasing R&D intensity with firm size, we document two main characteristics of advertising in the data. First, Table 4 shows that advertising intensity is decreasing in firm size measured by both sales and employment. In addition, we find that larger firms tend to use relatively more advertising than R&D to increase profits. Table 5 indeed shows that the ratio of R&D and advertising intensities is significantly decreasing in size. We also find that the effect of age on the relative use of R&D and advertising is not robustly significant. Overall, these results suggest that the interaction between R&D and advertising affects firm growth through a size effect (rather than an age effect), consistent with the advertising spillover that our model captures.

	(1) $\log\left(\frac{Adv.}{sales}\right)$	(2) $\log\left(\frac{Adv.}{sales}\right)$	(3) $\log\left(\frac{Adv.}{sales}\right)$	(4) logadve	(5) $\log\left(\frac{Adv.}{emp.}\right)$	(6) $\log\left(\frac{Adv.}{emp.}\right)$
$\log(sales)$	-0.0600*** (0.00212)	-0.0715*** (0.00235)	-0.0536*** (0.00250)			
$\log(emp.)$				-0.0171*** (0.00263)	-0.0303*** (0.00294)	-0.0295*** (0.00312)
<i>Age</i>		0.00523*** (0.000463)	0.00517*** (0.000470)		0.00528*** (0.000531)	0.00580*** (0.000545)
$\log\left(\frac{Assets}{Equity}\right)$			-0.0655*** (0.00749)			-0.0250*** (0.00885)
<i>Constant</i>	-5.240*** (0.0966)	-5.263*** (0.0965)	-5.251*** (0.0966)	-1.248*** (0.108)	-1.321*** (0.108)	-1.294*** (0.109)
Year dummies	Y	Y	Y	Y	Y	Y
Industry FE	Y	Y	Y	Y	Y	Y
Observations	66860	66860	62259	59920	59920	56090
R^2	0.20	0.20	0.21	0.19	0.20	0.20

t-Standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 4: Advertising intensity regressions

	(1) $\log\left(\frac{R\&D}{Adv.}\right)$	(2) $\log\left(\frac{R\&D}{Adv.}\right)$	(3) $\log\left(\frac{R\&D}{Adv.}\right)$	(4) $\log\left(\frac{R\&D}{Adv.}\right)$	(5) $\log\left(\frac{R\&D}{Adv.}\right)$	(6) $\log\left(\frac{R\&D}{Adv.}\right)$
$\log(sales)$	-0.116*** (0.00370)	-0.117*** (0.00413)	-0.111*** (0.00441)			
$\log(emp)$				-0.112*** (0.00424)	-0.110*** (0.00477)	-0.102*** (0.00503)
Age		0.000419 (0.000835)	-0.000405 (0.000854)		-0.00108 (0.000864)	-0.00177** (0.000882)
$\log\left(\frac{Assets}{Equity}\right)$			-0.0657*** (0.0146)			-0.0804*** (0.0154)
$Constant$	1.928*** (0.179)	1.926*** (0.179)	1.925*** (0.180)	1.432*** (0.179)	1.448*** (0.179)	1.476*** (0.181)
Year dummies	Y	Y	Y	Y	Y	Y
Industry FE	Y	Y	Y	Y	Y	Y
Observations	32169	32169	29933	30130	30130	28264
R^2	0.39	0.39	0.40	0.39	0.39	0.40

t-Standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 5: R&D to advertising expenditure regressions

4 An Endogenous Growth Model with Advertising

In this section, we build a theory of advertising into a model of endogenous firm growth. The basic structure follows the framework set up by Klette and Kortum (2004), and specifically allows for heterogeneous innovations as in Akcigit and Kerr (2015). The advertising theory, on the other hand, is based on empirical regularities that we take from the marketing literature. Our broader goal is to illustrate a novel interplay between innovation and advertising choices that critically shapes growth of firms and, by extension, that of the economy as a whole. Particularly, we shall analyze how this interaction can help rationalize some of the patterns that we have shown in Section 3, independently of technological differences in the innovative process regarding size and the radicality of certain types of innovation over others.

4.1 Environment

Households Time is continuous, infinite, and indexed by $t \in \mathbb{R}_+$. The economy is populated by a measure-one continuum of identical, infinitely-lived individuals with discount rate $\rho > 0$. A representative household has preferences:

$$U = \int_0^{+\infty} e^{-\rho t} \ln(C_t) dt$$

where C_t is consumption over the single final good. This household is endowed with one unit of time every instant, which is in turn supplied inelastically and split between a final-good sector and an intermediate-good sector in the form of labor. There is a common wage rate denoted by w_t , making individuals indifferent between the two types of work.

The final good is the numeraire, and we set $P_t = 1$ for all time periods with no loss of generality. The household carries wealth A_t intertemporally, which earns an instantaneous rate of return r_t . The flow budget constraint is, therefore:

$$\dot{A}_t = r_t A_t + w_t - C_t$$

with $A_0 \geq 0$ given.

Final-good sector The final good is produced by a representative final-good firm using labor and a measure-one continuum of intermediate good varieties, indexed by $j \in [0, 1]$. Technology is given by the Cobb-Douglas production function:

$$Y_t = \frac{1}{1-\beta} L_t^\beta X_t^{1-\beta}$$

where $\beta \in (0, 1)$, L_t is labor used in the production of the final good, and

$$X_t = \left(\int_0^1 \tilde{q}_{jt}^{\frac{1}{\epsilon}} y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$$

is an intermediate-good CES composite, where $\epsilon > 1$ is the elasticity of substitution between varieties. Here, y_{jt} is the quantity of intermediate good j that is produced in period t . Quantities are weighted by the term $\tilde{q}_{jt}$, which stands for the “quality” of the good that is being *perceived* by agents in the economy.

Product qualities The total perceived quality $\tilde{q}_{jt}$ of a certain good j at time t is defined by:

$$\tilde{q}_{jt} = q_{jt}(1 + d_{jt})$$

Total perceived quality includes two components. The first component is the *actual* (or *intrinsic*) quality of the product, denoted by $q_{jt} > 0$, which refers to the currently leading-edge technological efficiency of the product and is built upon over time through a process of technical innovation. In particular, this term advances on a ladder as in the models of [Grossman and Helpman \(1991\)](#) and [Aghion and Howitt \(1992\)](#), a phenomenon that is induced by expenditure into R&D at the intermediate sector level.

The second component in total perceived quality is the *subjective* (or *extrinsic*) quality of the product, given by the function $\phi(q_{jt}) \equiv q_{jt} \cdot d_{jt}$, where $d_{jt} \geq 0$ denotes advertising-induced quality. This component refers to the part of total perceived quality of a good that is induced by the producer's advertising efforts on that specific product. Besides depending upon the level of the good's own intrinsic quality, $\phi(\cdot)$ is also a function of the size of the firm as well as directed advertising expenditures, in a way that we specify below in detail.¹³ Henceforth, ϕ_{jt} may be referred to as the *effectiveness of marketing* product j in period t .

Note that intrinsic (q_j) and extrinsic (ϕ_j) quality are partial substitutes at the good level, as they enter additively into total quality (i.e., $\tilde{q}_j = q_j + \phi_j$). Namely, while spending on advertising (which increases ϕ_{jt}) is not indispensable for a good to be perceived with some positive quality by consumers, it does create an added value over and above the quality level that is purely intrinsic to the good. However, we also allow for a degree of complementarity coming from the fact that $\phi(\cdot)$ is itself an increasing function of intrinsic quality, q . Namely, advertising is effective in raising perceived quality only if the good has nonzero intrinsic quality, and it is more effective the higher the intrinsic quality of the good. This is consistent with the marketing literature reviewed in Section 2.

Moreover, by choosing their advertising expenditures, firms are effectively impacting consumer preferences. Indeed, as will become apparent later, the endogenous choice of ϕ_{jt} by firms serves as a consumer demand-curve shifter in equilibrium, and can thus be thought of as a microfoundation for the reduced-form approaches of [Becker and Murphy \(1993\)](#) and [Benhabib and Bisin \(2002\)](#), among others, which model advertising as directly affecting consumer preferences through product-specific taste parameters or as an explicit argument in the utility function. Specifically, we adopt the view that advertising is *persuasive*, and not purely *informative*. By this we mean that consumers cannot choose what information to be exposed to, but rather behave according to the shifts in tastes induced by advertisements, which they take as given.¹⁴ In our view, advertising is thus a flow variable (and per se not a state for the firm) that nonetheless embodies a general-equilibrium effect that critically changes the dynamic incentives of acquiring new product lines through R&D. We shall return to these effects once we have introduced the $\phi(\cdot)$ function and derived the key equilibrium conditions.

Intermediate-good sector At any instant, there is an endogenously determined set $\mathcal{F}$ (of measure $F > 0$) of active intermediate good incumbent producers operating in a monopolistically competitive product market. The representative household owns all intermediate-good firms in the economy. In turn, each intermediate good $j \in [0, 1]$ is owned by a single firm $f \in \mathcal{F}$. A firm is said to own a product if it can produce it at a higher intrinsic quality than any other firm. Hence, a typical firm f is summarized by the countable set of intrinsic product qualities for which it has the leading-edge

¹³We borrow the nomenclature “intrinsic”, “extrinsic” and “perceived” from the marketing literature (e.g., [Zeithaml \(1988\)](#)).

¹⁴For an extensive discussion of the two views on advertising, see [Becker \(1996\)](#).

technology, denoted $\mathcal{J}_f \equiv \{j : f \text{ owns } j\} \subseteq [0, 1]$. The number of active product lines owned by firm f is $n_f = |\mathcal{J}_f| \in \mathbb{Z}_+$. Henceforth, the variable n_f is referred to as the *size* of firm f . Since the only way for firms to acquire new product lines is through R&D, the variable n_f may be alternatively interpreted as a proxy for the firm's stock of knowledge. Finally, the product quality portfolio of firm f is given by $\mathbf{q}_f \equiv \{q_j : j \in \mathcal{J}_f\} \in \mathbb{R}_{++}^{n_f}$ (we assume $q_{j,0} > 0, \forall j \in [0, 1]$, to be taken as given by firms).

Each intermediate good variety j is produced with linear technology:

$$y_{jt} = \bar{Q}_t l_{jt}$$

where l_{jt} is labor input and $\bar{Q}_t \equiv \int_0^1 q_{jt} dj$ is the average intrinsic quality in the economy. This linear formulation implies that all intermediate good producers face the same marginal cost, $w_t/\bar{Q}_t$. Moreover, by making production scale with aggregate quality, we can ensure that output grows at the same rate as productivity, thereby ensuring the existence of balanced growth in the long-run equilibrium.

The sole engine of economic growth is thus fueled by intrinsic quality innovations at the intermediate sector level. Furthermore, firms can choose to advertise on their currently owned product lines in order to instantaneously increase demand. Advertising, therefore, does not have a direct impact on growth in that it does not generate direct increases in intrinsic quality, but it has an important indirect effect because it impacts further decisions on innovation.

4.2 Quality improvements

The total perceived quality of a good can be improved in both the intrinsic dimension, through R&D, and the extrinsic one, through advertising. We describe both of these margins next.

4.2.1 R&D technologies

Firms can invest into R&D to generate the possibility of intrinsically improving their own product lines (so-called *internal* innovation), as well as to creatively build upon the intrinsic quality of those goods owned by other incumbent monopolists (so-called *external* innovation). Firms then expand or contract in the product space whenever these innovations render successful.

Firstly, internal R&D is undertaken by incumbent firms in order to technologically advance upon their existing product lines. To create a Poisson flow rate $z_j \geq 0$ of improving the intrinsic quality of product $j \in \mathcal{J}_f$, firm f must spend $R_z(z_j)$ units of the final good. This cost function is assumed to be convex, so that

$$R_z(z_j) = \hat{\chi} q_j z_j^{\hat{\psi}}$$

where $\hat{\chi} > 0$ and $\hat{\psi} > 1$. If successful, internal R&D then improves intrinsic quality by a factor of

$(1 + \lambda^I) > 1$, so that

$$q_{j,t+\Delta t} = (1 + \lambda^I)q_{jt}$$

if there is only one innovation within an arbitrarily small time interval of size $\Delta t > 0$.¹⁵ Note that R_z increases linearly with q_j , meaning that more advanced technologies have higher R&D costs.

Secondly, external R&D is undertaken by incumbents and potential new entrants to obtain technological leadership over products that they do not currently own. For simplicity, external R&D is assumed to be undirected, so that the successful innovator uniformly draws a good from the set $[0, 1]$.¹⁶ Incumbents and entrants face, however, different innovation technologies.

On the one hand, entrants (firms with $n = 0$) must incur an expenditure of

$$R_e(x_e) = \nu \bar{Q} x_e$$

units of the final good in order to generate a Poisson flow rate of x_e , where $\nu > 0$ is a constant scale parameter. We assume that there is a measure-one mass of potential entrants, and determine x_e by a free-entry condition.¹⁷

On the other hand, in order to create a flow Poisson rate of X_n , an incumbent firm with $n \geq 1$ product lines must spend $R_x(X_n, n)$ units of the final good. In particular, the cost function is assumed to be convex in the rate of innovation, X_n , such that

$$R_x(X_n, n) = \tilde{\chi} \bar{Q} X_n^{\tilde{\psi}} n^\sigma$$

where $\tilde{\chi} > 0$, $\tilde{\psi} > 1$, and $\sigma \in \mathbb{R}$. If successful, external R&D then improves intrinsic quality by a factor of $(1 + \lambda^E) > 1$, so that

$$q_{j,t+\Delta t} = (1 + \lambda^E)q_{jt}$$

for a randomly selected product line $j \in [0, 1]$, if there is only one innovation within an arbitrarily small time interval of size $\Delta t > 0$.

Discussion The fact that a firm has built a stock of knowledge over time ($n \geq 1$) creates heterogeneity in external R&D returns when $\sigma \neq 0$. As mentioned in the Introduction, the degree to which these innovation costs scale with size plays a critical role in the size dependence of firm innovation

¹⁵In the continuous time limit as $\Delta \rightarrow 0$, the event of two or more innovations is probability zero.

¹⁶By law of large numbers, such good is not in the innovator's current portfolio almost surely.

¹⁷By normalizing the mass of potential entrants to unity, and because of linearity in the cost function, x_e denotes in equilibrium both the optimal Poisson rate of entry and the realized mass of entrants.

through R&D (i.e., absent our advertising margin). In the model of [Klette and Kortum \(2004\)](#), for example, constant returns to scale in innovation costs (i.e., $\tilde{\psi} + \sigma = 1$) ensures that ex-ante technological advantages do not exist with respect to differences in firm size. In other words, external innovation investment scales up on a one-for-one basis with added product lines: for any constant $\kappa > 0$, a firm of size n that wishes to increase the innovation rate by a factor of κ has the same total expenditure as that of κ firms of size n/κ , or $R_x(\kappa \cdot X, \kappa \cdot n) = \kappa \cdot R_x(X, n)$.¹⁸ This naturally delivers a theoretical version of Gibrat’s law: firm value is proportional to size by construction, and consequently firm growth is independent of n .

Because this specialization of the model is ill-suited to analyze, among other empirical regularities, the reasons why smaller firms are more innovation-intensive in the data, subsequent work has relaxed the homogeneity on returns. [Akcigit and Kerr \(2015\)](#), notably, extend the [Klette and Kortum \(2004\)](#) framework to incorporate decreasing returns to external R&D (i.e., $\tilde{\psi} + \sigma > 1$). This provides a technological advantage to small firms: for any constant $\kappa > 0$, a firm of size n that wishes to increase the innovation rate by a factor of κ has a *higher* total expenditure than that of κ firms of size n/κ , or $R_x(\kappa \cdot X, \kappa \cdot n) > \kappa \cdot R_x(X, n)$. As a result, external innovation scales weakly relative to internal innovation, and small firms optimally choose to use external innovations more intensively.

In contrast, we remain agnostic regarding the degree of scalability in R&D and the radicality of different types of innovation. In our model with advertising, we are able to qualitatively replicate empirical patterns of the data (e.g., deviations from Gibrat’s law, and decreasing R&D and advertising intensity across firm sizes) when there exist constant, and even increasing,¹⁹ returns to scale in R&D, and no type of innovation is more radical than any other.²⁰ Our formulation of advertising effectiveness, which relies on empirical patterns that we draw from the marketing literature, delivers as an equilibrium outcome that smaller firms are more concerned with innovation even when technological advantages in innovation might not be particularly tailored toward them. This is because those firms are the ones that marginally benefit the most from advertising, which leads them to prefer to expand their product portfolios and effectively be relatively more R&D-intensive. In doing so, we obtain as well that smaller firms invest more in advertising per dollar that they sell, in line with the evidence presented in [Section 3](#). This is a novel explanation that highlights the potential effects of firms’ advertising on growth.

¹⁸Equivalently, the Poisson rate faced by a firm with n product lines investing κ units of the final good into R&D is the same as the combined Poisson rate of κ firms of size n/κ investing one unit each into R&D.

¹⁹[Schumpeter \(1942\)](#) first hypothesized that “large-scale units of control” are preferred over small ones so all benefits from the nonrival nature of ideas can be reaped. Some studies argue that large firms possess a comparative advantage since R&D may involve large start-up costs ([Cohen, Levin, and Mowery \(1987\)](#)) and risks such as agency costs ([Holmstrom \(1989\)](#)), which larger firms are more willing to bear. A countervailing hypothesis can be made arguing that a business-stealing effect from product market rivals may operate in opposition to economies of scope. However, using a panel of U.S. firms, [Bloom, Schankerman, and Reenen \(2013\)](#) show that technological spillovers quantitatively dominate in the data.

²⁰In our benchmark calibration we assume constant returns ($\tilde{\psi} + \sigma = 1$), but we also report results for both decreasing and increasing returns as a robustness check in the comparative statics exercises of [Section 4.6.2](#). Regarding the radicality of innovations, we set $\lambda^I = \lambda^E$.

We next describe our exact specification for advertising effectiveness.

4.2.2 Advertising technology

Besides building upon the intrinsic quality component of their products, firms can undertake advertising expenditures in order to enhance extrinsic quality. A positive advertising expenditure translates instantaneously and almost surely into an increase in extrinsic quality.

Specifically, for a firm f of size $n \geq 1$, we assume the following marketing production function on good $j \in \mathcal{J}_f$:

$$\phi(m_j, n; q_j) = \theta \frac{q_j}{Q_f} \bar{Q}^{1-\zeta} m_j^\zeta n^\eta \quad (1)$$

where m_j is the number of units of the final good spent in advertising good j , $Q_f \equiv \left(\sum_{q_{j'} \in \mathbf{q}_f} q_{j'}^{1/\alpha} \right)^\alpha$ is a within-firm measure of aggregate quality, and $\theta > 0$ and $\alpha, \zeta, \eta \in (0, 1)$ are parameters.

Discussion Equation (1) embodies some of the main effects of advertising identified by the marketing literature reviewed in Section 2 that are necessary within our framework to confirm the main empirical findings discussed in Section 3.

First, everything else equal, the return to advertising a good j is increasing in advertising expenditure on that same good. In particular, it is Cobb-Douglas in expenditure m_j and aggregate quality in the economy, $\bar{Q}$. The parameter ζ is the elasticity of advertising returns to advertising expenditure. As will be made clear shortly, a condition on this parameter will ensure that, in equilibrium, aggregate return grows steadily in the long run and firms do not “outgrow” the economy.

Second, the marketing return increases in the object q_j/Q_f , which is a measure of the relative quality of the good with respect to the aggregate quality within the firm. The latter is expressed in the form of an aggregator across intrinsic quality levels within the firm’s portfolio, increasing in good qualities. In words, advertising is more effective for goods that are intrinsically of relatively higher quality within the firm.²¹ This circumstance is identified empirically by the marketing literature reviewed in Section 2, e.g. Nelson (1974), Archibald, Haulman, and Moody (1983), Caves and Greene (1996), Marquardt and McGann (1975), Rotfeld and Rotzoll (1976), Bagwell (2007) and Kirmani and Rao (2000).

Third, larger firms have an absolute marketing advantage over smaller firms in that, everything else equal, the return to advertising is greater for higher n . We interpret this as there being spillover

²¹One potential concern when modeling advertising is that, if large firms have an absolute advantage over small firms (like in our framework), they may choose to outgrow their competitors by keeping them out of product markets through an ever-reinforcing cycle of high advertising and high sales (a snowball effect). We avoid this in equilibrium by imposing that extrinsic quality can never outgrow aggregate intrinsic quality (since the exponent on $\bar{Q}$ is less than unity). Moreover, by assuming that advertising is increasing in the good’s intrinsic quality *relative* to other goods in the firm, we can keep firms from choosing to specialize on only one good in their portfolio, which in the aggregate would lead to a degenerate distribution of qualities.

effects across goods within the firm coming from good-specific advertising expenses: to the extent that conducting advertising increases demand and ultimately sales, owning multiple product lines is beneficial in that it increases revenues over *all* goods, even those that the firm might not be explicitly advertising. This effect is meant to capture the value of firm branding, as it is reminiscent of the “umbrella branding” effect discussed in the marketing literature also reviewed in Section 2, e.g. Lane and Jacobson (1995), Tauber (1981, 1988), Rangaswamy, Burke, and Oliver (1993), Smith and Park (1992), Balachander and Ghose (2003), Erdem (1998) and Erdem and Sun (2002). In our context, the fact that the firm owns multiple goods builds a brand value that is attributed evenly across all goods in its portfolio, even those that are not being explicitly advertised.

Moreover, because the elasticity of advertising effectiveness to firm size is assumed to be less than unity (or $\eta < 1$), the spillover effect is marginally stronger for smaller firms. In our context, this assumption has a critical general-equilibrium effect on the incentives to conduct external innovation in order to acquire new product lines, and is at the core of the differences in innovation intensities with respect to size that the model delivers in equilibrium. In the estimation of a calibrated model presented in Section 5 we verify that this restriction is needed to match the data.

Finally, θ is a parameter describing the component of advertising efficiency that is constant across products. It affects the equilibrium share of expenditure devoted to innovation versus advertising, as we shall see in comparative statics exercises later on.

4.3 Entry, exit, and pricing

There is no exogenous exit, and firms move on the size distribution endogenously solely through the external innovation margin. Schumpeterian creative destruction occurs whenever a good is taken over by a successful external innovator, and a firm exits when it loses its last remaining product.

We assume that a firm has the right to produce a good if it possesses a strictly technological leadership on it, that is, if it can produce the good at the highest *intrinsic* quality among all active firms.²² The last innovator in each product line then owns the leading patent and has monopolistic power until being displaced by another firm. There is free entry into research, and therefore any firm can undertake research on any of the product lines. Once a firm makes an innovation, it acquires a perpetual patent on it. However, this patent does not preclude other firms from investing into research to improve the intrinsic quality of this product. As it is standard in the literature, we assume that the new innovator is then able to price the old incumbent out of the market. Specifically, we assume that firms entering with new innovations can charge the unconstrained monopoly price, as opposed to having to price-compete with the current incumbent.

²²This assumption precludes firms from displacing technologically superior firms through intense advertising. We need to impose this because otherwise the firm distribution would not be well-defined.

The following summarizes our pricing assumption:

Assumption 1 (Monopoly pricing) *Any good is produced by the firm that can produce it at the highest intrinsic quality. Such firm is able to maintain its market leadership through monopolistic pricing.*

4.4 Resource constraints

The economy is closed and there is no government, so GDP equals aggregate consumption plus aggregate investment. The latter is split between aggregate R&D expenditure by entrants and incumbents, denoted Z_t , and aggregate advertising expenditure by incumbents, denoted M_t . The resource constraint at time t is

$$C_t + Z_t + M_t \leq Y_t \quad (2)$$

In the labor market, the unit supply of labor is shared between the demand for final-good workers, L_t , and the demand for intermediate-good workers, $\tilde{L}_t = \int_0^1 l_{jt} dj$, so that

$$L_t + \tilde{L}_t \leq 1$$

for all $t \in \mathbb{R}_+$.

4.5 Equilibrium

In this section we derive the Markov Perfect Equilibrium of the economy in any given period. Later on, we specialize the equilibrium to a balanced-growth path (BGP) in which all aggregates grow at a constant rate g .

Consumer's problem Taking initial wealth A_0 as given, the representative consumer chooses a path for consumption to maximize utility subject to the flow budget constraint and the No-Ponzi condition $\lim_{t \rightarrow +\infty} e^{-\int_0^t r_s ds} A_t \geq 0$. The optimality condition yields the standard Euler equation for consumption:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho \quad (3)$$

Intermediate-good firms are owned by the household, so the value of household wealth is equal to the value of corporate assets. Namely, $A_t = \int_{\mathcal{F}} V(\mathbf{q}_f) df$, where $V(\mathbf{q}_f)$ denotes the net present value of profits for a monopolist f who owns the quality portfolio $\mathbf{q}_f$ at time t . In equilibrium, the transversality

condition

$$\lim_{t \rightarrow +\infty} \left[e^{-\int_0^t r_s ds} \int_{\mathcal{F}} V_t(\mathbf{q}_f) df \right] = 0 \quad (4)$$

requires that the aggregate stock value of firms is zero in the limit of time.

Final-good firm's problem As it is standard in the literature, we set $\epsilon = \beta^{-1}$ for tractability.²³

Then, the representative final-good producer solves the following problem, taking qualities $\{\tilde{q}_j : j \in [0, 1]\}$ and input prices $\{p_j : j \in [0, 1]\}$ as given:

$$\begin{aligned} \max_{L_t, \{y_{jt} : j \in [0, 1]\}} & \left\{ Y_t - w_t L_t - \int_0^1 p_{jt} y_{jt} dj \right\} \\ \text{s.t. } & Y_t = \frac{1}{1 - \beta} L_t^\beta \int_0^1 \tilde{q}_{jt}^\beta y_{jt}^{1-\beta} dj \end{aligned}$$

This leads to:

$$p_{jt} = \left(\frac{y_{jt}}{\tilde{q}_{jt}} \right)^{-\beta} L_t^\beta \quad (5)$$

where recall that

$$\tilde{q}_{jt} = q_{jt} + \phi_{jt}$$

Equation (5) says that the inverse demand function for intermediate goods from final-good firms is iso-elastic, with β being the price-elasticity. This equation makes apparent that advertising, by increasing total quality through ϕ_{jt} , effectively impacts consumer preferences: it works as a perfect demand shifter that affects final-good production decisions, and thus consumption decisions.

Intermediate-good incumbent's problem The maximization problem of intermediate-good monopolist $j \in [0, 1]$ holding the patent for the leading-edge intrinsic quality of good j is to maximize static profits

²³This assumption is convenient because it allows us to find closed-form expressions for demand, sales and profits. It has been used by Acemoglu (2009), Acemoglu, Akcigit, and Celik (2014), Acemoglu and Cao (2015), and Akcigit and Kerr (2015), among others.

$$\begin{aligned}
\pi(\tilde{q}_{jt}) &= \max_{y_{jt}, l_{jt}} \left\{ p_{jt} y_{jt} - w_t l_{jt} \right\} \\
\text{s.t. } y_{jt} &= \bar{Q}_t l_{jt} \\
p_{jt} &= \left(\frac{y_{jt}}{\tilde{q}_{jt}} \right)^{-\beta} L_t^\beta
\end{aligned}$$

The optimality condition implies

$$p_{jt} = \left(\frac{1}{1-\beta} \right) \frac{w_t}{\bar{Q}_t} \quad (6)$$

meaning that all monopolists set the same price every period. Recall that our assumptions ensure that the unconstrained monopoly price given by (6) prevails for any intermediate producer until being displaced by a successful innovator. In combination with the labor demand by the final-good sector, we find the equilibrium wage

$$w_t = \tilde{\beta} [\bar{Q}_t + \bar{\Phi}_t]^\beta \bar{Q}_t^{1-\beta} \quad (7)$$

where $\tilde{\beta} \equiv \beta^\beta (1-\beta)^{1-2\beta}$ denotes the marginal cost of intermediate-good production, and $\bar{\Phi}_t \equiv \int_0^1 \phi_{jt} dj$ is aggregate extrinsic quality. This means that the monopoly price can be written as $p_{jt} = \frac{\tilde{\beta}}{1-\beta} \left(\frac{\bar{Q}_t + \bar{\Phi}_t}{\bar{Q}_t} \right)^\beta$. Aggregate output is

$$Y_t = \frac{\tilde{\beta}}{\beta} L_t [\bar{Q}_t + \bar{\Phi}_t]^\beta \bar{Q}_t^{1-\beta} \quad (8)$$

which, in combination with (7), means that β represents the labor-income share in the final-good sector. Imposing $L_t + \int_0^1 l_{jt} dj = 1$ by labor market clearing, with $l_{jt} = y_{jt}/\bar{Q}_t$, gives $L_t = L$ with

$$L = \frac{\beta}{(1-\beta)^2 + \beta} \quad (9)$$

a time-invariant cross-sectoral allocation of labor. Flow operating profits can then be written as

$$\pi_{jt} = \tilde{\pi}_t (q_{jt} + \phi_{jt}) \quad (10)$$

where

$$\tilde{\pi}_t \equiv \hat{\beta} \left(\frac{\bar{Q}_t}{\bar{Q}_t + \bar{\Phi}_t} \right)^{1-\beta} \quad (11)$$

is the time-varying component of profits that is independent of j , and $\hat{\beta} \equiv \frac{\beta^{1+\beta}(1-\beta)^{2(1-\beta)}}{(1-\beta)^2 + \beta}$ is a constant. Equation (10) shows that, in any given period, any difference in flow profits across goods

must be due to differences in either intrinsic or extrinsic quality, or both. Specifically, a firm can increase the flow profits for the goods that it currently possesses by having conducted internal R&D successfully in the past (thereby increasing intrinsic quality q_{jt}) and/or by having spent resources on advertising on that good (thereby increasing extrinsic quality ϕ_{jt}). Two goods of the same intrinsic quality can earn different profits if the firm chooses to advertise one more than the other. Therefore, R&D and advertising are substitutes, in the sense that they represent two non-mutually exclusive ways to increase profits. However, they are also partly complements as advertising is more effective for higher-quality goods (Equation (1)).

We next describe how the good-specific advertising decision is made optimally by the firm.

Advertising choices Henceforth, we drop time subscripts unless otherwise needed. When choosing advertising expenditures $\{m_j : j \in \mathcal{J}_f\}$ in any given period, a firm f of size n and portfolio of qualities $\mathbf{q}_f \in \mathbb{R}_+^n$ solves the static problem

$$\begin{aligned} V_n^{adv}(\mathbf{q}_f) &\equiv \max_{\{m_j : j \in \mathcal{J}_f\}} \sum_{q_j \in \mathbf{q}_f} \left[\tilde{\pi}(q_j + \phi(m_j, n; q_j)) - m_j \right] \\ \text{s.t. } \phi(m_j, n; q_j) &= \theta \frac{q_j}{\left(\sum_{q_{j'} \in \mathbf{q}_f} q_{j'}^{1/\alpha} \right)^\alpha} \bar{Q}^{1-\zeta} m_j^\zeta n^\eta \end{aligned}$$

We make the assumption that $\zeta = 1 - \alpha$ in order to reduce the dimensionality of the parameter space and be able to find closed-form expressions. The optimality condition gives good-specific advertising expenditures of

$$m_j = ((1 - \alpha)\theta\tilde{\pi})^{\frac{1}{\alpha}} \frac{q_j^{1/\alpha}}{\sum_{j'} q_{j'}^{1/\alpha}} \bar{Q} n^{\frac{\eta}{\alpha}} \quad (12)$$

and thus firm-level advertising expenditures of

$$M_n \equiv \sum_{q_j \in \mathbf{q}_f} m_j = ((1 - \alpha)\theta\tilde{\pi})^{\frac{1}{\alpha}} \bar{Q} n^{\frac{\eta}{\alpha}}$$

for a firm of size n . We in turn define optimal advertising intensity by M_n/n . In our empirical analysis, we have found that, after controlling for firm observables and time fixed-effects, advertising expenditures are increasing in size (as measured by either sales or employment), and that the ratio of advertising expenditures to sales (i.e., advertising intensity) is decreasing in size. To the extent that the number of products within the firm is a good proxy for size,²⁴ we then require the following parametric assumption for an equilibrium to be consistent with such empirical regularities:

²⁴For instance, [Plehn-Dujowich \(2013\)](#) show that there is a positive correlation between number of products and number of employees.

Assumption 2 $0 < \eta < \alpha < 1$.

Indeed, under Assumption 2, advertising intensity M_n/n is then an decreasing function of n , as seen in the data. This optimal behavior implies the total extrinsic quality at the good level to be

$$\phi_j = \tilde{\gamma} \frac{q_j^{1/\alpha}}{\sum_{j'} q_{j'}^{1/\alpha}} \bar{Q} n^{\frac{\eta}{\alpha}} \quad (13)$$

and the total firm-level extrinsic quality to be

$$\Phi_n \equiv \sum_{q_j \in \mathbf{q}_f} \phi_j = \tilde{\gamma} \bar{Q} n^{\frac{\eta}{\alpha}} \quad (14)$$

for a firm of size n , where $\tilde{\gamma} \equiv (\tilde{\pi}(1 - \alpha))^{\frac{1-\alpha}{\alpha}} \theta^{\frac{1}{\alpha}}$. Importantly, the last two equations show that the empirical findings that motivated our specification of the marketing structure in equation (1) feature in the optimal choice of firms as long as we impose Assumption 2, which is an empirically-motivated restriction on parameters.

At the good level (see equation (13)), (i) goods benefit more from advertising when the firm is larger (i.e., ϕ_j increases in n); (ii) through the umbrella-branding (or spillover) effect, it is for smaller firms that goods benefit the most from an innovation-driven n -to- $(n+1)$ transition (i.e., ϕ_j is concave in n); and (iii) for given size, relatively higher-quality goods within the firm benefit more from marketing (i.e., ϕ_j increases in q_j). The equilibrium size advantages survive at the firm level as well (see equation (14)): for a given portfolio of goods, firm-level extrinsic quality is higher for bigger firms, but when advancing the quality portfolio from n to $n+1$ goods through external innovation, smaller firms experience a relatively higher increase in extrinsic quality (as the marginal firm-level increase in extrinsic quality, given by $(\Phi_{n+1} - \Phi_n)$, is a decreasing function of n).

As a result, as will be formally shown in the context of the BGP solution below, smaller firms have a relatively stronger preference for both external R&D and advertising, while bigger firms have a higher preference to improve upon the products that they already own through internal R&D, rather than to open up to other products. What is more, as a firm grows larger, it tends to switch from R&D to advertising, in the sense that the firm-level R&D-to-advertising expenditure ratio is a decreasing function in size. These patterns are, once again, in line with the results from our empirical exploration from Section 3.

Combining returns and costs, we have that

$$V_n^{adv}(\mathbf{q}_f) = \tilde{\pi} \sum_{q_j \in \mathbf{q}_f} q_j + \gamma \bar{Q} n^{\frac{\eta}{\alpha}} \quad (15)$$

where $\gamma \equiv [\tilde{\pi}\tilde{\gamma} - (\theta\tilde{\gamma})^{\frac{1}{1-\alpha}}]$ is constant across firms. Equation (15) is the value of firm-level flow

operating profits net of instantaneous marketing expenditures. It has two components. The first one grows linearly with the aggregate intrinsic quality within the firm, with a factor of proportionality that does not vary across firms. The second one grows with the economy, and is increasing and concave in firm size. While the first term in the sum is standard in models of endogenous firm growth with scalable returns, the second term is new and critically alters the dynamic incentives to conduct research vis-à-vis advertising.

BGP equilibrium characterization A BGP equilibrium is defined as an equilibrium allocation in which output Y_t grows at a constant rate, which we denote by $g > 0$. To show the existence of such an equilibrium, we make use of the fact that economy-wide extrinsic quality $\bar{\Phi}$ in fact grows at the same rate as that of aggregate intrinsic quality, $\bar{Q}$. In this sense, R&D remains the only engine of growth in the model, as in the original formulation by [Klette and Kortum \(2004\)](#).

Formally, we have that

$$\bar{\Phi} = \Phi^* \bar{Q}$$

where Φ^* is a constant, both across time and firms. Under this result, it is clear from equation (8) that, if a BGP exists where output grows at rate g , then $g = \dot{\bar{Q}}/\bar{Q}$. Moreover, we can note from equation (11) that $\tilde{\pi}_t$ is time-invariant, and therefore flow operating profits can be written as $\pi_{jt} = \pi \tilde{q}_{jt}$ in BGP, where $\pi = \hat{\beta}(1 + \Phi^*)^{\beta-1}$ is a constant. In turn, $\tilde{\gamma}$ is then constant, and Φ_n becomes a linear function of $\bar{Q}$, from equation (14). By construction, we can express total aggregate extrinsic quality on the BGP as

$$\bar{\Phi} = \sum_{n=1}^{+\infty} F \mu_n \Phi_n$$

where μ_n is the invariant share of size- n firms (which we derive explicitly below), such that $\mu_n \in [0, 1]$ and $\sum_{n=1}^{+\infty} \mu_n = 1$. Similarly, $F \mu_n$ is the mass of active firms of size $n \geq 1$ at time t . Combining $\bar{\Phi} = \Phi^* \bar{Q}$ with the last equation and the equilibrium firm-level extrinsic quality found in (14), we then obtain that Φ^* is the fixed point of the following expression:

$$\Phi^* = \theta^{\frac{1}{\alpha}} \left(\frac{\hat{\beta}(1 - \alpha)}{(1 + \Phi^*)^{1-\beta}} \right)^{\frac{1-\alpha}{\alpha}} \sum_{n=1}^{+\infty} F \mu_n n^{\eta/\alpha} \quad (16)$$

Equation (16) delivers a unique solution $\Phi^* > 0$.²⁵ In our numerical algorithm described below, we exploit this fixed-point to solve for the unique BGP of the economy by iterating over Φ^* . A direct

²⁵This is because the right-hand side is continuous, strictly positive and decreasing in Φ^* , since $\alpha, \beta \in (0, 1)$.

consequence of the result is that aggregate marketing expenditures, M , grow at the rate g , since

$$M = \sum_{n=1}^{+\infty} F\mu_n M_n \quad (17)$$

and M_n has been shown to be linear in $\bar{Q}$. From the resource constraint in (2), if total R&D expenditures Z grow at the rate of $\bar{Q}$ (a result that we relegate to the end of this section), so does aggregate consumption, C . From the Euler equation (3), we then have that $g = r_t - \rho$, and therefore $r_t = r = g + \rho$, $\forall t$.

Value functions Denote by τ the endogenous rate of creative destruction along the BGP, i.e. the aggregate rate of external innovation coming from both entrants and incumbents. Taking (r, τ, g) as given, a firm f with a product portfolio with $1 \leq n = |\mathbf{q}_f|$ goods chooses external R&D intensity x , internal R&D intensities $\{z_j : j \in \mathcal{J}_f\}$, and advertising expenditures $\{M_j : j \in \mathcal{J}_f\}$ to maximize firm value $V_n(\mathbf{q}_f)$, written in the Hamilton-Jacobi-Bellman form:

$$\begin{aligned} rV_n(\mathbf{q}_f) = \max_{\substack{x \in [0, \bar{x}] \\ \{z_j \in [0, \bar{z}]\} \\ \{M_j\}}} \left\{ \sum_{q_j \in \mathbf{q}_f} \left[\pi(q_j + \phi_j) - \hat{\chi} z_j^{\hat{\psi}} q_j - M_j + z_j \left(V_n(\mathbf{q}_f \setminus \{q_j\} \cup_+ \{q_j(1 + \lambda^I)\}) - V_n(\mathbf{q}_f) \right) \right. \right. \\ \left. \left. + \tau \left(V_{n-1}(\mathbf{q}_f \setminus \{q_j\}) - V_n(\mathbf{q}_f) \right) \right] + nx \left(\mathbb{E}_j V_{n+1}(\mathbf{q}_f \cup_+ \{q_j(1 + \lambda^E)\}) - V_n(\mathbf{q}_f) \right) \right. \\ \left. - \tilde{\chi} n^{\sigma + \tilde{\psi}} x^{\tilde{\psi}} \bar{Q} \right\} + \dot{V}_n(\mathbf{q}_f) \end{aligned} \quad (18)$$

subject to equation (1), where $\cup_+$ and $\setminus_-$ are multiset union and difference operators.²⁶ The right-hand side of equation (18) is composed of multiple terms. The first three terms in the first line are good-specific flow operating profits net of internal R&D costs and advertising expenditures. The fourth term in the first line is the change in value due to the internal improvement over a currently held good. This event occurs at a Poisson flow rate of z_j , and increases the intrinsic quality of the good by a factor of $(1 + \lambda^I) > 1$. The first term in the second line is the change in the firm's value when losing a good j (and transitioning to $n - 1$) to another incumbent or an outside entrant through creative destruction, an event that occurs at the equilibrium Poisson rate τ . The second term on this line is the change in value for the firm when it successfully acquires a new product line through external innovation and transitions to $n + 1$, which occurs at the Poisson rate nx (recall that here x denotes intensity). The third line includes the flow resources spent in external R&D, and the change in firm value that occurs due to growth at the economy-wide level (the term $\dot{V}_n(\mathbf{q}_f)$).

²⁶These operators are defined $\{a, b\} \cup_+ \{b\} = \{a, b, b\}$, and $\{a, b, b\} \setminus \{b\} = \{a, b\}$, and they are needed because the set $\mathbf{q}_f$ can include more than one instance of the same element.

As for entrants, their value function is given by

$$rV_0 = \max_{x_e > 0} \{x_e [\mathbb{E}_j V_1(\{q_j(1 + \lambda^E)\}) - V_0] - \nu x_e \bar{Q}\} + \dot{V}_0 \quad (19)$$

Entrants are choosing the rate of entry x_e , in which event they randomly draw a good $j \in [0, 1]$ and, in case of success, become a size-one incumbent whose portfolio improves upon intrinsic quality by a factor of $(1 + \lambda^E) > 1$. The free-entry condition reads

$$V_0 = 0$$

Our first proposition shows that the growth rate of the economy is measured as the combined contributions of internal and external innovators, which advance aggregate productivity by λ^I and λ^E , respectively, at the frequencies of z and τ , respectively.²⁷

Proposition 1 (Growth rate) *The growth rate of the economy in the BGP equilibrium is given by*

$$g = \tau \lambda^E + z \lambda^I \quad (20)$$

Proof. All proofs are included in Appendix A.

Moreover, since innovation rates are independent, the rate of creative destruction τ is given by the aggregate Poisson rate of external innovation coming from entrants and incumbents. While x_e denotes both the realized rate of entry and the total equilibrium mass of entrants, we must take into account that there exist different shares of incumbents according to their size, and a total mass F of incumbents at any point in time. Therefore,

$$\tau = x_e + \sum_{n=1}^{+\infty} F \mu_n n x_n \quad (21)$$

In turn, we can find a closed-form expression for μ_n , the time-invariant share of firms of size n along the BGP:

Proposition 2 (Invariant firm-size distribution) *The invariant firm-size distribution is given by*

$$\mu_n = \frac{x_e}{F\tau} \frac{\prod_{i=1}^{n-1} x_i}{n\tau^{n-1}}; \quad \forall n \geq 1 \quad (22)$$

with $\mu_n \in [0, 1], \forall n \geq 1$, and $\sum_{n=1}^{+\infty} \mu_n = 1$.

Proof. See Appendix A.

²⁷To be precise, the proposition uses that $z_j = z, \forall j \in [0, 1]$, a result that we shall prove shortly.

We are now ready to solve for the value function and find optimal R&D intensities. To this purpose, we guess a linear functional form of the type:

$$V_n(\mathbf{q}_f) = \Gamma \sum_{q_j \in \mathbf{q}_f} q_j + \Upsilon_n \bar{Q}$$

for which we must verify the value of the constant coefficient $\Gamma \in \mathbb{R}$ and the sequence $\{\Upsilon_n\}_{n=1}^{+\infty}$.²⁸ This guess is informed by expression for the flow value of holding a given product, seen in (15). Namely, a firm derives value from the intrinsic quality of the products it is selling, independently of its size (the first term, whose coefficient Γ is constant across firm types), but also from marketing its goods in order to increase demand and revenue (the second term, whose coefficient Υ_n depends on firm type).²⁹ Further, we have shown that the latter source of value grows linearly with the economy and is influenced by firm size in a non-linear manner. Hence, the object Υ_n is a non-linear function of n .

The following proposition summarizes the solution:

Proposition 3 (Value functions) *Under Assumptions 1 and 2, and assuming that there is positive entry in equilibrium ($x_e > 0$), the value for a firm of size $n \geq 1$ is $V_n(\mathbf{q}_f) = \Gamma \sum_{q_j \in \mathbf{q}_f} q_j + \Upsilon_n \bar{Q}$, where*

$$\Gamma = \frac{\nu - \Upsilon_1}{1 + \lambda^E} \quad (23)$$

and Υ_n , for $n \geq 1$, is the solution to the second-order difference equation

$$\Upsilon_{n+1} - \Upsilon_n + \Gamma(1 + \lambda^E) = \tilde{\vartheta} \left(\rho \Upsilon_n n^{\frac{\sigma}{\psi-1}} - (\Upsilon_{n-1} - \Upsilon_n) \tau n^{\frac{\sigma + \tilde{\psi} - 1}{\psi-1}} - \gamma n^{\frac{\eta}{\alpha} + \frac{\sigma}{\psi-1}} \right)^{\frac{\tilde{\psi}-1}{\tilde{\psi}}} \quad (24)$$

where $\tilde{\vartheta} \equiv \tilde{\psi} \left(\frac{\tilde{\chi}}{(\tilde{\psi}-1)^{\tilde{\psi}-1}} \right)^{\frac{1}{\tilde{\psi}}}$ is a parameter, with boundary condition $\Upsilon_0 = 0$.

Proof. See Appendix A.

In the appendix we show that we obtain the following optimal R&D intensities:

$$z_j = \left(\frac{\lambda^I (\nu - \Upsilon_1)}{\tilde{\psi} \tilde{\chi} (1 + \lambda^E)} \right)^{\frac{1}{\tilde{\psi}-1}} \quad (25)$$

$$x_n = n^{\frac{1-\sigma-\tilde{\psi}}{\tilde{\psi}-1}} \left(\frac{\nu - \Upsilon_1 + \Upsilon_{n+1} - \Upsilon_n}{\tilde{\psi} \tilde{\chi}} \right)^{\frac{1}{\tilde{\psi}-1}} \quad (26)$$

for internal and external innovations, respectively, and all $n \geq 1$. For internal innovations, both

²⁸Throughout, we impose $\Upsilon_0 = 0$, as V_0 is fully characterized once x_e is found residually by free entry. See more details in the description of our algorithm below.

²⁹When $\tilde{\psi} + \sigma \neq 1$, Υ_n also captures the dependence of external R&D on firm size.

costs and returns are linear in q_j , and thus the optimal innovation effort is independent of the quality of the product line (namely, $z_j = z$). On the other hand, n affects innovation intensities through two channels: (i) the degree of scalability in external R&D technology (i.e, whether $\sigma + \tilde{\psi}$ is more or less than one); (ii) the degree of decreasing returns in advertising through the branding effect (i.e, the values for α and $\eta < \alpha$). The former channel features both explicitly in the first multiplicative term of (26), and implicitly within the Υ_n sequence in (24). The advertising channel, however, only features indirectly through Υ_n . Thus, we can obtain the desired inverse dependence of innovation expenditures to size as long as the advertising channel scales weakly enough to compensate any strong scalability coming from the R&D technology itself. This allows us to obtain finite growth in equilibrium even with constant returns to scale in external R&D. In fact, through weak enough returns to scale in advertising, we can obtain a steady state in which firms do not grow unboundedly even when they enjoy increasing returns in innovation.

To provide intuition, henceforth we focus on the case in which there are constant returns in external R&D (namely, internal and external scale proportionally with size), and relegate the analysis with decreasing and increasing returns to Section 4.6.2.

Assumption 3 $\sigma + \tilde{\psi} = 1$ (*constant returns to scale in external R&D*).

Under Assumption 3, equation (24) reads

$$\Upsilon_{n+1} - \Upsilon_n + \Gamma(1 + \lambda^E) = \tilde{\vartheta} \left(\frac{\rho \Upsilon_n}{n} - (\Upsilon_{n-1} - \Upsilon_n)\tau - \frac{\gamma n^{\eta/\alpha}}{n} \right)^{\frac{\tilde{\psi}-1}{\tilde{\psi}}}$$

This equation has an intuitive interpretation. The left side is the total change in value that is associated with an n -to- $(n+1)$ transition, and therefore it measures the marginal gain for a firm of size n of increasing in size through successful external innovation.³⁰ The right side of the equation includes counterpart marginal losses from external R&D. These losses come partly from the per-product decrease in value (the term $(\Upsilon_{n-1} - \Upsilon_n)\tau$) and the loss of the average value per good on the remaining goods (the term $-\rho \Upsilon_n/n$) for losing one product through creative destruction. We say that this combined change in value comes from a *creative destruction channel*. The remaining losses are the share of the net flow operating profit that is lost when the firm foregoes the advertising advantage over the good it is no longer producing (the term $\frac{\gamma n^{\eta/\alpha}}{n}$). This effect is therefore coming from the *advertising channel*.

In the absence of the advertising channel (the special case of $\theta = 0$), under Assumption 3 the economy collapses to that of Klette and Kortum (2004). In this case, Υ_n becomes the line $\Upsilon_n = n\Upsilon$ for some constant $\Upsilon > 0$, and we obtain the well-known theoretical results that (i) firm value scales perfectly with size, and (ii) consequently firm growth is independent of size (Gibrat's law holds). As

³⁰The term $\Gamma(1 + \lambda^E)$ is the change in value coming from the increase in the aggregate intrinsic quality of the firm's portfolio when acquiring one more product. Since this change in value is common across sizes, it does not play a direct role on innovation incentives across n , and we shall ignore it from our exposition.

we have already discussed, departure from Gibrat's law for small firms can be obtained if there are decreasing returns to external R&D, or $\sigma + \tilde{\psi} > 1$. When $\theta > 0$, however, Gibrat's law deviations can occur even when $\sigma + \tilde{\psi} \leq 1$: small firms might still be more innovative-intensive than large firms because they benefit marginally more from advertising the new product lines that they acquire through external R&D thanks to spillover effects across advertising different goods.

In particular, gains and losses depend non-linearly on n . As a small firm (low n) accumulates more products, marginal gains to external R&D decrease because the loss in value of marketing per good is decreasing faster than the decrease in the cost components of the creative destruction channel. This means that the value of external R&D for a firm of size n is increasing and concave, and thus the marginal gain of acquiring an additional product line for a firm of size n , given by $(\Upsilon_{n+1} - \Upsilon_n)$, is decreasing and convex in size. In the limit as n grows larger, the marginal gain becomes flat until marginal gains and losses associated with the advertising-innovation trade-off cancel each other out. In this sense, in the limit as size grows large, firms operate in a [Klette and Kortum \(2004\)](#) regime.

Equilibrium definition To close the optimality characterization and define an equilibrium, it remains to show three conditions that have been stated without proof in the above.

First, the assumption of positive entry ($x_e > 0$) stated in Proposition 3 relies on the equilibrium condition $\nu < \Upsilon_1$ (so that $\Gamma > 0$). Intuitively, this condition says that the cost of entry ν is lower than the gain, coming from the acquisition of one product line. This condition is always checked ex-post in our computation.³¹

Secondly, from (2) and the result that $g = \dot{Q}/\bar{Q} = \dot{C}/C = \dot{M}/M$, we need to check that aggregate R&D expenditure Z grows at the rate g , as well. Using z and $\{x_n\}_{n=1}^{+\infty}$ from (25) and (26), and the cost functions for R&D, we have that

$$Z = FR_z(z) + \sum_{n=1}^{+\infty} F\mu_n R_x(X_n, n) + R_e = \left(\hat{\chi} z^{\hat{\psi}} + \sum_{n=1}^{\infty} \mu_n \tilde{\chi} n x_n^{\tilde{\psi}} + \nu \frac{x_e}{F} \right) F\bar{Q} \quad (27)$$

The right-hand side includes aggregate R&D expenditures by incumbents attempting internal innovations, incumbents attempting external innovations, and outside firms attempting to enter into the market through external R&D, respectively. Indeed, we get that Z is linear in $\bar{Q}$, and hence $\dot{Z}/Z = g$. Aggregate consumption is computed as the residual $C = Y - Z - M$.

Finally, we must verify the transversality condition. Using $r_t = r$ and the solution to the value function inside equation (4) gives the standard requirement $r > g$, which by the Euler equation simply says that $\rho > 0$.

³¹A parametric closed-form condition equivalent to $\nu < \Upsilon_1$ does not exist because Υ_1 depends endogenously on nearly all equilibrium variables of the economy. Hence our ex-post approach.

We can now define a BGP equilibrium:

Definition 1 (BGP Equilibrium) *A BGP Markov Perfect Equilibrium is, for all $q_j \geq 0$, $j \in [0, 1]$, $n \in \mathbb{Z}_+$, $\bar{Q}$ and $t \in \mathbb{R}_+$, an allocation of expenditures y_j , z , x_n , x_e , and m_j ; extrinsic quality ϕ_j ; aggregates Y , C , Z , M , F , L , $\tilde{L}$, A , Γ , Υ_n , and μ_n ; prices w , p , and r ; and rates g and τ ; such that:*

- a. y_j (input quantity) and p (input prices) solve the intermediate sector problem and satisfy (6);*
- b. m_j (expenditure), ϕ_j (effectiveness), M (aggregate expenditure) solve the advertising problem and satisfy (12), (13) and (17), respectively;*
- c. Innovation flows z (internal for incumbents) and x_n (external for incumbents) solve the innovation problem and satisfy (25) and (26);*
- d. μ_n (the invariant measure of firms) satisfies (22);*
- e. x_e (entry flow) and F (mass of incumbent firms) solve the entry problem and satisfy the free-entry condition $V_0 = 0$ and the restriction $\sum_n \mu_n = 1$;*
- f. Γ (the value of a firm's portfolio) and Υ_n (the value of owning n products) solve the value function $V_n = \Gamma \sum_j q_j + \Upsilon_n \bar{Q}$ in (18), and satisfy (23) and (24) respectively;*
- g. g (growth rate) and τ (creative destruction rate) satisfy (20) and (21), respectively;*
- h. Aggregates Y (output), Z (R&D investment), C (consumption), L and $\tilde{L}$ (labor in final- and intermediate-good sectors) satisfy (2), (8), (9), (27), and $\tilde{L} = 1 - L$;*
- i. A (household wealth) satisfies $A = \sum_{n \geq 1} F \mu_n V_n$ and the transversality condition (4);*
- j. Prices r (interest rate) and w (wage) satisfy $r = g + \rho$ from (3), and (7), respectively.*

4.6 Computation

Before turning to the estimation of the model in Section 5, we discuss the computational implementation of the model and present a few comparative statics results for a given parametrization in order to illustrate our main advertising-innovation trade-off.

Throughout this section, we set $\lambda^E = \lambda^I$ so that neither type of innovation is more radical than the other. All the parameter values are set for expositional purposes, and we relegate the discussion on the calibration to Section 5.1.³² In particular, recall that we set $\sigma = 1 - \tilde{\psi}$, so that the model nests Klette and Kortum (2004) for $\theta = 0$.

4.6.1 The BGP Algorithm

To compute the equilibrium variables in the BGP, we loop over the fixed point of $\bar{\Phi}_t$ by using $\bar{\Phi}_t = \Phi^* \bar{Q}_t$, with Φ^* from equation (16). In light of the above discussion, we check for convergence by

³²Only in this section, the parameter values are $\rho = 0.02$, $\hat{\psi} = \tilde{\psi} = 1 - \sigma = 2$, $\lambda^E = \lambda^I = 0.05$, $\hat{\chi} = 0.1$, $\tilde{\chi} = 10$, $\nu = 2$, $\beta = 0.2$, $\eta = 0.5$, $\alpha = 0.9$, and $\theta = 1$.

imposing that the value of firm size, Υ_n , converges to a line as n grows larger. We truncate the size space at some large $N \in \mathbb{N}$. The following describes the steps:

1. Guess a number $\Phi^* > 0$.
2. Compute Υ_1 by imposing $\Upsilon_2 = 2\Upsilon_1$ in equation (24) at $n = 1$.
 - (a) Compute g from (20), τ implied by (21), Γ from (23), and z from (25).
 - (b) Compute $\{\Upsilon_n\}_{n=2}^{+\infty}$ using (24) forward from Υ_1 and $\Upsilon_0 = 0$, $\{x_n\}_{n=1}^{+\infty}$ from (26), and F/x_e using that $\sum_{n=1}^N \mu_n = 1$, where μ_n comes from (22).
3. Verify convergence by checking that $\mu_N \leq 1$. If there is no convergence by iteration $k \in \mathbb{N}$, go back to step 2 with the new guess for Υ_1 to be the solution to (24) at $n = 1$ when $\Upsilon_2 = 2\Upsilon_1 - \varepsilon_k$, for a small $\varepsilon_k > 0$.
4. Compute Φ^* as the solution to (16), and compare it to the initial guess. If it is too far, go back to step 1 using this solution as the new guess.

In step 2, we compute the maximum value for Υ_1 such that Υ_n can be weakly concave (i.e. $(\Upsilon_{n+1} - \Upsilon_n)$ decreasing). In particular, we force Υ_n to be straight line from $n = 1$ to $n = 2$ (note $\Upsilon_2 - \Upsilon_1 = \Upsilon_1 - \Upsilon_0$). If μ_n does not converge, it must be because $(\Upsilon_n - \Upsilon_{n-1})$ has not settled to a flat line as n has approached N , which means that the guess for Υ_1 was incorrect. Then, we iterate on new guesses for Υ_1 that parsimoniously allow for more concavity on the Υ_n sequence (indeed, note that in any iteration $k \geq 1$ we always start the Υ_n sequence at a Υ_1 such that $\Upsilon_2 - \Upsilon_1 < \Upsilon_1 - \Upsilon_0$). In step 3, we in turn bisect the new guess by a factor of ten on each new iteration, i.e. $\varepsilon_{k+1} = \varepsilon_k/10$. Finally, in step 4, we drop the old Φ^* guess altogether in case of no convergence, and use the resulting fixed-point as the new guess.

4.6.2 Comparative Statics on BGP

To illustrate the active forces in the model, the following presents a set of comparative statics exercises on outcomes of the invariant equilibrium with respect to the following parameters: advertising efficiency (θ), the strength of the spillover effect (η), and the degree of returns to scale in innovation ($\tilde{\psi} + \sigma$).

Advertising efficiency (θ) Figure 3 shows BGP results for the marginal gain of acquiring new product lines (or $\Upsilon_{n+1} - \Upsilon_n$), on the left panel, and the relative expenditure on external R&D (or R_x/n), on the right panel, as a function of firm size, for different values of θ . Figure 4 shows that our framework can generate an invariant firm-size distribution that resembles a Pareto.³³

³³Because firm growth scales imperfectly with size when $\theta > 0$, the shape coefficient of this distribution is less than unity, and Zipf's law does not typically hold. Namely, the fraction of firms with size greater than κ is less-than-proportional to $1/\kappa$.

Figure 3: Marginal gain for acquiring a new product line (left panel) and external R&D expenditure intensity (right panel) in BGP, for different values of advertising efficiency (θ).
Note: External R&D expenditure intensity is normalized by aggregate intrinsic quality, $\bar{Q}$.

We see that in the absence of advertising ($\theta = 0$), innovation incentives are constant in size. The gain of acquiring new products is proportional to size, or $\Upsilon_n = n\Upsilon$, and therefore expenditures are constant in the cross-section of firms. When we introduce a motive for advertising ($\theta > 0$), however, some of this expenditure is diverted toward improving quality along the extrinsic margin, so relative external R&D expenditure is lower for all levels of n . Moreover, because smaller firms benefit marginally more from expanding their product portfolios (left panel in Figure 3), they invest more in R&D relative to larger firms (right panel in Figure 3). Consequently, the firm-size distribution in Figure 4 is more right-skewed for any $\theta > 0$ than the one deliver by the special case of Klette and Kortum (2004): fewer firms achieve large scales because, as they keep growing, they become increasingly less concerned with expanding their size further.

As $\theta > 0$ increases (with all other parameters fixed), advertising becomes more efficient for all levels of firm size, and the aforementioned effects are reinforced. External R&D expenditure intensity shifts down for all sizes as a higher share of expenditure is now being devoted to advertising. Moreover, since marketing now delivers higher returns, in the margin small firms now benefit even more than they did before, relative to larger firms. This is reflected by the fact that $(\Upsilon_{n+1} - \Upsilon_n)$ becomes steeper as θ increases. As a consequence, expenditure decreases faster with size (R_x/n is also steeper), as firms switch faster from R&D into advertising. In this sense, these two technologies become more substitutes.

Figure 5 shows the effects of increasing advertising efficiency on the creative destruction rate τ (left panel), and the growth rate of the economy g (right panel). We also show the corresponding decomposition of these rates. For τ , the decomposition, given directly by (21), is the sum of the innovation rates by entrants and external incumbents. For the growth rate decomposition, we can combine (20) and (21) to obtain:

Figure 4: Firm size distribution in BGP, for different values of advertising efficiency (θ).

$$g = \lambda^E x_e + z\lambda^I + \lambda^E \sum_{n=1}^{+\infty} F\mu_n n x_n$$

The three terms show the contribution to growth by entrants, incumbents performing internal improvements, and incumbents performing external innovations, respectively. Figure 5 shows that outside firms are encouraged to enter more frequently when advertising becomes more efficient, as small firms are benefiting from advertising even more now than before with respect to larger firms. This increases the mass of firms with fewer goods, as seen in Figure 4. However, larger entrants have now a lower incentive to conduct R&D, which is more intensely done by small entering firms, and the equilibrium external innovation rate decreases with θ . All in all, entry is strong enough to make the creative destruction rate increase with θ , and the contribution to growth by entering external innovators increases as advertising becomes more efficient. Yet, the growth rate goes down because incumbent innovation intensity decreases sharply when advertising efficiency increases. This is because advertising and innovation are increasingly substitutable as the former becomes relatively more effective in raising profits, an effect that is especially present for larger firms. In sum, higher advertising efficiency has two opposing forces on growth: it encourages innovation by entrants but also discourages innovation by incumbents.

Interestingly, the final effect on growth depends on parameter values because the strength of these two opposing forces depends on the relative costs of different R&D technologies. Figure 6 provides an example for the same parameter values as before, except for $\hat{\chi} = 10$ (as opposed to $\hat{\chi} = 0.1$). Increasing the relative cost of internal innovation now fosters entry proportionally more than it deters innovation

Figure 5: Creative destruction rate (τ) and growth rate (g) decompositions, for different values of advertising efficiency (θ).

by incumbents, because in equilibrium external innovation is relatively more used by entering firms and small incumbents. Overall, the economy grows more rapidly as advertising becomes more and more efficient.

Figure 6: Same as Figure 5, for $\hat{\chi} = 10$.

The advertising spillover effect (η) Figure 7 shows the results for comparative statics on η . Since we maintain Assumption 2 throughout, we fix a value for α (at $\alpha = 0.9$, as before) and compare results by re-solving the model for various $\eta \in (0, \alpha)$. We see that as η increases, a relatively lower share of expenditure is diverted toward advertising for all firm size. Moreover, innovation intensity flattens, meaning that smaller firms have a relatively higher marginal advantage when η and α are far

apart. The reason is that the strength of decreasing returns in advertising is given precisely by the ratio η/α .³⁴ In contrast, when η/α is close to unity, smaller firms lose their marginal advantage, and differences in innovation and advertising expenditures across firm sizes vanish. Consequently, growth is higher when η is low because there is high entry, while it is lower as η approaches α because incumbents substitute internal for external innovation and outside firms find it more costly to enter (see Figure 8).

Figure 7: Comparative statics for η , with $\alpha = 0.9$.

Figure 8: Decomposition of growth and creative destruction rates, for different values of $\eta \in (0, \alpha)$, with $\alpha = 0.9$.

Returns to scale in innovation (σ) Finally, we show that the result that small firms are more innovation-intensive does not hinge on the relative degree of scalability in the returns of different types of innovation technologies. The solid (circled) line in Figure 9 shows the solution of the model for increasing (decreasing) returns, whereas the dashed line shows the case of constant returns to scale.

³⁴To see this, note from equation (1) with $\zeta = 1 - \alpha$ that $\phi(\kappa m_j, \kappa n; q_j) = \kappa^{1+\eta-\alpha} \phi(m_j, n; q_j)$ for any constant κ , so there are decreasing returns if $\eta < \alpha$. The farther apart η is from α , the stronger decreasing returns are.

First, decreasing returns lowers innovation incentives for all sizes with respect to constant returns, but makes it even more profitable for small firms to invest more intensely into R&D (i.e., the R_x/n line steepens). Symmetrically, having increasing returns in innovation increases optimal intensity for all n . Interestingly, it is still the case that smaller firms optimally choose to invest relatively more, even if the marginal benefit is lower than before (i.e., the R_x/n line flattens but it is still decreasing). This is because, for this specific parametrization, the advertising channel is strong enough to overturn marginal advantages to large firms coming from strong scalability in the R&D technology. On the other hand, though the effects are quantitatively small for this particular parametrization, we obtain in Figure 10 that the growth and creative destruction rates are higher and the entry rate is lower when returns scale more strongly with size.

Figure 9: Comparative statics for $\sigma + \tilde{\psi} = 1.05 > 1$ (decreasing returns), $\sigma + \tilde{\psi} = 1$ (constant returns), and $\sigma + \tilde{\psi} = 0.95 < 1$ (increasing returns).

Figure 10: Comparative statics for $\sigma + \tilde{\psi} = 1.05 > 1$ (decreasing returns), $\sigma + \tilde{\psi} = 1$ (constant returns), and $\sigma + \tilde{\psi} = 0.95 < 1$ (increasing returns).

The possibility of solving the model when there exist increasing returns in R&D is a strength of our framework, given that the special case of [Klette and Kortum \(2004\)](#) (i.e., $\theta = 0$) has no solution when $\sigma + \tilde{\psi} < 1$. In fact, we know of no other model in this vein that can accommodate increasing returns to R&D and still deliver R&D intensities across sizes that are in line with the data.

5 Estimation [PRELIMINARY]

We calibrate and estimate the model combining the micro data used in [Section 3](#) with aggregate moments on GDP for the U.S. in the period 1981 - 2015. We impose the same data restrictions as in [Section 3](#). Namely, we use firms with strictly positive R&D and advertising expenditures, and exclude firms that report negative sales or employment, as well as those experiencing year-on-year growth of more than 1000%.

5.1 Baseline Calibration

There are 13 parameters to identify: $(\rho, \lambda^E, \lambda^I, \tilde{\chi}, \tilde{\psi}, \hat{\chi}, \hat{\psi}, \sigma, \nu, \beta, \eta, \alpha, \theta)$. Some of these parameters are externally identified, while some others are internally calibrated via an indirect inference approach by matching simulation-implied moments and regression coefficients to those seen in the data. This calibration strategy is fairly standard in the endogenous growth and firm dynamics literature (e.g. [Lentz and Mortensen \(2008\)](#)).

In our baseline calibration, we assume constant returns to external innovation ($\sigma = 1 - \tilde{\psi}$) and assume that no innovation is more radical than the other ($\lambda^E = \lambda^I \equiv \lambda$), though we conduct robustness checks later on in both of these dimensions. Once again, we do this to isolate the effects of our advertising-R&D interaction on firm growth.

External Identification The parameters $(\rho, \hat{\psi}, \tilde{\psi}, \beta, \alpha)$ are externally calibrated. We set $\rho = 0.02$, which is equivalent to a discount factor of approximately 98% annually. We impose $\hat{\psi} = \tilde{\psi} = 2 \equiv \psi$, following [Akcigit and Kerr \(2015\)](#) and prior empirical literature estimating the cost curvature of different types of R&D. To find a value for β , we use the fact that the labor income share of the economy is given by $w/Y = \beta/L$, where $L = \beta/(1 - \beta + \beta^2)$ by equation (9). In turn, the labor-income share can be computed by netting out the shares of income going to R&D (on average 2.56%), advertising (on average 2.21%) and profits (on average 10%).³⁵ This implies that $\beta = 0.178$.

Finally, we use that α is in equilibrium directly related to the elasticity of sales to advertising expenditures. To see this, firstly note from equation (1) that $\frac{\partial \log d_j}{\partial \log m_j} = 1 - \alpha$, where recall that

³⁵The data on average R&D-to-income and advertising-to-income shares comes from the OECD and the Coen Structured Advertising Expenditures databases, respectively. For an evolution of these variables over time, see [Figure 1](#). The data on the profit share is computed using our Compustat sample.

$d_j = \phi_j/q_j$. Also, per-product sales $p_j y_j$ are linear in $\tilde{q}_j = q_j(1 + d_j)$, so

$$\frac{\partial \log(p_j y_j)}{\partial \log m_j} = \frac{\partial \log(p_j y_j)}{\partial \log(1 + d_j)} \frac{\partial \log(1 + d_j)}{\partial \log m_j} \approx \frac{\partial \log d_j}{\partial \log m_j} = 1 - \alpha$$

where we have used $\frac{\partial \log(p_j y_j)}{\partial \log(1 + d_j)} = 1$, and the approximation is valid in BGP because d_j grows unboundedly large as time goes by. Thus, the contemporaneous elasticity of sales to advertising expenditures is equal to $1 - \alpha$ in the limit of time. From the empirical marketing literature, we know this elasticity is on average approximately 10% at the product level, so we set $\alpha = 0.9$.³⁶

Internal Identification We are left with the parameters $(\lambda, \tilde{\chi}, \hat{\chi}, \nu, \eta, \theta)$, which are internally calibrated by means of an indirect inference approach. We solve and simulate the economy and find parameter values that match model moments and simulated-implied regression coefficients to those observed in the data. The invariant equilibrium is computed using the algorithm described in Section 4.6.1, whereas the simulation of the model uses 2,000 firms and discretizes time to $T = 500$ periods of length $\Delta t = 0.01$.

In particular, to find values for these six parameters we target the following moments for our period of interest. In the aggregate, we target the firm entry rate, the average growth rate of the economy, and the average ratios of R&D-to-sales and R&D-to-advertising expenditures in Compustat. Moreover, we use our simulation-implied regression results to target the coefficients on the regressions on R&D intensity and advertising intensity presented in Tables 3 and 4, respectively, of the Introduction. Table 6 shows the full set of calibrated parameter values.

Parameter	Value	Description	Source
ρ	0.02	Time discount rate	Standard
ψ	2	Curvature in R&D cost	Akcigit and Kerr (2015)
β	0.178	Labor share in FG sector	OECD, CSAE and Compustat
$1 - \alpha$	0.1	Sales-advertising elasticity	Tellis (2009)
λ	0.0344	Innovation step	Internally estimated
$\tilde{\chi}$	0.9045	External R&D cost coef.	Internally estimated
$\hat{\chi}$	9.4943	Internal R&D cost coef.	Internally estimated
ν	0.9187	Entry cost coefficient	Internally estimated
η	0.8907	Advertising spillover	Internally estimated
θ	0.7783	Advertising efficiency	Internally estimated

Table 6: Calibrated parameters in baseline estimation. *Note:* CSAE refers to the Coen Structured Advertising Expenditure data source.

³⁶In particular, we take this number from Tellis (2009), who in turn estimates it as the mean elasticity over a sample of over 260 estimates gathered from prior studies.

5.2 Estimation Results

This section presents the results for the above calibration. Section 5.2.1 shows the moments implied by the model, and how they compare to the data. Section 5.2.2 shows plots from the model’s simulation.

5.2.1 Moments

Table 7 shows the results for targeted moments, compared to the values in the data. In general, we do well on the average growth and entry rates. The two regression-based moments have the correct sign, but we miss the magnitude in the R&D intensity regression. Finally, the average R&D-to-sales and R&D-to-advertising ratios are roughly of the correct magnitude, although the model attributes a little bit too much innovation investment to firms. In our future calibration we will combine other moments to try and improve on the dimensions that we are missing.

Moment	Model	Data	Data Source
Entry rate	9.05%	10%	Gourio, Messer, and Siemer (2014)
R&D intensity coefficient	-0.1102	-0.2671	Col.(3) in Table 3
Advertising intensity coefficient	-0.0482	-0.0536	Col.(3) in Table 4
Average growth rate	1.62%	1.72%	World Bank
Average R&D-Sales ratio	0.3254	0.298	Compustat
Average R&D-Advertising ratio	29.97	32.65	Compustat

Table 7: Targeted moments: model vs. data.

Table 8 shows results for some of the moments that we have not targeted. We match all three moments relatively well and in particular obtain that small firms grow relatively faster, even though our estimation has not targeted this aspect of the data. We also see that, both in the data and in the estimated model, firms tend to switch from R&D to advertising as they grow larger. Finally, we underestimate slightly the average advertising expenditures per dollar of sales in the cross-section of firms.

Moment	Model	Data	Data Source
Gibrat’s law coefficient	-0.1577	-0.0599	Col.(3) in Table 2
R&D/Advertising coefficient	-0.0619	-0.111	Col.(3) in Table 5
Average Advertising-Sales ratio	0.0108	0.045	Compustat

Table 8: Untargeted moments: model vs. data.

Once again, we intend to improve upon these results by bringing in new moments to the estimation, such as those from the quintile analysis in Figure 2. Moreover, we will explore the possibility of relaxing some of our assumptions in the baseline calibration, such as $\lambda^E = \lambda^I$ or $\sigma = 1 - \tilde{\psi}$. Finally, one possible

improvement is to measure R&D and advertising intensities relative to employment as opposed to sales. We will explore all of these aspects in our upcoming calibrations.

5.2.2 Simulation Results

Based on the above calibration, this section offers a graphical interpretation of the above results by presenting the outcome of the simulation of the model. Each circle in the plots corresponds to a different firm. Figures 11, 13 and 14 show that the empirical regularities identified in Section 3 are delivered by the model.

Firstly, Figure 11 depicts how R&D and advertising expenditure intensity relative to sales change with the level of sales.³⁷ We confirm that, on average, smaller firms are both more innovative- and more advertising-intensive, as in the data shown in Tables 3 and 4. Figure 12 shows that, as firms grow large, they tend to substitute R&D expenditure for advertising expenditure, as seen in the data results of Table 5. Figure 13 shows that the model delivers deviations from Gibrat's law: smaller firms are growing on average faster than larger firms, like in Table 2. Moreover, the dispersion in growth rates among small firms is higher. Finally, Figure 14 shows the distribution of sales, normalized by their mean, and we see that (i) the size distribution is right-skewed, resembling an exponential (left panel), and (ii) there is a high positive correlation between the number of product owned by a firm and the level of sales for that firm (right panel).

Figure 11: R&D and advertising expenditure intensity as a function of sales.
Note: Parameter values as in Table 6.

³⁷This is informative because sales will become our measure of firm size once we put the model to an empirical test in Section 5.

Figure 12: Ratio of R&D and advertising expenditure intensities as a function of sales.
Note: Parameter values as in Table 6.

Figure 13: Sales growth on sales (Gibrat's law).
Note: Parameter values as in Table 6.

Figure 14: Distribution of normalized sales (histogram, and as a function of n).
Note: Parameter values as in Table 6.

6 Conclusion [PRELIMINARY]

In this paper, we have presented a model of firm dynamics and endogenous growth through product innovation that incorporates explicitly advertising decisions by firms. In modeling advertising we have been inspired by observations coming from the empirical marketing literature. We have shown that the model can qualitatively match empirical regularities across sizes in sales growth, R&D intensity and advertising intensity even in the presence of increasing returns to scale in R&D. Our mechanism incorporates an interaction between advertising and R&D that implies that smaller firms spend more in both types of investments (relative to sales). Finally, we have presented a baseline calibration that shows that the model can match some of these patterns also quantitatively. Our work is now focused on improving upon this estimation.

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A Proofs

Proof of Proposition 1 Assume that $z_j = z, \forall j \in [0, 1]$, a result that we prove independently in Proposition 3. Using that $g = \frac{\dot{\bar{Q}}_t}{\bar{Q}_t}$, we note that after any small interval Δt of time, aggregate quality is

$$\bar{Q}_{t+\Delta t} = \underbrace{\tau \Delta t \cdot (1 + \lambda^E) \bar{Q}_t}_{\text{One external innovation}} + \underbrace{z \Delta t \cdot (1 + \lambda^I) \bar{Q}_t}_{\text{One internal innovation}} + \underbrace{(1 - \tau \Delta t - z \Delta t) \cdot \bar{Q}_t}_{\text{No innovations}} + \underbrace{o(\Delta t)}_{\text{2 or more}}$$

where we have used that the fact that, for a sufficiently small time step, Poisson rates approximate probabilities. Subtracting $\bar{Q}_t$ from both sides, dividing through by Δt , taking the limit as $\Delta \rightarrow 0$ and using that $\lim_{\Delta \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$, gives $\dot{\bar{Q}}_t = \tau \lambda^E \bar{Q}_t + z \lambda^I \bar{Q}_t$. Therefore,

$$g = \tau \lambda^E + z \lambda^I$$

as we sought to show. $\square$

Proof of Proposition 2 Let μ_n denote the equilibrium share of incumbent firms that own $n \geq 1$ product lines, such that $\mu_n \in [0, 1], \forall n$, and $\sum_{n=1}^{+\infty} \mu_n = 1$. The invariant distribution depends upon the following flow equations:

<u># products</u>	<u>Inflows</u>	<u>Outflows</u>
$n = 0 :$	$F \mu_1 \tau$	$= x_e$
$n = 1 :$	$F \mu_2 2\tau + x_e$	$= F \mu_1 (x_1 + \tau)$
$n \geq 2 :$	$F \mu_{n+1} (n+1) \tau + F \mu_{n-1} (n-1) x_{n-1}$	$= F \mu_n (n x_n + n \tau)$

The left-hand (right-hand) side of these equalities describes the mass of firms that enters into (exists out of) the state identified by the first column. For instance, for $n \geq 2$, inflows are given by the mass of size- $(n+1)$ incumbent firms that lost a product through creative destruction, and the mass of size- $(n-1)$ incumbent firms that gained one through external innovation. Similarly, outflows are given by the mass of size- n incumbents that either lost a product to another incumbent, or gained one thanks to external innovation. Because innovation allows firms to gain or lose only one product at a time, potential entrants (entering at rate x_e) only feature as a flow out of $n = 0$, or into $n = 1$.

We guess-and-verify that the solution for μ_n is of the form

$$\mu_n = \frac{1}{n!} C D_n$$

with C and $\{D_n\}_{n=1}^{+\infty}$ to be determined. The first and second equations say that

$$\mu_1 = \frac{x_e}{F\tau} = CD_1; \quad \mu_2 = \frac{\mu_1 x_1}{\tau} = \frac{x_e}{2F\tau^2} x_1 = \frac{CD_2}{2}$$

We can re-write the third equation as:

$$(n+1)\mu_{n+1}\tau + (n-1)\mu_{n-1}x_{n-1} = n\mu_n x_n + n\mu_n \tau$$

Evaluated at $n = 2$, this can be shown to give $CD_3 = \frac{x_e}{F\tau} \left(\frac{2x_1 x_2}{\tau^2} \right)$, and at $n = 3$ we get that $CD_4 = \frac{x_e}{F\tau} \left(\frac{6x_1 x_2 x_3}{\tau^3} \right)$. Therefore, $C = \frac{x_e}{F\tau}$ and $D_n = \frac{(n-1)! \prod_{i=1}^{n-1} x_i}{\tau^{n-1}}$, and by induction we get

$$\mu_n = \frac{x_e}{F\tau} \frac{\prod_{i=1}^{n-1} x_i}{n\tau^{n-1}}; \quad \forall n \geq 1$$

our desired result. $\square$

Proof of Proposition 3 We find Γ and $\{\Upsilon_n\}_{n=1}^{+\infty}$ using the method of undetermined coefficients. Plugging the guess $V_n(\mathbf{q}_f) = \Gamma \sum_{q_j \in \mathbf{q}_f} q_j + \Upsilon_n \bar{Q}$ into (18), we get that

$$\begin{aligned} r\Gamma \sum_{q_j \in \mathbf{q}_f} q_j + r\Upsilon_n \bar{Q} &= \max_{\substack{x \in [0, \bar{x}] \\ \{z_j \in [0, \bar{z}]\}_{\mathcal{J}_f}}} \left\{ \sum_{q_j \in \mathbf{q}_f} \left[z_j \Gamma \lambda^I q_j + \pi q_j + \tau \left((\Upsilon_{n-1} - \Upsilon_n) \bar{Q} - \Gamma q_j \right) - \widehat{\chi} z_j^{\widehat{\psi}} q_j \right] \right. \\ &\quad \left. + nx \left(\Gamma \bar{Q} (1 + \lambda^E) + (\Upsilon_{n+1} - \Upsilon_n) \bar{Q} \right) - \widetilde{\chi} n^{\sigma + \widetilde{\psi}} x^{\widetilde{\psi}} \bar{Q} + \gamma \bar{Q} n^{\frac{n}{\alpha}} \right\} + \Upsilon_n \bar{Q} g \end{aligned}$$

Equating the terms with q_j and $\bar{Q}$ gives the following pair of optimization problems:

$$\begin{aligned} (q_j) : \quad r\Gamma &= \max_{z_j} \left\{ \pi + z_j \Gamma \lambda^I - \tau \Gamma - \widehat{\chi} z_j^{\widehat{\psi}} \right\} \\ (\bar{Q}) : \quad (r - g)\Upsilon_n &= \max_{x_n} \left\{ (\Upsilon_{n-1} - \Upsilon_n) n\tau + nx_n \left(\Gamma(1 + \lambda^E) + \Upsilon_{n+1} - \Upsilon_n \right) - \widetilde{\chi} n^{\sigma + \widetilde{\psi}} x_n^{\widetilde{\psi}} + \gamma n^{\frac{n}{\alpha}} \right\} \end{aligned}$$

The first-order conditions are:

$$\begin{aligned} z_j &= \left(\frac{\Gamma \lambda^I}{\widehat{\psi} \widehat{\chi}} \right)^{\frac{1}{\widehat{\psi}-1}} \\ x_n &= n^{\frac{1-\sigma-\widetilde{\psi}}{\widetilde{\psi}-1}} \left(\frac{\Gamma(1 + \lambda^E) + \Upsilon_{n+1} - \Upsilon_n}{\widetilde{\psi} \widetilde{\chi}} \right)^{\frac{1}{\widetilde{\psi}-1}} \end{aligned}$$

respectively. Assuming that there is positive entry in equilibrium ($x_e > 0$), we can exploit the

free-entry condition $V_0 = 0$ in (19) to get that

$$\Gamma = \frac{\nu - \Upsilon_1}{1 + \lambda^E}$$

This means that the optimal internal R&D investment by incumbents is

$$z_j = \left(\frac{\lambda^I(\nu - \Upsilon_1)}{\widehat{\psi}\widehat{\chi}(1 + \lambda^E)} \right)^{\frac{1}{\widehat{\psi}-1}}$$

so $z_j = z, \forall j \in [0, 1]$. Back into the optimality condition for z , we can obtain the implied rate of creative destruction:

$$\tau = \frac{1}{\nu - \Upsilon_1} \left[\pi - \widehat{\chi} \left(\frac{\lambda^I(\nu - \Upsilon_1)}{\widehat{\chi}\widehat{\psi}(1 + \lambda^E)} \right)^{\frac{\widehat{\psi}}{\widehat{\psi}-1}} \right] - \frac{\rho}{1 + \lambda^E}$$

where we have used that $g = r - \rho$ from the Euler equation, and $g = \tau\lambda^E + z\lambda^I$. It remains to find an expression for Υ_n . Using free-entry, we know

$$x_n = n^{\frac{1-\sigma-\widetilde{\psi}}{\widetilde{\psi}-1}} \left(\frac{\nu - \Upsilon_1 + \Upsilon_{n+1} - \Upsilon_n}{\widetilde{\psi}\widetilde{\chi}} \right)^{\frac{1}{\widetilde{\psi}-1}}$$

Back into the second maximization problem, we get the second-order difference equation

$$(r - g)\Upsilon_n = (\Upsilon_{n-1} - \Upsilon_n)n\tau + \gamma n^{\eta/\alpha} + \widetilde{\chi}(\widetilde{\psi} - 1)n^{-\frac{\sigma}{\widetilde{\psi}-1}} \left[\frac{\nu - \Upsilon_1 + \Upsilon_{n+1} - \Upsilon_n}{\widetilde{\psi}\widetilde{\chi}} \right]^{\frac{\widetilde{\psi}}{\widetilde{\psi}-1}}$$

Solving for Υ_{n+1} gives (24). Using $V_0 = 0$ and the guess $V_n = \Gamma \sum_j q_j + \Upsilon_n \bar{Q}$, it is clear that the boundary condition for this difference equation must then be $\Upsilon_0 = 0$. $\square$