Speculative Trade under Ambiguity *

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Abstract: Ambiguous beliefs may lead to speculative trade and speculative bubbles. We demonstrate this by showing that the classical Harrison and Kreps (1978) example of speculative trade among agents with heterogeneous beliefs can be replicated with agents having common but ambiguous beliefs. More precisely, we show that the same asset prices and pattern of trade can be obtained in equilibrium with agents' having recursive multiple-prior expected utilities with common set of probabilities.

While learning about the true distribution of asset dividends makes speculative bubbles vanish in the long run under heterogeneous beliefs, it may not do so under common ambiguous beliefs. Ambiguity need not disappear with learning over time, and speculative bubbles may persist forever.

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1. Introduction.

Ambiguous beliefs may lead to speculative trade and speculative bubbles. We demonstrate this by showing that the classical Harrison and Kreps (1978) example of speculative trade among agents with heterogeneous beliefs can be replicated with agents having common ambiguous beliefs. More precisely, we show that the same asset prices and speculative pattern of trade can be obtained in equilibrium with agents' having multiple-prior expected utilities with common set of probabilities.

The key question of Harrison and Kreps (1978) was whether equilibrium prices in asset markets can persistently exceed all agents' valuations of the asset where valuation is defined by what an agent would be willing to pay if obliged to hold the asset forever. If price exceeds all valuations, then agents who buy the asset must intend to sell it in the future. Agents trade for short-term gain and hence engage in speculation.

Harrison and Kreps considered a model of infinite-time asset markets where risk-neutral agents have heterogeneous beliefs about asset dividends and short selling is prohibited. Agents' beliefs exhibit perpetual switching (see Morris (1996)): there is no single agent who is more optimistic at all future dates and states than other agents about next period dividends of the asset. In equilibrium, the agent who has the most optimistic belief buys the asset and agents with less optimistic beliefs sell the asset if they have some holdings from previous date. As beliefs keep switching, persistent speculative trade emerges.

Equilibrium asset price p_t at date t in the Harrison and Kreps (1978) model satisfies the relationship

$$p_t = \max_i \beta E_t^i[p_{t+1} + x_{t+1}], \quad (1)$$

where x_{t+1} denotes dividend at date $t+1$, E_t^i stand for date- t conditional expectation under agent's i belief, and β is a discount factor. The maximizing belief in (1) is the belief of the agent who is most optimistic about next period dividend and holds the asset. Less optimistic agents have zero holdings and their beliefs give expected value of next-period price plus dividend lower than price p_t . Because of risk-neutrality, agents' valuations of the asset are simply sums of discounted expected future dividends under their beliefs, that is $\sum_{\tau > t} \beta^{\tau-t} E_t^i[x_\tau]$. It follows

that asset price p_t exceeds every agent's valuation at every date and state. The difference between the asset price and the maximal valuation is speculative bubble.

A model of asset markets in continuous time exhibiting the same features as Harrison and Kreps (1978) has been studied by Scheinkman and Xiong (2003). Heterogeneous beliefs in their model are generated by agents' overconfidence in informativeness of signals about future dividends. As signals arrive over time, agents' beliefs about future dividends exhibit perpetual switching. This gives rise to speculative trade and speculative bubbles.

We consider the same model of asset markets as Harrison and Kreps (1978) except for the specification of agents' beliefs. Instead of heterogeneous but exact beliefs, agents in our model have common ambiguous beliefs described by sets of one-period-ahead probabilities. Their decision criterion is the recursive multiple-prior expected utility - an extension of Gilboa and Schmeidler (1989) maxmin criterion to dynamic setting due to Epstein and Schneider (2003). Our key observation is that equilibrium pricing relationship (1) continues to hold with expectation E_t^i of agent i being now the one-period-ahead belief that minimizes expected value of that agent's date- $(t + 1)$ continuation utility over the set of multiple probabilities. We call those probabilities effective one-period-ahead beliefs, and they feature in the valuation of agents' willingness to pay for the asset if obliged to hold it forever. If there is sufficient heterogeneity of agents' equilibrium consumption plans, then effective beliefs have the switching property and - as in Harrison and Kreps (1978) - equilibrium prices exceed all agents' asset valuations. Further, there is speculative trade. Heterogeneity of equilibrium consumption is generated by heterogeneous endowments. Initial endowments play a critical role under common ambiguous beliefs, in contrast to the case of heterogeneous beliefs where they do not matter.

Two objections have been frequently raised against Harrison and Kreps' model of speculative trade. The first is that it departs from the common-prior assumption. This objection does not apply to our model. Agents in our model have common priors, or more precisely, common set of priors. The second is that agents have dogmatic beliefs and do not learn from observations of realized dividends over time. In response to this objection, Morris (1996) introduced learning in the model of speculative trade. He considered an i.i.d dividend process parametrized by a single parameter of its distribution (probability of high dividend) that is unknown

to the agents. Agents have heterogeneous prior beliefs about that parameter. Morris (1996) showed that, as the agents update their beliefs over time, their posterior beliefs will exhibit switching property that leads to speculative trade. It is a standard result in Bayesian learning (Blackwell and Dubins' (1962) merging of opinions) that agents posterior beliefs converge to the true parameter of the dividend distribution. This implies that asset prices converge to agents' valuations as time goes to infinity.

A slightly different model of speculative trade under heterogeneous beliefs with learning has been studied in Slawski (2008). In his model the dividend process is a Markov chain. Agents don't know the true transition matrix and have heterogeneous priors on a set of possible transition matrices. Slawski (2008) provides conditions under which there is speculative trade in equilibrium when agents update their beliefs. Under fairly general conditions, posterior beliefs converge to true parameter as time goes to infinity, and asset prices converge to agents' valuations.

Learning and updating of beliefs can be significantly different under ambiguity than with no ambiguity.¹ Depending on the interaction between random experiments that generate dividends ambiguity about dividends may or may not fade away in the long run. The dividend process in our version of Harrison and Kreps' model can be thought of as resulting from sequences of indistinguishable but unrelated experiments, see Epstein and Schneider (2003b). Such experiments give rise to persistent ambiguity that is unaffected by learning. Morris' (1996) model of speculative trade under heterogeneous beliefs with learning can be replicated with common ambiguous beliefs, too. Agents have common set of prior beliefs about a parameter of probability distribution of an i.i.d process of dividends. They update their beliefs prior-by-prior in Bayesian way. If agents' consumption plans exhibit sufficient heterogeneity, effective posterior beliefs have the switching property and there results speculative trade. The critical condition is heterogeneity of agents' initial endowments. In this case, ambiguity about dividends fades away in the long run because posterior beliefs converge to true probability. As in Morris (1996) asset prices converge to valuations and the speculative bubble converges to zero.

In their comprehensive study of the dot.com bubble of 2000-2001 Ofek and

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¹Epstein and Schneider (2007) propose a model of learning with multiple priors that may leave some ambiguity remaining in the long run.

Richardson (2003) concluded that short-sales restrictions and heterogeneity of investors' beliefs were the main reasons for the dramatic rise and fall of prices of internet stocks during that period. Short sales restrictions on internet stocks were particularly stringent because of the so-called lockups. Their main argument in support of belief heterogeneity was relatively low level of institutional holdings of internet stocks. Individual investors tend to have more diverse beliefs. Yet, it is hard to believe that investors could have so diverse beliefs over a relatively long period of time. The argument of merging of opinions implies that diverse beliefs should quickly disappear. Our findings offer a different interpretation of Ofek and Richardson's analysis. Instead of being diverse, investors beliefs could have been ambiguous but common. Majority of dot.com stocks were new to the market justifying potential ambiguity of investors beliefs. Ambiguity of beliefs could persist over a long period of time.

The results of this paper are in stark contrast to the existing literature on implications of ambiguous beliefs (or ambiguity aversion) on equilibrium in asset markets. Inspired by the portfolio-inertia result of Dow and Werlang (1992), the literature has strived to demonstrate that non-participation in trade by some agents with ambiguous beliefs may arise in equilibrium. In Cao, Wang and Zhang (2005) agents have heterogeneous ambiguity and those with the highest degree of ambiguity opt out of trading risky assets in equilibrium.² Mukerji and Tallon (2001) show that ambiguous beliefs concerning idiosyncratic risk may lead to break down of trade of some assets.

The paper is organized as follows. In Section 2 we review the model of Harrison and Kreps (1978) of speculative trade under heterogeneous beliefs. In Section 3 we show how the same asset prices and asset holdings can be obtained in equilibrium with agents having recursive multiple-prior expected utilities with common set of probabilities. In Section 4 we extend the model and identify the property of switching beliefs that is shown to give rise to speculative bubbles. A model of speculative trade with learning under ambiguity is presented in Section 5.

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²Further studies of limited participation in trade under ambiguity are Easley and O'Hara (2009), Illeditsch (2011), and Ozsoylev and Werner (2011).

2. Speculation under Heterogeneous Beliefs.

The following example is due to Harrison and Kreps (1978). There is a single infinitely-lived asset with uncertain dividend equal to either 0 (low dividend) or 1 (high dividend) at every date $t \geq 1$. Date-0 dividend is equal to zero. Let $S = \{0, 1\}$ be the set of states for every date. There are two agents who perceive the dividend process $\{x_t\}$ as Markov chain with different transition probabilities, that is, they have *heterogeneous beliefs*. Transition probabilities are specified by probabilities of next-period high dividend conditional on current state, $q^i(s)$, with the respective probability of low dividend being $1 - q^i(s)$. Suppose that

$$q^1(0) = \frac{1}{2} \quad q^1(1) = \frac{1}{3}, \quad (2)$$

for agent 1, and

$$q^2(0) = \frac{1}{3}, \quad q^2(1) = \frac{3}{4} \quad (3)$$

for agent 2. The key feature of transition probabilities (2 - 3) is the property of switching beliefs: Agent 1 is more optimistic than agent 2 about next-period high dividend when current dividend is zero, while it is the other way round when current dividend is one.

Agents are risk-neutral with utility functions over infinite-time consumption plans given by the discounted expected value

$$u^i(c) = E^i\left[\sum_{t=0}^{\infty} \beta^t c_t\right], \quad (4)$$

where E^i denotes expectation under the unique probability measure on S^∞ derived from transition probabilities q^i . The common discount factor is $\beta = 0.75$. Consumption endowments don't matter and are left unspecified. The asset supply is normalized to one share which is initially held by agent 1. Short selling of the asset is prohibited in that there is zero short-sales constraint.

In equilibrium, the agent who is more optimistic at any date and state holds the asset and the price reflects his one-period valuation of the payoff. The less optimistic agent wants to sell the asset short and ends up with zero holding because of the short-sales constraint. There exists a stationary equilibrium with asset prices

that depend only on the current dividend. Equilibrium prices, denoted by $p(0)$ and $p(1)$, obtain from the first-order conditions for the respective optimistic agent,

$$p(0) = \beta[(1 - q^1(0))p(0) + q^1(0)(p(1) + 1)] \quad (5)$$

$$p(1) = \beta[(1 - q^2(1))p(0) + q^2(1)(p(1) + 1)] \quad (6)$$

They are

$$p(0) = \frac{24}{13}, \quad p(1) = \frac{27}{13}. \quad (7)$$

Security holdings are

$$h^1(0) = 1, \quad h^1(1) = 0, \quad h^2(0) = 0, \quad h^2(1) = 1. \quad (8)$$

The first-order conditions for agent 2 when the dividend is zero and for agent 1 when the dividend is one hold as strict inequalities. Transversality conditions hold, too.

The discounted expected value of the asset's future dividends at date t under agent's i beliefs is

$$V_t^i(s) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E^i[x_\tau | x_t = s] \quad (9)$$

for $s = 0, 1$. The value $V_t^i(s)$ does not depend on t and we drop subscript t from the notation. Because of linearity of utility functions, discounted expected value of dividends is the agent's willingness to pay for the asset if obliged to hold forever. Therefore $V^i(s)$ may be called the *fundamental value*.³ If asset price $p(s)$ strictly exceeds every agent's fundamental value, then there is *speculative bubble*. An agent who buys the asset at price $p(s)$ must be planning to sell it at a future date. Thus, the agent engages in speculative trade. The difference between the price and the maximum of fundamental values is termed *speculative premium*.

Elementary algebra shows that

$$V^1(0) = \frac{4}{3}, \quad V^1(1) = \frac{11}{9}, \quad (10)$$

$$V^2(0) = \frac{16}{11}, \quad V^2(1) = \frac{21}{11}. \quad (11)$$

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³Needless to say, this is a different notion of fundamental value than the one used in the literature on rational price bubbles, see for example LeRoy and Werner (2014).

It holds

$$p(0) > V^i(0) \text{ and } p(1) > V^i(1)$$

for $i = 1, 2$. There is speculative bubble in every state at every date.

3. Speculation under Ambiguous Beliefs.

Consider the same asset as in Section 2 with dividends equal to 0 or 1 at every date, and no short selling. The specification of agents' endowment is important here. Suppose that the endowment e_t^1 of agent 1 can take two possible values 10 and 5 for every date $t \geq 1$. Agent's 2 endowment is $e_t^2 = 15 - e_t^1$. The joint process (x_t, e_t^1) takes four possible values $(0, 5), (0, 10), (1, 5), (1, 10)$ for every date $t \geq 1$. Those four values are the set of states S . Date-0 state is $(0, 10)$.

Agents have common ambiguous beliefs about the dividend-endowment process that are described by sets of transition (or one-period-ahead) probabilities on S . These sets depend only on current dividend and are constructed using the probabilities from Section 2 in the following way. If the current dividend is zero, the set of transition probabilities is the convex hull of two probability vectors, $((1 - q^1(0)), 0, q^1(0), 0)$ and $(0, (1 - q^2(0)), 0, q^2(0))$, where $q^1(0)$ and $q^2(0)$ are as in eqs. (2 - 3). Thus

$$\mathcal{P}(0) = co \left\{ \left(\frac{1}{2}, 0, \frac{1}{2}, 0 \right), \left(0, \frac{2}{3}, 0, \frac{1}{3} \right) \right\}. \quad (12)$$

Equivalently, $\mathcal{P}(0)$ can be described as the set of probability vectors $(1/2\lambda, 2/3(1 - \lambda), 1/2\lambda, 1/3(1 - \lambda))$ over all $\lambda \in [0, 1]$. Note that under the specification (12) the upper probability of high dividend next period is $1/2$ or $q^1(0)$ while the lower probability is $1/3$ or $q^2(0)$.⁴ Upper and lower probabilities of high endowment (equal to 10) next period are 1 and 0, respectively. Thus there is maximal ambiguity about high or low endowment. For an example of an experiment with two Ellsberg urns that leads to a set of probabilities of the form (12), see Couso et al (1999), Example 5.

If the current dividend is one, the set of transition probabilities is

$$\mathcal{P}(1) = co \left\{ (2/3, 0, 1/3, 0), (0, 1/4, 0, 3/4) \right\}. \quad (13)$$

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⁴Upper probability of an event $A \subset \Omega$ for a set of probability measures $\mathcal{P}$ on Ω is defined as $\max_{P \in \mathcal{P}} P(A)$. Lower probability of A is the minimum over the same set.

Here, the upper probability of high dividend next period is $3/4$ or $q^2(1)$ while the lower probability is $1/3$ or $q^1(1)$.

We proceed now to describe agents' recursive multiple-prior expected utilities with sets of transitions probabilities $P(0)$ and $P(1)$. Consumption plans are bounded and adapted processes on the state-space S^∞ of infinite histories of dividends and (agent's 1) endowments equipped the usual information structure of finite-time histories. Date- t history of dividends and endowments is denoted by s_t and referred to as date- t event. A typical consumption plan is denoted by $c = \{c_t\}_{t=0}^\infty$. The common recursive multiple-prior expected utility, with linear period utility, is defined by

$$u_t(c, s_t) = c_t(s_t) + \beta \min_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c)|s_t], \quad (14)$$

where the set of transition probabilities $\mathcal{P}(s_t)$ is either $\mathcal{P}(0)$ or $\mathcal{P}(1)$ depending on the current dividend in event s_t . The discount factor is $\beta = 0.75$. Date-0 utility function implied by the recursive relation (14) is

$$u_0(c) = \min_{\pi \in \Pi} E_\pi \left[\sum_{t=0}^{\infty} \beta^t c_t \right], \quad (15)$$

where Π is a set of probabilities on S^∞ such that conditional one-period-ahead probabilities at any date t are $\mathcal{P}(s_t)$, see Epstein and Schneider (2003).

An equilibrium in this economy is characterized as follows. Equilibrium asset prices p satisfy the relation

$$p_t(s_t) = \beta \max_i E_{P^i(s_t)} [p_{t+1} + x_{t+1} | s_t] \quad (16)$$

for every s_t , where probability vector $P^i(s_t)$ minimizes the expected value of next-period continuation utility of equilibrium consumption plan c^i , that is,

$$P^i(s_t) \in \operatorname{argmin}_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c^i)|s_t], \quad (17)$$

for $i = 1, 2$. Beliefs $P^i(s_t)$ satisfying (17) are called *effective beliefs* at c^i . The agent whose effective belief is the maximizing one on the right-hand side of (16) holds the asset in s_t while the other agent whose belief gives lower expectation has zero holding (see Appendix).

We claim that asset prices (7) and asset holdings (8) derived in Section 2 together with the implied consumption plans are an equilibrium. Consider a date- t event with current dividend equal to zero. The transition probabilities in $\mathcal{P}(0)$ that minimize the expected value of agent's 1 next period endowment are $(\frac{1}{2}, 0, \frac{1}{2}, 0)$. Calculations in the Appendix show that the same probability vector minimizes the expected value of agent's 1 next period continuation utility of consumption plan c^1 . Thus, $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ is the effective belief of agent 1 if current dividend is zero. Similarly, the transition probabilities that minimize the expected value of agent's 2 next period continuation utility of c^2 over the set $\mathcal{P}(0)$ is the same as the probability vector minimizing the expected value of her endowment. That probability vector is $(0, \frac{2}{3}, 0, \frac{1}{3})$ and is agent's 2 effective belief. It follows that asset prices (7) satisfy relation (16) which in turn implies that asset holdings (8) are optimal if current dividend is zero.

Consider next a date- t state with current dividend equal to one. Calculation in the Appendix show that agent's 1 effective belief in that state at c^1 is $(\frac{2}{3}, 0, \frac{1}{3}, 0)$ while agent's 2 effective belief at c^2 is $(0, \frac{1}{4}, 0, \frac{3}{4})$. Again, those effective belief are the same the effective beliefs at their respective endowments. Asset prices (7) satisfy relation (16) if current dividend is one. Consequently, asset holdings (8) are optimal as well. The equilibrium prices $p(0)$ and $p(1)$ are equal to the maximum discounted expected one-period payoff over all probabilities in the respective set of transition probabilities $\mathcal{P}(0)$ or $\mathcal{P}(1)$.

An agent's willingness to pay for the asset if obliged to hold forever is the discounted expected value of future dividends under probabilities that minimize the expected value of the agent's next period continuation utility. Therefore, fundamental values of the asset remain the same $V^i(0)$ and $V^i(1)$ for $i = 1, 2$ as in eq. (11). Equilibrium prices $p(0)$ and $p(1)$ strictly exceed both agents' fundamental valuations and there is speculative bubble. Speculative premium is time independent.

Figure 1 in the Appendix provides graphical intuition (albeit with two states) for the type of equilibrium we obtained. It makes clear that heterogeneity of agents endowments, which appears here in a strong form of negative comonotonicity between e_t^1 and e_t^2 , is crucial for speculative trade under ambiguity.

4. Belief Switching and Speculation.

In this section we extend the results of Section 3 to more general specification of transition probabilities satisfying a condition of perpetual switching. Let the set of transition probabilities $\mathcal{P}_t(s_t)$ in event s_t be a convex hull of two probability vectors, $Q_t^1(s_t) = ((1 - q_t^1(s_t)), 0, q_t^1(s_t), 0)$ and $Q_t^2(s_t) = (0, (1 - q_t^2(s_t)), 0, q_t^2(s_t))$, which depend on the history of dividends but not on agents' endowments. We have

$$\mathcal{P}_t(s_t) = co \{Q_t^1(s_t), Q_t^2(s_t)\}. \quad (18)$$

Equivalently, $\mathcal{P}_t(s_t)$ can be described as $(\lambda(1 - q_t^1(s_t)), (1 - \lambda)(1 - q_t^2(s_t)), \lambda q_t^1(s_t), (1 - \lambda)q_t^2(s_t))$ for all $\lambda \in [0, 1]$. Note that the upper probability of high dividend next period is $\max\{q_t^1(s_t), q_t^2(s_t)\}$ while the lower probability is $\min\{q_t^1(s_t), q_t^2(s_t)\}$. The upper and lower probabilities of high endowment next period are 1 and 0, respectively. Thus there is maximal ambiguity about high or low endowment.⁵ Agents have recursive multiple-prior expected utilities with linear period utility specified by (14) with sets of one-period-ahead probabilities given by (18).

Consider an equilibrium consisting of asset prices p , consumption allocation (c^1, c^2) , and asset holdings (h^1, h^2) . Let $P_t^1(s_t) \in \mathcal{P}_t(s_t)$ and $P_t^2(s_t) \in \mathcal{P}_t(s_t)$ be agents' effective beliefs as in (17) at equilibrium consumption plans and such that the pricing relation (16) holds. The agent whose effective belief is the maximizing one on the right-hand side of (16) holds the asset in event s_t while the other agent whose belief gives lower expectation has zero holding. Let $\hat{P}_t(s_t)$ denote the maximizing probability in (16). Further, let π^i for $i = 1, 2$ and $\hat{\pi}$ be probability measures on S^∞ derived from one-period-ahead probabilities P_t^i and $\hat{P}_t$, respectively. Note that $\hat{\pi}$ is the risk-neutral pricing measure (or state-price process) for p . It follows from Theorem 3.3 of Santos and Woodford (1997) that equilibrium price of the asset is equal to the infinite sum of discounted expected dividends under the risk-neutral measure. That is,

$$p_t(s_t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\hat{\pi}}[x_\tau | s_t], \quad (19)$$

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⁵This extreme ambiguity is not essential for our results, but the presence of some ambiguity about endowments is essential.

for every s_t . The fundamental value of the asset is the sum of discounted expected dividends under the agent's effective beliefs. That is,

$$V_t^i(s_t) = \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} E_{\pi^i}[x_\tau | s_t], \quad (20)$$

for every s_t . It follows from eq. (16) that $p_t(s_t) \geq V_t^i(s_t)$ for every i .

Following Morris (1996), we say that transition probabilities $\{q_t^1(s_t)\}$ and $\{q_t^2(s_t)\}$ exhibit *perpetual switching*, if for every $i = 1, 2$ and event s_t there exists $\tau > t$ and event s_τ which is a successor of s_t such that $q_t^i(s_\tau) < \hat{q}_t(s_\tau)$, where $\hat{q}_t(s_t) = \max\{q_t^1(s_t), q_t^2(s_t)\}$.

We have the following:

Theorem 1: *Suppose that probabilities $q_t^i(s_t)$ depend only on the number of high dividends from date 1 through t , for $i = 1, 2$, and that β is relatively small. If $\{q_t^1(s_t)\}$ and $\{q_t^2(s_t)\}$ exhibit perpetual switching, then in equilibrium*

$$p_t(s_t) > V_t^i(s_t), \quad (21)$$

for $i = 1, 2$ and for all s_t .

PROOF: Consider asset prices that depend only on history of dividends and are defined by the recursive relation

$$p_t(s_t) = \beta[(1 - \hat{q}_t(s_t))p_{t+1}(s_t, 0) + \hat{q}_t(s_t)(p_{t+1}(s_t, 1) + 1)], \quad (22)$$

where, with slight abuse of notation, $(s_t, 0)$ and $(s_t, 1)$ are the two continuation histories of dividends at date $t+1$. We claim that so defined prices are equilibrium prices. First, we show that the effective belief of agent 1 at s_t is $Q_t^1(s_t)$. That is, $P_t^1(s_t) = Q_t^1(s_t)$. Similarly, $P_t^2(s_t) = Q_t^2(s_t)$. The proof of this step is in the Appendix. Second, we show that relation (16) holds for prices given by (22) with the maximizing belief being the effective belief of the agent who assigns higher probability of next period high dividend, that is, $\hat{P}_t(s_t) = \hat{Q}_t(s_t)$. This implies that holding one share of the asset is optimal for that agent while zero holding is optimal for the other agent guaranteeing that markets clear. The proof that $\hat{P}_t(s_t) = \hat{Q}_t(s_t)$ is the Appendix. Lastly, we show that if there is perpetual switching, then (21)

holds so that the speculative bubble is strictly positive. The argument for this last step is as follows. Eq. (16) implies that

$$p_t(s_t) \geq \sum_{\tau=t+1}^T \beta^{\tau-t} E_{\pi^i}[x_\tau|s_t] + \beta^{T-t} E_{\pi^i}[p_T|s_t], \quad (23)$$

for every i and every s_t and $T > t$. The assumption of perpetual switching implies that there is s_τ , a successor of s_t , such that $\hat{q}_\tau(s_\tau) > q_\tau^i(s_\tau)$. Then $\hat{P}_\tau(s_\tau) \neq Q_\tau^i(s_\tau)$, and for every $T \geq \tau$ inequality in (23) is strict. Since the right-hand side of (23) is non-increasing in T and it converges to $V_t^i(s_t)$ as T goes to infinity, we obtain (21). $\square$

Theorem 1 is a counterpart of Theorem 2 in Morris (1996) for heterogeneous beliefs (see also Proposition 2.8 in Slawski (2008)). Harrison and Kreps (1978) (see also Slawski (2008)) provide a method of calculating solutions to recursive pricing equation (22). Similar recursive relations for fundamental values of the asset are

$$V_t^i(s_t) = \beta[(1 - q_t^i(s_t))V_{t+1}^i(s_t, 0) + q_t^i(s_t)(V_{t+1}^i(s_t, 1) + 1)]. \quad (24)$$

5. Speculation and Learning.

The setting of repeated trade of an asset where agents observe dividends and prices period after period offers ample opportunities to learn the true distribution of dividends. Yet, agents in the Harrison and Kreps' model never update their beliefs even though at least one of these beliefs is wrong. The reason is that agents have dogmatic beliefs - that should be viewed as represented by Dirac measures on the space of parameters of transition probabilities - leaving no room for updating.

Morris (1996) considered a variation of the Harrison and Kreps' model of speculative trade where agents with heterogeneous prior beliefs learn from realized dividends. The dividend process is taken to be an i.i.d process with the probability of high dividend unknown to agents. Agents have different priors about that probability, with full support on $[0, 1]$. Morris (1996) showed that, as the agents update beliefs over time, their posterior beliefs exhibit perpetual switching for a large class of heterogeneous priors. Perpetual switching of posterior beliefs leads to speculative trade. Thus learning generates speculative trade. It is a standard result

in Bayesian learning (see Blackwell and Dubins (1962)) that conditional posterior beliefs converge to each other over time. This implies that speculative premium under learning converges to zero over time. In contrast, speculative premium in the Harrison and Kreps' example is time independent.

Learning and updating of beliefs can be significantly different under ambiguity than with no ambiguity. For example, ambiguity may or may not fade away in the long run in repeated experiments depending on the interaction between those experiments. Epstein and Schneider (2007) propose a general model of learning with multiple priors that may leave some ambiguity remaining in the long run (see also Marinacci (2002)). The dividend-endowment process in our version of Harrison and Kreps' model has sets of transition probabilities that do not change over time. Time-invariant sets may arise in repeated experiments that are indistinguishable but have unknown relationship as in the I.I.D. process of Epstein and Schneider (2003b).

The model of Morris (1996) of speculative trade under heterogeneous beliefs with learning can be replicated with common ambiguous beliefs. Suppose that the dividend process x_t is i.i.d taking values 0 and 1. The true probability of high dividend $\pi \in [0, 1]$ is not known to the agents. They have prior beliefs about π . Prior beliefs are probability measures on $[0, 1]$, that is, elements of the set of measures $\mathcal{M}([0, 1])$. Because of ambiguity, agents do not have unique prior but instead they have multiple prior beliefs. More specific, there are two priors $\mu^1 \in \mathcal{M}([0, 1])$ and $\mu^2 \in \mathcal{M}([0, 1])$ assumed to have density functions (twice differentiable and bounded below) on $[0, 1]$. The endowment process e_t^1 taking values 5 and 10 is an I.I.D. process (see Epstein and Schneider (2003b)) with time- and state-independent set of one-period-ahead probabilities $\mathcal{P} = co\{(0, 1), (1, 0)\}$, that is, $\mathcal{P} = \Delta^2$.

Agents update their prior beliefs about the dividend process using observations of realized dividends. Let $\theta_t^i(s_t)$ be the posterior probability of next-period high dividend conditional on observed history of dividends in event s_t at date t derived from prior μ^i . No updating of beliefs about endowment process takes place since the underlying random experiments at different dates are considered unrelated but indistinguishable. The interaction between processes x_t and e_t^1 is characterized by independence in the selection, see Couso et al (1999). That is, the set of one-

period-ahead probabilities for the joint process (x_t, e_t^1) at date t is

$$\mathcal{P}_t(s_t) = co \{(1 - \theta_t^1(s_t), 0, \theta_t^1(s_t), 0), (0, 1 - \theta_t^2(s_t), 0, \theta_t^2(s_t))\}. \quad (25)$$

Agents have recursive multiple-prior expected utilities (14) with sets of one-period-ahead probabilities given by (25). Morris (1996) shows that under fairly general conditions sequences of posterior probabilities $\{\theta_t^1(s_t)\}$ and $\{\theta_t^2(s_t)\}$ exhibit perpetual switching, that is, for every $i = 1, 2$ and event s_t there exists $\tau > t$ and event s_τ which is a successor of s_t such that $\theta_\tau^i(s_\tau) < \hat{\theta}_\tau(s_\tau)$, where $\hat{\theta}_t(s_t) = \max\{\theta_t^1(s_t), \theta_t^2(s_t)\}$. For example, if one prior is the uniform (or “ignorance”) prior while the second is Jeffreys’ prior, then there is perpetual switching (see Example 1 in Morris (1996)). It follows from Theorem 1 that there is strictly positive speculative premium and speculative trade in equilibrium.

Because prior beliefs about dividends are absolutely continuous with respect to each other, posterior probabilities of high dividend converge to the true probability π . This implies that speculative premium converges to zero as time goes to infinity. In contrast, the marginal probability of high endowment of agent 1 lies between 0 and 1 at every date. Ambiguity about endowments does not fade away. Endowment process has unknown relationship between subsequent experiments and hence is an I.I.D. process with persistent ambiguity. Needless to say, this persistent ambiguity about endowments has no effect on speculative premium in this model.

6. Appendix.

6.1 Portfolio choice under ambiguity with risky endowments.

We first explain the features of optimal portfolio choice under ambiguity that the results of Section 3 rely on in a two-period model.

Consider an agent whose preferences over date-1 state-dependent consumption plans are described by multiple-prior expected utility with the set of probabilities $\mathcal{P}$ and linear utility function. Date-1 endowment $\tilde{e}$ is risky. There is a single asset with date-1 payoff $\tilde{x}$ and date-0 price p . At first, we assume that short-sales are unrestricted.

The investment problem is

$$\begin{aligned} \max_{c_0, h} & \left[c_0 + \beta \min_{P \in \mathcal{P}} E_P(\tilde{e} + \tilde{x}h) \right], \\ \text{subject to} & \quad c_0 + ph = w_0, \end{aligned} \tag{26}$$

where w_0 is date-0 wealth.

For any date-1 consumption plan $\tilde{c}$ we denote the set of minimizing probabilities at $\tilde{c}$ by $\mathcal{P}(\tilde{c}) = \operatorname{argmin}_{P \in \mathcal{P}} E_P[\tilde{c}]$. A necessary and sufficient condition for h^* to be a solution to (26) is that

$$\min_{P \in \mathcal{P}(c^*)} \beta E_P[\tilde{x}] \leq p \leq \max_{P \in \mathcal{P}(c^*)} \beta E_P[\tilde{x}] \tag{27}$$

where $c^* = \tilde{e} + \tilde{x}h^*$. Note that (27) can be equivalently written as $p = \beta E_P[\tilde{x}]$ for some $P \in \mathcal{P}(c^*)$. The left-hand and the right-hand sides of (27) are equal if $\mathcal{P}(c^*)$ is singleton which is exactly when the multiple-prior utility is differentiable at c^* .

The proof of (27) is a simple application of superdifferential calculus. We sketch the argument for completeness. Define function $g : R \rightarrow R$ by $g(h) = \min_{P \in \mathcal{P}} E_P[\beta \tilde{e} + (\beta \tilde{x} - p)h]$. Function g is concave. A necessary and sufficient condition for h^* to be a solution to (26) is that $0 \in \partial g(h^*)$, that is, 0 lies in the superdifferential of g at h^* . It holds $\partial g(h^*) = \{\phi \in R : \phi = \beta E_P[\tilde{x}] - p \text{ for some } P \in \mathcal{P}(c^*)\}$. We have $0 \in \partial g(h^*)$ if and only if (27).

We focus now on the possibility of the solution being $h^* = 0$. It follows from (27) that $h^* = 0$ is a solution to (26) if and only if

$$\min_{P \in \mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}] \leq p \leq \max_{P \in \mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}] \tag{28}$$

If (28) holds with strict inequalities, then $h^* = 0$ is the unique solution.

If there is zero short-sales constraint in the investment problem (26), then the optimal investment is $h^* = 0$ if and only if $p \geq \min_{P \in \mathcal{P}(\tilde{e})} \beta E_P[\tilde{x}]$. If the inequality is strict, then $h^* = 0$ is the unique optimal investment.

These observations extend the portfolio-inertia result of Dow and Werlang (1992). If date-1 endowment $\tilde{e}$ is risk-free, then $\mathcal{P}(\tilde{e}) = \mathcal{P}$ and it follows from (28) that the optimal investment is zero for all asset prices in the interval between the minimum and the maximum discounted expected payoff over all beliefs in $\mathcal{P}$.

Figure 1 illustrates the optimal investment under ambiguity with background risk and zero short-sales constraint. There are two states and consumption takes place only at date 1. The set of probabilities is $\mathcal{P} = \{(\pi, 1 - \pi) : 0.4 \leq \pi \leq 0.6\}$. The unique probability measure in $\mathcal{P}$ that minimizes the expected value of endowment $e_1 = (10, 5)$ is $\pi_1 = (0.4, 0.6)$. For asset price $p = E_{\pi_1}[\tilde{x}]$, holding one share of the asset - which results in consumption equal to $e_1 + (\tilde{x} - p)$ - is an optimal investments (as is zero holding). At this price p , the optimal investment for initial endowment $e_2 = (5, 10)$ and subject to the short-sales constraint is zero. This is so because $p > E_{\pi_2}[\tilde{x}]$ where $\pi_2 = (0.6, 0.4)$ is the probability minimizing the expected value of e_2 over $\mathcal{P}$.

In a two-agent economy where both agents have the set of probabilities $\mathcal{P}$ but one has endowment e_1 while the other has e_2 and there is unitary supply of the asset traded under zero short-sales constraint, the equilibrium price of the asset is p . The first agent holds the asset. Note that $p = \max_{P \in \mathcal{P}} E_P[\tilde{x}]$.

Dynamic portfolio choice under ambiguity.

Consider the recursive multiple-prior expected utility (14). That is

$$u_t(c, s_t) = c_t(s_t) + \beta \min_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c)|s_t]. \quad (29)$$

The portfolio choice problem can be described by the following Bellman equation:

$$V(h, s_t) = \max_{c, \tilde{h}} \left\{ c + \min_{P \in \mathcal{P}(s_t)} E_P[V(\tilde{h}, \cdot)|s_t] \right\} \quad (30)$$

$$\text{s.t } c + p(s_t)\tilde{h} \leq e(s_t) + [p(s_t) + x(s_t)]h, \quad h \geq 0. \quad (31)$$

If $c = \{c_t\}$ and $h = \{h_t\}$ are sequences of consumption and asset holdings satisfying the Bellman equation (31), then $V(h_{t-1}, s_t) = u_t(c, s_t)$. The first-order condition for optimal asset holding $h(s_t) > 0$ in (31) is

$$p_t(s_t) = \beta E_{P(s_t)}[(p_{t+1} + x_{t+1})|s_t], \quad (32)$$

for some probability measure $P(s_t) \in \mathcal{P}(s_t)$ such that

$$P(s_t) \in \operatorname{argmin}_{P \in \mathcal{P}(s_t)} E_P[u_{t+1}(c)|s_t]. \quad (33)$$

Probability measure $P(s_t)$ is the agent's effective belief at c is (see (17)). Condition (32) can be alternatively written as

$$\min_{P \in \mathcal{P}(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s_t] \leq p_t(s_t) \leq \max_{P \in \mathcal{P}(s_t)} \beta E_P[(p_{t+1} + x_{t+1})|s_t] \quad (34)$$

The respective condition for $h(s_t) = 0$ to be solution to (31) is

$$p_t(s_t) \geq \beta E_{P(s_t)}[(p_{t+1} + x_{t+1})|s_t] \quad (35)$$

for some $P(s_t) \in \mathcal{P}(s_t)$ such that (33) holds.

6.2 Proofs.

Example of Section 3: We prove that agents' effective beliefs at consumption plans c^i implied by asset holdings (8) and prices (7) are as claimed in Section 3. We have $c^i(s_t) = e^i(s_t) + x(s_t)h^i(s_{t-1}) + p(s_t)[h^i(s_{t-1}) - h^i(s_t)]$. According to (29), continuation utility equals current consumption plus discounted conditional expectation of next period continuation utility. If the dividend in s_t is zero, agent's 1 asset holding is 1 and next period continuation utilities in the four successor events are

$$(5 + \Delta^1(0), 10 + \Delta^1(0), 6 + p(1) + \Delta^1(1), 11 + p(1) + \Delta^1(1)) \quad (36)$$

where $\Delta^1(0)$ and $\Delta^1(1)$ denote discounted expected values of next period continuation utility which depend only on current dividend. If the dividend is one and asset holding is zero, next period continuation utilities are

$$(5 - p(0) + \Delta^1(0), 10 - p(0) + \Delta^1(0), 5 + \Delta^1(1), 10 + \Delta^1(1)). \quad (37)$$

The values of $\Delta^1(0)$ and $\Delta^1(1)$ can be calculated from two recursive equations

$$\Delta^1(0) = \beta \left[\frac{1}{2}(5 + \Delta^1(0)) + \frac{1}{2}(6 + p(1) + \Delta^1(1)) \right] \quad (38)$$

$$\Delta^1(1) = \beta \left[\frac{2}{3}(5 - p(0) + \Delta^1(0)) + \frac{1}{3}(\Delta^1(1) + 5) \right], \quad (39)$$

where we assumed (to be verified later) that effective beliefs at c^1 are $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ if dividend is zero and $(\frac{2}{3}, 0, \frac{1}{3}, 0)$ if dividend is one. For $\beta = \frac{3}{4}$, the solutions are $\Delta^1(0) = 16\frac{11}{13}$ and $\Delta^1(1) = 15$.

Because there is relatively small variability in Δ^1 and in net gains from asset trade across states, the probability in $\mathcal{P}(0)$ that minimizes the expected value of (36) is the same as for endowment $(5, 10, 5, 10)$, as can be easily verified. Thus $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ is indeed the effective belief if dividend is zero. Similarly, the probability in $\mathcal{P}(1)$ that minimizes the expected value of (37) is $(\frac{2}{3}, 0, \frac{1}{3}, 0)$. Calculations of agent's 2 effective beliefs are omitted as they are very similar.

Proof of Theorem 1: First, we prove that the effective belief of agent 1 in event s_t is $Q^1(s_t)$. The argument is similar to the example of Section 3. Probability vector $Q^1(s_t)$ minimizes the expected value of next-period endowment e_{t+1}^1 over all probabilities in $\mathcal{P}_t(s_t)$. As long as β is relatively small, there is small variability in Δ^1 and in net gains from asset trade across states, implying that the probability in $\mathcal{P}(s_t)$ that minimizes the expected value of next period continuation utility of agent 1 is the same as for her next-period endowment. Therefore, $Q^1(s_t)$ is the effective belief of agent 1. The same argument implies that $Q^2(s_t)$ is the effective belief of agent 2.

For step 2 of the proof, consider asset prices defined by the recursive relation (22), that is

$$p_t(s_t) = \beta \left[(1 - \hat{q}_t(s_t))p_{t+1}(s_t, 0) + \hat{q}_t(s_t)(p_{t+1}(s_t, 1) + 1) \right], \quad (40)$$

Prices $p(s_t)$ depend only on the number of high dividends from date 1 through t . It is sufficient to show that

$$p_{t+1}(s_t, 1) + 1 > p_{t+1}(s_t, 0), \quad (41)$$

because inequality (41) implies that the maximizing probability in (16) is $Q_t^i(s_t)$ for i such that $q_t^i(s_t) = \hat{q}_t(s_t)$. This in turn implies that p is an equilibrium price.

To prove (41) we follow Harrison and Kreps (1978) and define $p_t^n(s_t)$ inductively by

$$p_t^{n+1}(s_t) = \beta[(1 - \hat{q}_t(s_t))p_{t+1}^n(s_t, 0) + \hat{q}_t(s_t)(p_{t+1}^n(s_t, 1) + 1)], \quad (42)$$

with $p_t^0(s_t) = 0$. Using an inductive argument, one can show that $p_t^n(s_t)$ is bounded by $\beta/(1-\beta)$ for every n and every s_t , and non-decreasing in n . Therefore $\lim_n p_t^n(s_t) = p_t(s_t)$, and

$$p_t(s_t) \leq \frac{\beta}{1-\beta} \quad (43)$$

We claim that (41) holds for p^n for every n . The proof is by induction. Assume that (41) holds for p^n . We obtain

$$p_{t+1}^{n+1}(s_t, 1) + 1 - p_{t+1}^{n+1}(s_t, 0) \geq 1 + \beta p_{t+2}^n(s_t, 1, 0) - \beta(1 + p_{t+2}^n(s_t, 0, 1)), \quad (44)$$

where we used (42). Since $p_{t+2}^n(s_t, 0, 1) = p_{t+2}^n(s_t, 1, 0)$, we obtain

$$p_{t+1}^{n+1}(s_t, 1) + 1 - p_{t+1}^{n+1}(s_t, 0) \geq 1 - \beta > 0.$$

Taking limits, we obtain (41).

Figure 1: Equilibrium under ambiguity

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