Cascading Failures in Production Networks*

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Abstract

I show how the extensive margin of firm entry and exit can greatly amplify idiosyncratic shocks in an economy with a production network. I show that input-output models with entry and exit behave very differently to models without this margin. In particular, in such models, sales provide a very poor measure of the systemic importance of firms or industries. I derive a new notion of systemic influence called *exit centrality* that captures how exits in one industry will affect equilibrium output. I show that exit centrality need not be related to an industry's sales, size, or prices. Unlike the relevant notions of centrality in standard input-output models, exit centrality depends on the industry's role as both a supplier and as a consumer of inputs, as well as market structure. I show that, unlike competitive models, vanishingly small industries can have arbitrarily large effects on equilibrium outcomes. In this sense, the network can amplify shocks in a way that standard input-output models cannot.

Keywords: Propagation, networks, macroeconomics, amplification, entry and exit, market structure

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1 Introduction

In this paper, I investigate how the entry and exit of firms can act to amplify and propagate shocks in production networks. I model cascades of failures among firms linked through a production network, and show which firms and industries are systemically important for aggregate outcomes. I show that the extensive margin of entry is a powerful mechanism for propagating shocks through production networks. Furthermore, the industrial organization of different industries has important macroeconomic implications. This paper connects the literature on the macroeconomic importance of networks, as typified by Acemoglu et al. (2012), to the literature on the macroeconomic importance of product entry and exit, as typified by Bilbiie et al. (2012).

The idea that key consumers or key suppliers can have outsized effects on an economy if they enter or exit has long been popular with policy makers and the public. Policies as diverse as agricultural subsidies, industrial policy, and government bail-outs frequently have been justified on the grounds of *systemic* importance. In academic economics, such ideas have been around since at least the time of Hirschman (1958) and Rosenstein-Rodan (1943). However, despite their popularity with policy makers, these ideas have not been as influential in macroeconomics. Results like those of Hulten (1978) suggest that the systemic importance of firms and industries can be approximated by their sales, even in the presence of linkages between firms, and therefore appeals to systemic importance are at best misguided and at worst dishonest.

However, in recent times, and motivated by current events, these ideas have experienced a resurgence in macroeconomics. After the failure of Lehman Brothers in the United States, the existence of systemically important financial institutions, or SIFIs, became well-accepted in the finance and macro-finance literatures.

Furthermore, recent work has also highlighted that systemic importance need not be limited to financial relationships. A memorable example of this is case of the US automobile manufacturing industry, as discussed by Acemoglu et al. (2012). In the fall of 2008, the president of Ford Motor Co., Alan Mulally, requested government support for General Motors and Chrysler, but not for Ford. The reason Mullaly wanted government support for his company's rivals was that the failure of GM and Chrysler were predicted to result in the failure of many of their suppliers. Since Ford relied on many of those same suppliers, Mulally feared that the knock-on effect on its suppliers could put Ford itself out of business. Mulally's actions go against the standard expectation that Ford would benefit from the failure of its fiercest rivals, and the sales of GM did not tell us the full story about the potential consequences of GM's failure.

While a government bailout of GM and Chrysler prevented a tide of failures in the United States,¹ a scenario similar to the one Mulally described did play out in Australia. In May 2013, Ford Australia announced it they would stop manufacturing cars in 2017. Seven months later, GM Australia announced it would also stop manufacturing cars in 2017. Three months after that,

¹See Goolsbee and Krueger (2015).

in February 2014, Toyota Australia also announced that it would close its manufacturing plants at the same time. This effectively ended automobile manufacturing in Australia. The Australian government predicted that this would result in the loss of over 30,000 jobs, a figure they arrived at by adding the number of people directly employed by the three automakers and the Australian car parts industry.

Empirical work by Carvalho et al. (2014) provides further support for the idea that firm entry and exit have important spill-overs on other firms through supply and demand chains. Carvalho et al. (2014) use the 2011 Tohoku earthquake to predict firm exits, and then show that firm exits have important negative spill-over effects on both upstream and downstream firms, even when these firms are not directly connected to the exiting firm. They show that exits upstream or downstream from a firm reduce the firm's profits and make the firm more likely to exit.

Despite the importance of these spill-over effects, standard macroeconomic models, even ones with production networks, do not allow for such forces. In this paper, I explicitly incorporate the extensive margin of firm entry into an input-output model of production, and show that the extensive margin alters the quantitative and qualitative properties of the model. This paper contributes to the growing literature on the microeconomic origins of aggregate fluctuations. Two recent strands of this literature are the *granular origins* hypothesis of Gabaix (2011) and the *network origins* hypothesis of Acemoglu et al. (2012). The former posits that shocks to large, or granular, firms can have non-trivial effects on real GDP. The latter posits that shocks to highly-connected firms can have non-trivial effects on real GDP. In practice, both theories emphasize a firm or industry's size, as measured by its share of sales, as being the relevant measure of that player's importance. The intuition here is that if a firm is systemically important, then in equilibrium, it will have a large fraction of total sales. However, as discussed above, we might expect interconnectedness to matter in ways that are not captured by prices or total quantities. In this paper, I show that allowing for entry and exit means that the network structure of production can amplify shocks and determine systemic importance in ways that are not captured by size.

Broadly speaking, the paper makes two points. First, in the presence of imperfect competition and entry, the sales of an industry are no longer a good guide to an industry's significance. For instance, industries at the end of long supply chains will be too small in equilibrium as a result of double-marginalization. This means that the observed sales share of the industries no longer summarize the information relevant about how microeconomic shocks will affect the macroeconomy. In particular, this opens the door to the theoretical possibility that a vanishingly small industry can have an arbitrarily large effect on the economy. Furthermore, aggregating such an economy does not allow us to abstract away from this issue. The responses of the aggregate network economy to aggregate shocks can be very different depending on the micro-structure of the underlying network.

Secondly, with non-unitary elasticities of substitution, the pattern of the diffusion of shocks can

be much more complex. Two industries that are substitutes in their product market can become complements if external economies are sufficiently powerful in their upstream markets. In other words, unlike standard input-output models, shocks can reverse directions as they travel from industry to industry. This means that shocks can have both demand-side and supply-side effects, and the causes of a recession (or negative shock) can be different depending on exactly which agents are being studied.

On a methodological level, I show how the Cobb-Douglas analysis of Long and Plosser (1983) and Acemoglu et al. (2012) can be extended to the case with non-unitary elasticities of substitution. In particular, I show that the equilibrium sales distribution for a production network with non-unitary elasticity of substitution is given as the product of a supplying centrality and a consuming centrality. The supplier centrality only takes into account the role of the industry as a supplier of inputs to other industries, while the consumer centrality only takes into account the role of the industry as a consumer of inputs from other industries. In previous work, which was restricted to studying Cobb-Douglas economies, the consuming centrality measure was absent due to the knife-edge income and substitution effects of the Cobb-Douglas functional form. Despite this, I show that many of the intuitions of Acemoglu et al. (2012) continue to hold in the more general case, and their results are not colored by this assumption. However, it turns out that once non-unitary elasticities are combined with external economies of scale, then the behavior of the model departs dramatically from that of Acemoglu et al. (2012).

This paper also contributes to the wider literature on diffusion in network theory, by bridging the gap between two alternative modelling traditions. Loosely speaking, there are two popular approaches to modelling diffusion on social networks. First, there are continuous models of diffusion like Katz (1953). Here, nodes influence each other in continuous ways – shocks travel away from their source like waves and slowly die out. The strength of the connections between the nodes controls the rate of decay. Such shocks, sometimes called pulse processes, are characterized by geometric sums. The other tradition consists of models that behave discontinuously, as typified by Morris (2000) or Elliott et al. (2012). In this class of models, sometimes called threshold models, each node has a threshold and is either active or inactive. When a node crosses its threshold it changes states and, by changing states, pushes its neighbors closer to their thresholds. Such models are frequently used to study the spread of epidemics, products, or even ideas. One of the earliest and most influential threshold models is the Schelling (1971) model of segregation. Threshold models do not have wave-like properties since the rate at which shocks decay is not geometric. Crucially, these models are capable of generating local amplification – that is, shocks can be amplified as they travel through the network. Unfortunately, these models are notoriously difficult to analyze.

Intuitively, in the first class of models, at least locally, shocks are transmitted from nodes to other nodes via a matrix that captures the strength of connections between the nodes. The properties

of this class of economic models is characterized in generality by Acemoglu et al. (2015). In the second class of models, although the nodes are still connected to one another by way of a matrix, this matrix changes depending on the state of the nodes. For instance, in Elliott et al. (2013), who model banking failures, when a bank fails, its corresponding row and column are removed from the adjacency matrix. In this paper, I consider a model that bridges the gap between these two traditions. Specifically, I explicitly account for the mass of firms in a given industry. This allows to think of the forces present in Elliott et al. (2013) as though they are continuous. If we were to restrict the mass of firms to be non-infinitesimal, then we would have exactly the same kind of threshold rules present in Elliott et al. (2013).

This paper also belongs to the literature seeking to derive aggregate fluctuations by combining microeconomic shocks with local interactions. Empirical work on this front includes Foerster et al. (2011), Di Giovanni et al. (2014), Barrot and Sauvagnat (2015), Carvalho et al. (2014), and Acemoglu et al. (2015). Theoretical work can be roughly divided into two categories. The first category are papers that build on the multi-sector neoclassical model of Long and Plosser (1983), like Horvath (2000), Acemoglu et al. (2012), Atalay (2013), and Jones (2011). These are log-linear models of propagation, and follow the tradition of Leontief (1936) in having linear-geometric diffusion. As shown by Hulten (1978), the propagation of TFP shocks in such models is, at least locally, tied to the distribution of firm sizes. This means that in order for a firm to be systemically important, it must have high sales. The second category are models that do not satisfy the requirements of Hulten's theorem. These models, like Jovanovic (1987), Durlauf (1993), and Scheinkman and Woodford (1994), feature strong nonlinearities, their sales distribution is not a decisive sufficient statistic, and they do not feature the kind of geometric patterns of diffusion found in input-output models. However, these models depart dramatically from the standard assumptions of macroeconomics and can be difficult to work with. This paper has elements in common with both of these literatures. The individual ingredients in my model are very familiar to macroeconomists – monopolistic competition, free-entry, and input-output linkages via constant-elasticity-of-substitution production functions. However, the presence of all three ingredients together results in strongly nonlinear interactions and an uncoupling of systemic importance from the distribution of sales.

The structure of paper is as follows. In section 2, I set up the model and define its equilibrium. In section 3, I characterize the equilibrium conditional on the mass of entrants in each industry and define some key centrality measures. In section 4, I study how the model behaves when the extensive margin of firm entry and exit is shut down. I prove results showing that the network structure can be summarized by sufficient statistics related to size. I also show that we can think of these models as non-interconnected models with different parameters. In section 5, I allow for firm entry and exit. First, I characterize the model's responses to shocks in the limit where all firms are massless. I show that sufficient statistics are no longer available, and I derive an

endogenous measure of influence called *exit centrality*. Then, I consider the case when firms can have positive mass, and show that with atomistic firms, shocks can be locally amplified. I prove an approximability result showing conditions under which we can approximate an intractable discontinuous model using a tractable continuous model. Motivated by Ford's request for a bailout of GM, I also consider whether efficient bail-outs can be identified by surveying firms. Specifically, when can we trust one firm's testimony about whether or not another firm should be bailed out. I conclude in section 6.

2 Model

In this section, I spell out the structure of the model and define the equilibrium. There are three types of agents: households, firms, and a government. Each firm belongs to an industry, and there are N industries. There are two periods: entry decisions happen simultaneously in period 1, production and consumption take place in period 2.

Household's Problem

The households in the model are homogenous with a unit mass. The representative household maximizes utility

$$U(c_1, \dots, c_N) = \left(\sum_{k=1}^N \beta_k^{\frac{1}{\sigma}} c_k^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

where c_k represents composite consumption of varieties from industry k and $\sigma > 0$ is the elasticity of substitution across industries. The weights $\beta_k \geq 0$ determine household tastes for goods and services from the different industries. The composite consumption good produced by industry k is given by

$$c_k = \left(M_k^{-\varphi_k} \int_{M_k} c(k, i)^{\frac{\varepsilon_k-1}{\varepsilon_k}} di \right)^{\frac{\varepsilon_k}{\varepsilon_k-1}},$$

where $c(k, i)$ is household consumption from firm i in industry k and $\varepsilon_k > 1$ is the elasticity of substitution across firms within industry k . Here, M_k is the mass of product varieties offered by industry k . Finally, the term $M_k^{-\varphi_k}$ controls the strength of the love-of-variety effect. If we let $\varphi_k = 1/\varepsilon_k$, then there is no love-of-varieties, and if we let $\varphi_k = 0$, we get the standard Dixit and

Stiglitz (1977) demand system.² The household's budget is given by

$$\sum_{k=1}^N \int_{M_k} p(k, i) c(k, i) di = wl + \sum_{k=1}^N \int_{M_k} \pi(k, i) di,$$

where $p(k, i)$ is the price of firm i in industry k and $\pi(k, i)$ is firm i in industry k 's profits. The wage is w and labor is inelastically supplied at l . For the rest of the paper, and without loss of generality, we take labor to be the numeraire so that $w = 1$, and fix the supply of labor $l = 1$.

Firms' Problem

Let firm i in industry k have profits

$$\pi(k, i) = p(k, i) y(k, i) - \sum_{l=1}^N \int_{M_l} p(l, j) x(k, i, l, j) dj - wh(k, i) - wf_k,$$

where $p(k, i)$ is the price and $y(k, i)$ is the output of the firm. Inputs from firm j in industry l are $x(k, i, l, j)$ and labor inputs are $h(k, i)$. Finally, in order to operate, each firm must pay an overhead cost f_k in units of labor. The firm's production function (once the fixed cost has been paid) is constant-returns-to-scale

$$y(k, i) = \left(\alpha_k^{\frac{1}{\sigma}} (z_k l(k, i))^{\frac{\sigma-1}{\sigma}} + \sum_{l=1}^N \omega_{kl}^{\frac{1}{\sigma}} x(k, i, l)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

Here, $\sigma > 0$ is again the elasticity of substitution among inputs, and ω_{kl} is the share parameter for how intensively firms in industry k use composite inputs from industry l . The $N \times N$ matrix of ω_{kl} , denoted by Ω , determines the network-structure of this economy. One can think of this matrix as the adjacency matrix of a weighted directed graph. The parameter $\alpha_k > 0$ gives the intensity with which firms in industry k use labor (net of fixed costs). We can think of f_k as the entrepreneurial labor required to create a new product, and $l(k, i)$ as the worker labor required to produce additional units of this product. For simplicity, I treat worker types as perfect substitutes, so that any unit of labor can be allocated cordlessly to either product creation or product production.

The composite intermediate input from industry l used by firm i in industry k is

$$x(k, i, l) = \left(M_l^{-\varphi_l} \int_{M_l} x(k, i, l, j)^{\frac{\varepsilon_l-1}{\varepsilon_l}} di \right)^{\frac{\varepsilon_l}{\varepsilon_l-1}},$$

²See section 5.1 for more discussion about the significance of the love-of-varieties effect in this model. Love of varieties can be thought of as a reduced form for a more complex model where increasing product variety improves the match between products and consumers. Anderson et al. (1992) provide a microfoundation for this via a discrete choice model where each consumer consumes only a single variety, but there exists a CES representative consumer for the population of consumers.

where ε_l is the elasticity of substitution across different firms within industry l . As before, the term $M_l^{-\varphi_l}$ affects the strength of the love-of-varieties effect in industry l . Letting $\varphi_l = 0$ gives us the traditional Dixit-Stiglitz formulation. Note that the elasticities of substitution are the same for all users of an industry's output.

Notation

Let Ω be the $N \times N$ matrix whose ij th element is equal to ω_{ij} . Let α and β be the $N \times 1$ vectors consisting of α_i s and β_i s. Let M be the $N \times N$ diagonal matrix whose i th element is M_i , and let $\tilde{M}$ be the $N \times N$ diagonal matrix whose i th diagonal element is equal to $M_i^{\frac{1-\varphi_i\varepsilon_i}{\varepsilon_i-1}}$. Let $\circ$ denote the element-wise or Hadamard product. Let e_i denote the i th standard basis vector, and $\text{diag} : \mathbb{R}^N \rightarrow \mathbb{R}^{N^2}$ be the operator that maps a vector to a diagonal matrix, or more formally, $\text{diag}(X) = \sum_i x_i e_i e_i'$. To simplify the notation, when a variable appears without a subscript, this represents the matrix of all such variables, and the object v^s , where v is a vector and s is a scalar, should be interpreted as the vector v raised to the power of s elementwise.

Definition 2.1. An economy E is defined by the tuple $E = (\beta, \Omega, \alpha, \varepsilon, \sigma, f, \varphi)$. The vector β contains household taste parameters, Ω captures the input-output share parameters, α contains the industrial labor share parameters, $\varepsilon > 1$ is the vector of industrial elasticities of substitution, $\sigma > 0$ is the cross-industry elasticity of substitution, f is the vector of fixed costs, and φ is the parameter controlling the love-of-variety effect.

Equilibrium

We study the subgame perfect Nash equilibrium. In period 1, potential entrants make simultaneous entry decisions. In period 2, firms play general equilibrium conditional on period 1's entry decisions. Define a markup function $\mu : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ to be a function that maps the mass of entrants $\{M_k\}_k$ and productivity shocks z_k to markups for each industry. Now we can define general equilibrium as follows. I assume that firms set their prices to equal the markup function times their marginal cost.

Definition 2.2. A general equilibrium of economy E is a collection of prices $p(i, k)$, wage w , and input demands $x(i, k, l, j)$, outputs $y(i, k)$, consumptions $c(i, k)$ and labor demands $l(i, k)$ such that for masses of varieties $\{M_k\}_{k=1}^N$, vector of productivity shocks z , and markup function $\mu(M, z)$:

- (i) Each firm minimizes its costs subject to demand for its goods given the markup function,
- (ii) the representative household chooses consumption to maximize utility,
- (iii) the government runs a balanced budget,

(iv) markets for each good and labor clear.

Note that prices and quantities in a general equilibrium depend on the number of entrants $\{M_k\}$, the vector of productivity shocks, and the assumed industry-level markups μ . Conditions (ii)–(iv) are standard, but condition (i) merits further discussion. Condition (i) is deliberately vague. It simply requires that the firms minimize their costs, given the demand they face and leaves their choice over prices unspecified – it simply requires that there be some function μ that determines their markups as a function of the mass of varieties $\{M_k\}$ and the productivities z .

The exact nature of the markup function μ , which will depend on what we assume about competition in each industry, will play an important role in our analysis. However, for now, we can leave the market structure relatively general.

Before analyzing the model, it helps to define some key statistics. These are standard definitions from the literature on monopolistic competition. See, for example, Bettendorf and Heijdra (2003).

Definition 2.3. The price index for industry k is given by

$$p_k = \left(M_k^{-\varphi_k \varepsilon_k} \int^{M_k} p(k, i)^{1-\varepsilon_k} di \right)^{\frac{1}{1-\varepsilon_k}},$$

and the total composite output of industry k is given by

$$y_k = \left(M_k^{-\varphi_k} \int^{M_k} y(k, i)^{\frac{\varepsilon_k-1}{\varepsilon_k}} di \right)^{\frac{\varepsilon_k}{\varepsilon_k-1}}.$$

The consumer price index, which represents the price level for the household, is given by

$$P_c = \left(\sum_k^N \beta_k p_k^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

and total consumption by the household is given by

$$C = \left(\sum_{k=1}^N \beta_k^{\frac{1}{\sigma}} c_k^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

which is just the utility of the household.

These are the “ideal” price and quantity averages for each industry. The reason we do not simply average prices to get a price index or add outputs to get total output is because even within each industry, each firm is producing a slightly different product. When the elasticity of substitution $\varepsilon_k = 0$, then the price index for that industry is simply the sum of all the prices, since there is no substitution and a consumer of this industry must buy all the varieties. When

the elasticity of substitution $\varepsilon_k \rightarrow \infty$, the price index for an industry is the minimum price, since households will only purchase from the cheapest firm in each industry. An important special case is when $\varepsilon_k \rightarrow 1$, where the price index is a geometric average of the industrial prices.

3 General Equilibrium Subgame

In this section, I characterize the equilibrium in the general equilibrium subgame of period 2, conditional on the number of entrants in each industry. I introduce two related notions of centrality, the supplier and consumer centrality, and I show that together, they determine the size distribution of industries in the economy.

Let us define supply-side centrality measure.

Definition 3.1. The supplier centrality is

$$\tilde{\beta}' = \beta' \Psi_s = \beta' \sum_{n=0}^{\infty} \left(\tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega \right)^n, \quad (1)$$

where

$$\Psi_s = (I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1} = \sum_{n=0}^{\infty} \left(\tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega \right)^n,$$

is the supply-side influence matrix.

We can think of $\tilde{\beta}$ as the *network-adjusted* consumption share of the industries. The k th element of $\tilde{\beta}$ captures demand from the household that reaches industry k , whether directly or indirectly through other industries who use k 's products. Each term of the sum in (1) is interpretable. The first term captures the direct intensity with which the household buys from an industry, the second term captures the indirect effect when the household consumes something that used the industry as an input, and so on. In other words, supplier centrality captures the importance of the industry as a supplier to the household. The following helps establish why we might care about $\tilde{\beta}$

Lemma 3.1. In equilibrium,

$$\tilde{\beta}_i = \left(\frac{p_i^\sigma y_i}{P_c^\sigma C} \right),$$

where P_c is the ideal price index for the household, C is total consumption by the household, p_i is industry i 's ideal price index and y_i is industry i 's composite output.

Note that in the Cobb-Douglas limit, where the elasticity of substitution across industries σ is equal to one, $\tilde{\beta}_i$ is precisely an industry's share of sales. In the Cobb-Douglas case, $\tilde{\beta}$ coincides with the *influence measure* defined by Acemoglu et al. (2012).

There are two reasons to think of $\tilde{\beta}$ as a supplier centrality. First, since it measures the intensity with which the household consumes goods from an industry (both directly and indirectly through

intermediaries), $\tilde{\beta}$ is high the more frequently an industry appears in the households' supply chains. So, all else equal, star suppliers, who supply many firms, have higher $\tilde{\beta}$. Secondly, the value of $\tilde{\beta}_k$ depends only on who the industry k sells to, and not on who it buys from.

Proposition 3.2. *Consider two industries k and l such that*

$$\omega_{jk} = \omega_{jl}, \quad (j = 1, \dots, N), \quad \beta_k = \beta_l, \quad \mu_k = \mu_l, \quad M_k = M_l,$$

then $\tilde{\beta}_k = \tilde{\beta}_l$. In other words, if two industries have the same immediate customer base, their supplier-centralities are the same.

Note that with entry and exit,

$$\tilde{\beta}' = \beta'(I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1},$$

is endogenous. In particular, a change in the mass of firms affects both the love-of-variety effect $\tilde{M}$ and potentially the markups μ . These in turn change the value of $\tilde{\beta}$ in highly nonlinear ways. Intuitively, to get interesting results, entry and exit have to affect the strength of connections between industries. The centrality of industries in the network must change as firms enter and exit the market. The connections encapsulated in Ω have to become stronger or weaker depending on the number of entrants in each industry. If we make expenditure shares exogenous, by shutting down love-of-varieties and imposing constant markups, then $\tilde{\beta}$ will not respond to extensive margin shocks.

Now let us define the analogous demand-side consumer centrality.

Definition 3.2. The consumer centrality is

$$\tilde{\alpha} = \Psi_d(\alpha \circ z^{\sigma-1}),$$

where

$$\Psi_d = (I - \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega)^{-1} \mu^{1-\sigma} \tilde{M}^{\sigma-1} = \sum_{n=0}^{\infty} (\mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega)^n \mu^{1-\sigma} \tilde{M}^{\sigma-1},$$

is the demand-side influence matrix.

This is the flip-side to the supply-side centrality measure. It captures how frequently an industry appears downstream from the factors of production (in this case labor).³ Whereas $\tilde{\beta}_k$ depended only on who bought from k , the consumer centrality $\tilde{\alpha}_k$ depends only on who k buys from.

³As discussed before, to simplify notation, we can simply subsume the effect of z into α , writing only $\tilde{\alpha} = \Psi_d \alpha$.

Proposition 3.3. *Consider two industries k and l such that*

$$\omega_{kj} = \omega_{lj}, \quad (j = 1, \dots, N), \alpha_k = \alpha_l, \mu_k = \mu_l, M_k = M_l,$$

then $\tilde{\alpha}_k = \tilde{\alpha}_l$. In other words, if two industries have the same immediate supplier base, their consumer centralities are the same.

As with the supplier centrality, note that with entry and exit,

$$\tilde{\alpha} = (I - \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega)^{-1} \mu^{1-\sigma} \tilde{M}^{\sigma-1} \alpha,$$

is endogenous. In particular, a change in the mass of firms affects both the love-of-variety effect $\tilde{M}$ and potentially the markups μ . These in turn change the value of $\tilde{\alpha}$ in highly nonlinear ways. As with the supplier centrality, if we make expenditure shares exogenous, by shutting down love-of-varieties and imposing constant markups, then consumer centrality $\tilde{\alpha}$ does not respond to extensive margin shocks.

Lemma 3.4. *In equilibrium,*

$$\tilde{\alpha}_i = \left(\frac{p_i}{w} \right)^{1-\sigma},$$

where p_i is industry i 's price index and w is the nominal wage.

The intuition is that $\tilde{\alpha}_k$ captures all the ways industry k uses labor, both directly, and indirectly through its suppliers, suppliers' suppliers, and so on. In other words, we can think of $\tilde{\alpha}_k$ as the *network-adjusted* factor costs of k . The higher $\tilde{\alpha}_k$ is, the longer, more competitive, and more value-intensive the supply chains of k are, and therefore, the lower its marginal costs and prices. This is why $\tilde{\alpha}_k$ depends on industry k 's supply-chain, since it is suppliers and not consumers, who contribute to marginal costs.

A key result of this paper, and one that delivers much of the intuition for the results, is the following characterization of active firms' profit functions in terms of supplier and consumer centrality measures:

Theorem 3.5. *The profit of firm i in industry k are equal to*

$$\pi(k, i) = \underbrace{\frac{\mu_k - 1}{\mu_k} \frac{1}{M_k}}_{\text{industrial competition}} \times \underbrace{P_c C}_{\text{aggregate demand}} \times \underbrace{\left(\frac{w}{P_c} \right)^{1-\sigma}}_{\text{aggregate supply}} \times \underbrace{\tilde{\beta}_k}_{\text{supplier centrality}} \times \underbrace{\tilde{\alpha}_k}_{\text{consumer centrality}} - \underbrace{w f_k}_{\text{fixed cost}}.$$

The expression in theorem 3.5 tells us that the profits of a firm are determined by a few intuitive key statistics. The product of $\tilde{\beta}_k$ and $\tilde{\alpha}_k$, which are the supply-side and demand-side centrality of the industry, give us an industry's share of sales. The nominal GDP term $P_c C$ is an economy-wide

shifter of all industry's profits, akin to aggregate demand. The terms $\frac{\mu_k - 1}{\mu_k}$ and M_k convert an industry's sales into firm-level gross profit. Finally, we arrive at a firm's net profits by subtracting the fixed costs of entry from its gross profit.

Path Example

Before moving on to an analysis of the equilibrium, first let us demonstrate the intuition so far using a simple example of a vertical economy, shown in figure 1.

Figure 1: Vertical supply chain, where the solid arrows represent the flow of goods and the dashed arrows represent the flow of money. Only industry N directly employs labor and only industry 1 sells to the household. The flow of profits from firms to households has been suppressed.

Begin by computing the supplier-centrality for industry k in this chain:

$$\begin{aligned}\tilde{\beta}_k &= \beta'(I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1} e_k, \\ &= \tilde{M}_1^{\sigma-1} \mu_1^{-\sigma} \times \dots \times \tilde{M}_{k-1}^{\sigma-1} \mu_{k-1}^{-\sigma}.\end{aligned}$$

Note that, for this example, $\tilde{\beta}_k$ is a product. Therefore, if any downstream industry $i < k$ disappears, so that $\tilde{M}_i = 0$, then the supplier centrality of industry k drops to zero. This is intuitive, since if a downstream industry collapses, that cuts all upstream industries off from any demand. Observe that, as implied by proposition 3.2, the k th industry's supplier centrality depends purely on its customers, customers' customers, and so on. It does not depend on its suppliers.

Now, compute the consumer-centrality for node k in this chain:

$$\begin{aligned}\tilde{\alpha}_k &= e'_k (I - \tilde{M}^{\sigma-1} \mu^{1-\sigma} \Omega)^{-1} \tilde{M}^{\sigma-1} \mu^{1-\sigma} \alpha, \\ &= \tilde{M}_k^{\sigma-1} \mu_k^{1-\sigma} \times \tilde{M}_{k+1}^{\sigma-1} \mu_{k+1}^{1-\sigma} \times \dots \times \tilde{M}_N^{\sigma-1} \mu_N^{1-\sigma}.\end{aligned}$$

This depends on the industries upstream from k . Once again, if any upstream industry collapses, this cuts industry k off from labor, and therefore causes its marginal cost to diverge to infinity. Note that, as implied by proposition 3.3, consumer centrality of a firm depends only on who it buys from, and not on who it sells to.

4 Equilibrium without the Extensive Margin

Before moving onto the case with the extensive margin, let us consider the behavior of this model when the extensive margin is shut down. This makes the model into a generalization of the canonical Cobb-Douglas input-output model of Acemoglu et al. (2012). They show that, with unitary elasticity of substitution, an industry's influence on other industries and on the aggregate economy is exogenously determined, and only depends on the industry's role as a supplier of inputs (and not as a consumer of inputs). In this section, we see that this insight is not an artefact of assuming Cobb-Douglas, and in fact holds more generally with any elasticity of substitution. Furthermore, as with the Cobb-Douglas case, a firm's size is an important determinant of a firm's influence.

Standard models that lack an entry margin are typically perfectly competitive, with no product creation, so that the representative firm in each industry makes zero profits and firm size is indeterminate. To get to this benchmark, let $f_i = 0$ for every industry so that product creation is costless, and let $\varepsilon_i \rightarrow \infty$ so that products are perfect substitutes within every industry. Then, due to constant returns to scale, the size of firms in each industry is indeterminate. Without loss of generality, we can fix $M = I$ so that there is a unit mass of firms in each industry. Technically, there may be entry or exit, but since firms in each industry are completely homogenous, entry and exit are observationally equivalent to firms getting larger or smaller. That is, the extensive and intensive margins are indistinguishable. Lastly, given the lack of product differentiation and free entry, there are no markups in this economy $\mu = I$. Note that in this special case, the supply-side and demand-side influence matrices are equal $\Psi^d = \Psi^s = (I - \Omega)^{-1}$.

Let us consider real GDP $C(z|E)$ as a function of productivity shocks z for an economy E .

Theorem 4.1. *For every household share parameter vector β and input-output share parameter matrix Ω , the non-interconnected economy with household share parameter vector $\tilde{\beta}$ and degenerate input-output share parameter matrix $\mathbf{0}$, we have*

$$C(z|\beta, \Omega) = C(z|\tilde{\beta}, \mathbf{0}) = \left(\sum \tilde{\beta}_k \alpha_k z_k^{\sigma-1} \right)^{\frac{1}{\sigma-1}}.$$

This result means that the existence of input-output connections is isomorphic to a change in household share parameters. This underlies the earlier claim that, it is not the interconnections themselves, but how intensively the household ultimately consumes goods from each industry that determines the model's equilibrium responses to shocks. It is important to note that proposition 4.1 is not implied by, nor implies, Hulten's theorem. Although this environment satisfies the requirements of Hulten (competitive, constant returns to scale), the type of shocks under consideration are labor productivity shocks (not TFP shocks), and the result is global rather than local.

Proposition 4.1 implies an important insight about the diffusion of labor productivity shocks.

Namely, shocks only diffuse from suppliers to consumers in this economy. By lemma 3.1, we know that if two industries k and l have the same customer base, then $\tilde{\beta}_k = \tilde{\beta}_l$, regardless of who k and l buy from. This means that industry k and industry l must have the same effect on GDP. In fact, we can go further, from a micro perspective, we have the following result

Corollary. *Suppose that there exists a directed walk from industry k to industry l but not vice-versa. In other words, industry l appears somewhere upstream from industry k but not vice-versa. Then*

$$\frac{d \log(p_l y_l)}{d \log(z_k)} = (1 - \sigma) \frac{d \log(w/P_c)}{d \log(z_k)}.$$

In other words, the only effect of a shock on upstream industries, from a shock to a customer, is through changes in the real wage. In this sense, in this environment, shocks only diffuse from suppliers to consumers. An important distinction to keep in mind here is that we are dealing with labor productivity shocks only (or more generally, value-added technology shocks). Total factor productivity shocks that affect intermediate inputs directly *would* propagate upwards from consumers to suppliers. The intuition for this is that such a shock is implicitly making upstream firms more productive in supplying to the affected industry. As an example, a negative gross TFP shock to General Motors would directly affect the sales of the upstream parts industry, however, such a negative shock would still not propagate downstream to hit Ford (besides the general equilibrium effect on the real wage).

Proposition 3.1 shows that the influence of each industry is proportional to a weighted product of the industry's price and the industry's total output. The next proposition shows that, for the special case where all industries are equally productive, supplier centrality is proportional to an industry's total sales.

Proposition 4.2. *Around the steady-state, where $z_k = 1$ for all k , we have that*

$$\frac{\tilde{\beta}_k}{\tilde{\beta}_l} = \frac{p_k y_k}{p_l y_l},$$

so although $\tilde{\beta}$ is not equal to share of sales everywhere, at the steady-state, it does reflect an industry's sales.

5 Equilibrium with Extensive Margin

Now that we understand the properties of the model without the extensive margin, let us consider how entry and exit changes the model's properties. In this case, the network structure is endogenous since the number of firms in each industry is endogenous. This means that our notions of centrality $\tilde{\beta}$ and $\tilde{\alpha}$ are determined in equilibrium, and they change in response to shocks. It will no longer be the case that an industry's influence is exogenous, nor will its influence depend solely

on its role as a supplier. With entry, an industry's influence depends both on its in-degrees (arrows pointing in) and its out-degrees (arrows pointing out), and these are determined in equilibrium depending on the mass of firms in other industries. Two firms with identical roles as suppliers can have markedly different effects on the equilibrium depending on their roles as consumers.

Our analysis will proceed in steps. First, I discuss the market structure and the mechanisms through which entry can affect the equilibrium. Then, in section 5.2, I consider the Cobb-Douglas special case, and look at how the model can potentially amplify fluctuations and decouple systemic influence from size. However, unlike the competitive case, where arguably Cobb-Douglas is not as restrictive of an assumption, it turns out is a very restrictive assumption once we allow for entry. Therefore, in section 5.4, we investigate the properties of the general model.

To foreshadow the formal model, consider an extremely stylized example in figure 2. The household HH is served by three industries: A_1 , A_2 , and the unspecified *rest of the economy*. A_1 and A_2 share a common supplier S . To make the intuition transparent, suppose that each industry consists of a single firm. Recall that theorem 3.5 implies that in order for a firm to remain profitable, its gross profits

$$\frac{\mu_k - 1}{\mu_k} \tilde{\beta}_i \tilde{\alpha}_i P_c^\sigma C$$

have to be greater than an exogenous threshold f_i . For this example, suppose that markups are constant. Each term of $\tilde{\beta}$, $\tilde{\alpha}$, and $P_c^\sigma C$ can respond to a shock. Therefore, the shock to a firm can travel via network connections $\tilde{\beta}$ and $\tilde{\alpha}$ or it can travel through household demand $P_c^\sigma C$.

The example in figure 2 illustrates the various ways shocks can propagate. In panel 2a, I show a negative shock to the rest of the economy that reduces the aggregate demand term $P_c^\sigma C$. Then, because of this change in $P_c^\sigma C$, it can be the case that A_1 is forced to exit, and this is shown in panel 2b. A_1 exiting will cause S 's supplier centrality $\tilde{\beta}_S$ to fall, and so S can be forced out, shown in panel 2c. Finally, once S is gone, this can cause A_2 's consumer centrality $\tilde{\alpha}$ to drop, which can cause A_2 to also exit. Of course, this is not a realistic calibration of the model, but it does illustrate the forces operating in the model. In a canonical input-output model, like the one in section 4, a change to $P_c^\sigma C$ would have no network effects since $\tilde{\beta}$ and $\tilde{\alpha}$ would be exogenous.

Finally, this example shows that the model is capable of formalizing the intuition for why a bail-out of GM may have been crucial. As pointed out by Goolsbee and Krueger (2015), in this scenario, it might be “essential to rescue GM to prevent an uncontrolled bankruptcy and the failure of countless suppliers, with potentially systemic effects that could sink the entire auto industry.”

Figure 2: A toy representation of an economy where the direction of arrows denote the flow of goods. The node HH denotes the household. This shows how the shock can travel in both directions on the extensive margin.

Definition 5.1. The **exit centrality** v is given by

$$\frac{d \log(C)}{d \log(f)} \equiv \tilde{v}$$

5.1 External Economies of Scale

So far, we have not explicitly defined the structure of each market. Instead, by working with the markups directly, we have avoided the need to specify the nature of competition. In fact, for many of the results that are to follow, all we require is that the markups $\mu_k(M)$ in industry k depend only on the number of entrants in industry k . This assumption is enough to deliver all the qualitative results we need.

One of the novel contributions of this paper is in combining external economies of scale with input-output networks. As an industry experiences more entry, it becomes more productive, which in turn stimulates the surrounding industries, and they too can experience more entry, thereby making them more productive, and so on. This requires that more entry makes the industry more productive. There are potentially two reasons why there are external economies of scale in this model: (1) product differentiation means that more entry results in more product variety, which is valued by consumers; (2) variable markups mean that markups fall as an industry experiences

more entry, which is valued by consumers. These two forces are qualitatively similar.

To see this, note that the influence matrices Ψ_s and Ψ_d , which capture the strength of the connections are of the form

$$\Psi_s = (I - S\Omega)^{-1} \quad \text{and} \quad \Psi_d = (I - D\Omega)^{-1}D,$$

where S and D are diagonal matrices. Specifically, S and D are products of the diagonal matrix consisting of the product-variety effect $\tilde{M}$ and the diagonal matrix of markups μ .

For simplicity, I assume the Dixit-Stiglitz market structure with $\psi_k = 0$, so that markups are constant and all the effects come through the love of product-variety. As stated before, this does not color the qualitative nature of the results. In appendix A, I show what the formulas would look like for the general case.⁴

5.2 Cobb-Douglas

Consider a very special case – the case where the elasticity of substitution is equal to one. Since we are assuming the Dixit-Stiglitz structure, markups are constant and do not respond to increased entry. Therefore, supplier centrality $\tilde{\beta}$ does not respond to shocks. In this case, we have the following result:

Proposition 5.1. *When the elasticity of substitution is equal to one, and markups are constant,*

$$\log(C(z, f)) = \beta'(I - \Omega)^{-1} \left(\alpha \circ \log(z) - \frac{1}{\varepsilon - 1} \circ \log(f) \right) + \text{const},$$

so that

$$\tilde{v} = \beta'(I - \Omega)^{-1} \circ \frac{1}{\varepsilon - 1},$$

which is only a function of the exogenous parameters of the mode. Furthermore, the equilibrium mass of entrants is given by

$$M_k = \left(\frac{\tilde{\beta}}{f_k \varepsilon_k} \right)^{\frac{1}{\varepsilon_k - 1}}.$$

Before discussing this result, it helps to notice that the mass of products M in each industry depends only on the productivity of labor used in product creation f and not on the productivity of labor used in production z_k . The reason is that product creation costs f are paid in units of labor, and not in units of the consumption good. Therefore, an increase in the productivity of

⁴There are different ways to get variable markups in this context. The simplest is to assume that each industry contains an atomistic incumbent firm, with an infinitesimal periphery. As more firms enter, the large firm loses market power, and therefore industry level markups decline continuously as a function of entry. Alternatively, one could model within-industry demand as being translog so that the elasticity of substitution is increasing in the mass of products. This would give rise to declining markups due to demand-side pricing complementarities as in Krugman (1979) and Zhelobodko et al. (2012).

production labor z_k does not make product creation f_k cheaper. Furthermore, Cobb-Douglas shuts down expenditure substitution across industries, which means that changes in the productivity of production labor do not alter the profitability of industries (since sales, and therefore, profits do not depend on marginal costs).

There are three insights we can glean from this proposition. First, note that an industry's sales are no longer a good measure of the industry's importance. An industry's sales as a fraction of GDP is given by

$$\frac{p_k y_k}{P_c C} = \tilde{\beta}_k = \beta'(I - \mu^{-\sigma} \Omega)^{-1} e_k,$$

which differs from that industry's share of sales due to double-marginalization. This difference can be dramatic if the industry is at the end of many long and uncompetitive supply chains. Crucially, the market structure of industry k does *not* drive a wedge between the industry's social value and the industry's size.⁵ In other words, the decoupling of size from influence requires both the existence of markups *and* the existence of network connections.

Second, the effect of extensive margin shocks depend on both the network structure and the external economies of scale. In this case, external economies are determined by within-industry elasticity of substitution ε . Specifically, the lower the elasticity of substitution, the larger the impact to GDP. In this knife-edge case, where $\tilde{\beta}$ and $\tilde{\alpha}$ are both exogenously determined (due to constant markups and unitary elasticity of substitution), the way external economies interacts with the network is quite simple. If we relax either the assumption of constant markups, or the assumption of unitary elasticities of substitution, then much more complex patterns of diffusion can be generated.

Third, aggregating the economy does not diminish the importance of the underlying network structure. To see this, consider an aggregate shock to worker productivity in this economy. In other words, let the labor force be given by $L = A\bar{L}$, where $\bar{L}$ is the baseline labor force measured in efficiency units. Then,

$$\log(GDP(A)) = \beta'(I - \Omega)^{-1} \alpha \log(A) + \beta'(I - \Omega)^{-1} \circ \frac{1}{\varepsilon - 1} \log(A) + \text{const},$$

where the first term is the effect of the aggregate shock on the productivity along the intensive margin and the second term is the effect of the aggregate shock on productivity along the extensive margin. However, $\beta'(I - \Omega)^{-1} \alpha = 1$, so that

$$\log(GDP(A)) = \log(A) + \beta'(I - \Omega)^{-1} \circ \frac{1}{\varepsilon - 1} \log(A) + \text{const},$$

where $\beta'(I - \Omega)^{-1} > 1$, if the network structure is not degenerate. So, while the effect of the aggregate shock does not depend on the network structure along the intensive margin, its effect

⁵As long as that industry does not use its own output as an input.

on GDP is amplified and *does* depend on the network structure along the extensive margin.

However, as in the competitive case described in proposition 4.1, the diffusion of shocks here is unidirectional. That is, the exit-centrality of an industry depends only on its role as a supplier, and not on its role as a consumer of inputs. This means that a change in product creation will diffuse to downstream industries but not upstream industries. Once again, this rules out the auto-industry pattern of diffusion which requires shocks to travel upstream from General Motors to a supplier, and then back down again from the supplier to Ford.

5.3 Amplification in the Vertical and Horizontal Economy

Consider the example where the economy is completely vertical, so that $\beta = (1, 0, \dots, 0)'$, and $\alpha = (0, \dots, 0, 1)'$, depicted in figure 1. For this example σ does not matter. This means that we can set $\sigma = 1$ and use proposition 5.1 to easily derive the exit centralities of the industries

Proposition 5.2. *Exit centralities for the economy depicted in figure 1 is given by*

$$\tilde{v} = \left(\frac{1}{\varepsilon_1 - 1}, \dots, \frac{1}{\varepsilon_N - 1} \right)',$$

while sales as a share of GDP is given by

$$\tilde{\beta} = \left(1, \mu_1^{-1}, \dots, \prod_{i=1}^{N-1} \mu_i^{-1} \right)',$$

where μ_i are the markups of industry i .

In this case, the influence of the N th industry is given by $1/(\varepsilon_N - 1)$, which can be unboundedly large $\varepsilon_N \rightarrow 1$. On the other hand, the sales of industry N can be arbitrarily close to zero, either if a downstream industry charges high markups or if the chain is sufficiently long. This simple example serves as an important qualification to Gabaix (2016), who wrote that “networks are a particular case of granularity rather than an alternative to it: if all firms had small sales, the central limit theorem would hold and idiosyncratic shocks would all wash out.” In this example, where we have deviated only slightly from the perfectly competitive benchmark, an arbitrarily small player can have an arbitrarily large impact on GDP.

On the other hand, consider the horizontal economy, depicted in figure 3. This is a standard multi-sector model without an input-output structure, so that $\Omega = \mathbf{0}$ and $\alpha_i = 1$ for all i . For this case, choosing the cross-industry elasticity of substitution is no longer innocuous, and the derivations are more involved and shown in the appendix, but the exit centrality of the industries, evaluated at $M = 1$ is

$$\frac{d \log C}{d \log f_i} = -\frac{1}{\varepsilon_i - 1} \frac{\mu_i^{1-\sigma} \left(\frac{\varepsilon_i - 1}{\varepsilon_i - \sigma} \right)}{\sum_j \mu_j^{1-\sigma} \left(\frac{\varepsilon_j - 1}{\varepsilon_j - \sigma} \right)},$$

Figure 3: Horizontal supply chain, where the solid arrows represent the flow of goods.

which takes the industry's returns to scale $\frac{1}{\varepsilon_i - 1}$ and reweights it. If $\sigma < 1$, then the weighting increases the weights for industries whose elasticity of substitution is lower than the rest (since industries are complements). On the other hand, if $\sigma > 1$, the weighting increases the weights for the industries whose elasticity of substitution is higher than the rest (since industries are substitutes). For the Cobb-Douglas case, where $\sigma = 1$, we have the following result.

Proposition 5.3. *Suppose that $\sigma = 1$. Exit centralities for the economy depicted in figure 3 is given by*

$$\tilde{v} = -\frac{1}{N} \left(\frac{1}{\varepsilon_1 - 1}, \dots, \frac{1}{\varepsilon_N - 1} \right)',$$

while sales as a share of GDP is given by

$$\tilde{\beta} = \frac{1}{N} (1, \dots, 1)',$$

where μ_i are the markups of industry i .

In this example, both the sales and the exit centrality go to zero as we increase N , and the relationship between the sales share and the exit centrality is much tighter. This example illustrates how the relationship between sales, exit-centralities, and elasticities of substitution depend crucially on the network structure and the cross-industry pattern of substitution.

5.4 Non-unitary Elasticity of Substitution

In section 5.2, we saw how the model behaves when we set the elasticity of substitution across industries equal to 1. In this case, the centrality measures $\tilde{\beta}$ and $\tilde{\alpha}$ are both exogenous, and so the model loses some of the interesting patterns of diffusion that we would like to see. For one, Cobb-Douglas imposes the same pattern of diffusion as the competitive model – shocks can only travel from upstream suppliers to downstream consumers. The role of a given industry as a consumer of inputs is irrelevant. This means that General Motors should be equally important whether it buys parts from the same supply chain as other firms or not. In this section, we relax the Cobb-Douglas

Figure 4: Example inspired by the auto-makers.

assumption, and allow for the possibility that shocks can travel in both directions. As a first step, it helps to begin with a simple example inspired by the auto-makers.

5.4.1 Example Inspired by the Automakers

Consider the economy depicted in figure 7. To simplify the algebra, consider the case where A_3 and A_2 are perfectly competitive industries, but A_1 and S are not. Let $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$. Also, let $\beta = (1/3, 1/3, 0, 1/3)$.

Proposition 5.4. *Let $\sigma > 1$. Then there exists a critical value ε_3^* such that $dsales_{A_2}/df_{A_1} < 0$ if and only if $\varepsilon_3 < \varepsilon_3^*$.*

If the supplying industry A_3 is sufficiently competitive, then A_2 benefits from the bad shock to A_1 , but otherwise, it is hurt by a shock. Intuitively, a negative shock to A_1 reduces the number of varieties in A_1 . This has two effects: (1) household demand substitutes away from A_1 towards A_2 and A_3 (beneficial to A_2), (2) household demand substitutes away from S reducing the number of upstream varieties (harmful to A_2). Which effect dominates depends on the size of ε_3 .

It is important to make three observations here. First, we need $\sigma > 1$, so that industry A_1, A_2, A_3 are symmetric substitutes. If the industries were complements, then a negative shock to one would negatively affect the others, since they would lose demand to the industry that experienced the negative shock. Furthermore, in the case where $\sigma = 1$, in other words Cobb-Douglas, there would be no effect on A_2, A_3 , and A_4 in response to a shock to A_1 , because there would be no substitution effects.

Second, not only do we need $\sigma > 1$, we need external economies to be sufficiently strong in the upstream industry A_3 , so that the negative supply effect dominates the positive demand effect for A_2 . Recall that in the general framework, external economies of scale occur due to the presence of variable markups and love-of-variety effects. In this specific example, where markups are set to

the constant, it is only the love-of-variety effect that matters, and the strength of the variety effect is decreasing in ε_3 .

Finally, knowing everything about the nature of competition in the downstream industries A_1, A_2 , and A_4 is not sufficient to tell us how one industry is affected by a negative shock to the other. This example illustrates that in order to understand these effects, we need to have a more complete picture that takes into account the entire ecosystem that these firms operate in.

5.4.2 Out-of-Equilibrium Effect

Now let us turn to the properties of this model in general. To get intuition for the model's properties, let's consider the following out-of-equilibrium comparative statics: how does industry k 's supplier centrality change when there is entry in industry $i \neq k$? This comparative static holds fixed the mass of firms in all industries except industry i . In equilibrium, of course, the masses in other industries would respond to the increased entry, but it helps our understanding if we first isolate the partial equilibrium effect. If we had lags in entry and exit, these partial equilibrium results would be relevant for understanding short-run effects. Once we have characterized the out-of-equilibrium effects, we turn our attention to the equilibrium responses of the model.

Lemma 5.5. *The derivative of $\tilde{\beta}_k$ with respect to a percentage change in the mass of firms in industry i , holding fixed the mass of firms in all other industries is given by*

$$\frac{\partial \tilde{\beta}_k}{\partial \log M_i} = \left(\frac{\sigma - 1}{\varepsilon_i - 1} \right) \tilde{\beta}_i (\Psi_{ik}^s - 1(i = k)).$$

This expression is very intuitive. The impact to industry k 's centrality as a supplier depends on i 's importance as a supplier $\tilde{\beta}$, and on how much i buys from k (whether directly or indirectly) Ψ_{ik}^s . So big effects are felt if i is a key supplier *and* i is also a key consumer for k . For ease of notation, denote the matrix of these partial derivatives by

$$\frac{\partial \tilde{\beta}}{\partial \log(M\mathbf{1})} = \Psi_1.$$

Similar results apply to the consumer centrality measure.

Lemma 5.6. *The derivative of $\tilde{\alpha}_k$ with respect to a percentage change in the mass of firms in industry i , holding fixed the mass of firms in all other industries is given by*

$$M_i^\sigma \frac{\partial \tilde{\alpha}_k}{\partial \log M_i} = \left(\frac{\sigma - 1}{\varepsilon_i - 1} \right) \left(\frac{\varepsilon_i}{\varepsilon_i - 1} \right)^{\sigma-1} \tilde{\alpha}_i \Psi_{ki}^d.$$

This expression is the demand-side analogue to lemma 5.5. Recall that $\tilde{\alpha}$ is a measure of prices. Lemma 5.6 shows that the impact to industry k 's prices as depends on i 's price $\tilde{\alpha}_i$, and on how

much i sells to k (whether directly or indirectly) Ψ_{ki}^d . So big effects are felt if i is a key consumer of inputs *and* i is a key supplier of inputs to k . Once again, the impact to industry k 's consumer centrality depends on industry i 's suppliers and i 's customers. For ease of notation, denote the matrix of partial derivatives by

$$\frac{\partial \tilde{\alpha}}{\partial \log(M\mathbf{1})} = \Psi_2.$$

5.4.3 Path Example

Before moving on to an analysis of the general equilibrium, let us first demonstrate the intuition of our partial equilibrium results using a simple example of a production chain depicted in figure 5.

Figure 5: The arrows represent the flow of goods and services. Let $\omega_{k-1,k}$ be constant for all k , and β_k , α_k and ε_k be constant for all k .

In this example, each industry k sells some goods directly to the household, and some goods to the industry below it $k - 1$. For ease of exposition, consider $\tilde{\beta}$ and $\tilde{\alpha}$ at the point where all industries have a unit mass of firms. In this example, $\tilde{\beta}_k$ is increasing in k and $\tilde{\alpha}_k$ is decreasing in k . That is, industry N is the most central supplier and least central consumer, and industry 1 is the most central consumer and least central supplier.

Now consider changing the mass of firms in industry N . Note that although industry N is the most central supplier, so $\tilde{\beta}_N > \tilde{\beta}_k$ for $k \neq N$, it has no effect on the supplier centrality of other industries

$$\frac{\partial \tilde{\beta}_k}{\partial M_N} = 0.$$

This is because industry N buys from no other industries. Therefore, despite having the most demand, its direct impact on how much demand reaches other industries is zero.

Now consider changing the mass of firms in industry 1. Industry 1 is the most central consumer so $\tilde{\alpha}_1 > \tilde{\alpha}_k$ for $k \neq 1$. However,

$$\frac{\partial \tilde{\alpha}_k}{\partial M_1} = 0 \quad (k > 1),$$

because industry 1 sells to no other industries. Therefore, despite having the longest supply chain, its direct impact on the supply chains of other industries is zero.

This simple example illustrates why both the in-degrees and the out-degrees will matter for

how the centrality measures will respond to a change in the mass of firms in each industry. Being a central supplier does not mean that entry or exit in your industry will have any effects on the supplier centralities of other industries. Similarly, being a central consumer does not imply that entry or exist in your industry will have any effects on the consumer centralities of other industries. To be influential, a firm must be central both as a consumer and as a supplier.

5.4.4 Equilibrium Impact of Shocks

Using lemmas 5.5 and 5.6 together, we can figure out the general equilibrium impact of a shock to an industry. For ease of notation, let

$$\Lambda = \frac{\partial \log \text{share of sales}}{\partial \log(M)} = \frac{\partial \log(\tilde{\beta}) + \log(\tilde{\alpha})}{\partial \log(M)} = \text{diag}(\tilde{\beta})^{-1}\Psi_1 + \text{diag}(\tilde{\alpha})^{-1}\Psi_2,$$

which is the partial-equilibrium elasticity of the sales distribution of industries with respect to the mass of entrants M . Note that if k is an isolated industry, that is, an industry with connections only to the household, then $\Psi_{ik}^s = \Psi_{ki}^d = 0$ for $i \neq k$. Lemmas 5.5 and 5.6 then imply that $\Lambda_{ik} = \Lambda_{ki} = 0$ for all $i \neq k$. In other words, even though k can have very high supplier centrality $\tilde{\beta}$, consumer centrality $\tilde{\alpha}$, and sales $\tilde{\beta} \times \tilde{\alpha}$, its importance to other industries through the extensive margin can be zero. Interconnectedness matters directly in ways not captured by standard sufficient statistics.

To isolate the effect of the shocks through just the extensive margin, I consider shocks to the fixed costs f_k rather than to labor productivity. Such shocks only propagate through the network due to the change in masses, and therefore give rise to cleaner analytical expressions. However, to keep the results comparable to standard models, I show the effect of labor productivity shocks (which have both intensive and extensive margin effects) on output in Appendix D.

Proposition 5.7. *Let e_k denote the k th standard basis vector. Then, in equilibrium, we have*

$$\begin{aligned} \left(\frac{d \log(M\mathbf{1})}{d \log(f_k)} \right) &= -(I - \Lambda)^{-1} \left(e_k - \mathbf{1} \frac{dP_c^\sigma C / df_k}{P_c^\sigma C} \right) \\ &= - \underbrace{\left(\sum_{t=0}^{\infty} \Lambda^t e_k \right)}_{\text{network effect}} + \underbrace{\text{const}_k}_{\text{general equilibrium}}. \end{aligned}$$

The interpretation of this expression is reminiscent of a shock in traditional input-output models. A fixed cost shock to industry k affects the mass of firms in industry k , which in turn causes entry in upstream and downstream industries, which in turn causes even more entry, and so on ad infinitum. The term $(I - \Lambda)^{-1}$ captures the cumulative effect. The crucial difference here is that Λ is not the input-output matrix anymore. Once again, if an industry k is isolated, then all the non-diagonal elements of the k th row and k th column of $(I - \Lambda)^{-1}$ are zero.

We can also investigate how these fixed cost shocks to an industry affect aggregate output, or welfare.

Proposition 5.8. *The exit centrality v is given by*

$$\tilde{v} \equiv \frac{d \log C}{d \log P} \frac{d \log P}{d \log M} \frac{d \log M}{d \log f} = \frac{1}{\sigma - 1} \frac{1}{\beta' \tilde{\alpha} + \beta' \Psi_2 (I - \Lambda)^{-1} \mathbf{1}} \beta' \Psi_2 (I - \Lambda)^{-1} \propto \underbrace{\beta'}_{\text{tastes}} \underbrace{\Psi_2}_{\text{costs}} \underbrace{(I - \Lambda)^{-1}}_{\text{extensive margin}}.$$

This vector $\tilde{v}$ is the relevant notion of an industry's centrality with respect to small changes to the number of entrants in that industry. This measure has an intuitive interpretation. The effect on output, of a change in the mass of entrants, can be boiled down to the effect on the prices of goods and services the household buys. The chain rule means that this can be broken down into how consumption responds to price changes (*tastes* term), how prices respond to entry (*prices* term), and how entry responds to shocks (*extensive margin* term). Since we have independently characterized each of these terms, it's a matter of combining them. Observe that the exit centrality does *not* capture the effect on total sales, or the effect on total entrants, rather it translates these effects into its ultimate effect on the consumption of the household.

The formula for exit centrality is complicated and hard to interpret intuitively, however, we know that unlike the centrality measures of canonical input-output models, this measure depends directly on interconnectedness, and on each industry's role as both a supplier and as a consumer. Furthermore, it responds endogenously to shocks. It is also worth pointing out that the only endogenous quantities in this formula are the mass of entrants in each industry (which we have not solved for in closed form). Conditional on the mass of entrants, we have closed form expressions for all equilibrium objects. Since, at least in principle, we can observe the mass of products or the mass of firms in each industry in equilibrium, the flexibility of the model allows us to calibrate the model, and to compute exit centralities. This provides a partial response to a critique by Krugman (1997) that systemically important industries, with many forward and backward linkages as described by Hirschman (1958), cannot be deduced from input-output data. This formula gives us such a quantification, albeit it requires more data than can just be learned by looking at input-output tables.

Proposition 5.9. *With entry,*

$$\frac{d \log(C)/d \log(f_k)}{d \log(C)/d \log(f_j)} = \frac{\tilde{v}_k}{\tilde{v}_j}.$$

$$\frac{d \log(C)/d \log(z_k)}{d \log(C)/d \log(z_j)} = \frac{(\tilde{v} + \beta)' \text{diag}(\tilde{\alpha})^{-1} \Psi_d \alpha_k e_k}{(\tilde{v} + \beta)' \text{diag}(\tilde{\alpha})^{-1} \Psi_d \alpha_j e_j}.$$

Unlike in the Cobb-Douglas case, where shocks to production-labor productivity z_k had no extensive margin implications, with $\sigma \neq 1$, changes in production labor productivity now trigger substitutions in the total sales of different industries, and this results in adjustments along both the

extensive and intensive margins. With the intensive margin shocks z_k , the effect can be decomposed fairly cleanly into

$$d \log(C)/d \log(z_k) \propto \underbrace{\tilde{v}' \text{diag}(\tilde{\alpha})^{-1} \Psi_d \alpha_k e_k}_{\text{extensive margin}} + \underbrace{\beta' \text{diag}(\tilde{\alpha})^{-1} \Psi_d \alpha_k e_k}_{\text{intensive margin}},$$

since the latter term is the effect on GDP holding fixed the mass of entrants, and the former takes into account the extensive margin response from the intensive margin shock.

Finally, it is worth noting the following intuitive property of exit-centralities, which holds in the general case.

Corollary. *Exit centrality goes to zero as the industry becomes perfectly competitive*

$$\tilde{v}_k \rightarrow 0, \quad \text{as} \quad \varepsilon_k \rightarrow \infty.$$

5.5 The Macroeconomic Importance of Microeconomic Structure

In general, the properties of exit centrality can be difficult to intuit. However, there is a special case of symmetric economies where the formulas are more easily interpretable, and allow us to demonstrate some insights.

Definition 5.2. We call an economy **regular** if $\sum_j \omega_{ij} = \sum_i \omega_{ij}$, $\alpha_i = \alpha_j$, $\varepsilon_i = \varepsilon_j$, $\beta_i = 1/N$, and $f_i = f_j$ for all i and j .

A regular economy is a symmetric one, and is called regular by analogy to regular graphs from graph theory. Examples would include the circle, the completely connected network, and the network where each industry is only connected to itself. In this context, we can interpret $1 - \alpha$ as an overall measure of connectedness – the lower is α , the higher is the intermediate input share of the economy. On the other hand, we can think of ε as an overall measure of the returns to scale.

Proposition 5.10. *For any regular economy with parameters α, β, f, σ , we have*

$$\tilde{\beta}_i = \tilde{\beta}, \quad \tilde{\alpha} = \tilde{\alpha}, \quad \tilde{v}_i = \tilde{v},$$

and when $\sigma = 1$,

$$\tilde{\beta} = \frac{1}{N}, \quad \tilde{\alpha} = 1, \quad \tilde{v} = -\frac{1}{N\alpha(\varepsilon - 1)}.$$

There are two insights here. First, for the Cobb-Douglas case, the exit centrality depends on the degree of connectivity α whereas the intensive margin centrality $\tilde{\beta}$ does not. So although all regular Cobb-Douglas economies are equivalently stable along the intensive margin, they are not

equivalently stable along the extensive margin. This intuition can be more formally stated as follows.

Corollary. *Consider three different shocks: shocks to the productivity of production workers z_k , shocks to the productivity of creation workers f_k , and shocks to the productivity of all workers A_k . Then for a regular Cobb-Douglas economy with the extensive margin*

$$\log(GDP) = \frac{1}{N} \sum_i (\log(z_k) + \log(A_k)) + \frac{1}{N\alpha(\varepsilon - 1)} \sum_i (\log(A_k) - \log(f_k)).$$

Suppose that $\log(z_k)$, $\log(A_k)$, and $\log(f_k)$ are all identically and independently distributed with variance σ^2 . Then, if $\alpha \rightarrow 0$ at the same rate as $N \rightarrow \infty$, then even in the limit, the regular economy will feature aggregate volatility due to fluctuations along the extensive margin. However, if we shut down the extensive margin by letting $\varepsilon \rightarrow \infty$, then the volatility of the regular economy will decrease by $1/N$.

The second implication of proposition 5.10 is that any two regular economies with the same parameters are equivalently stable to extensive and intensive margin shocks. This confirms that the result established by Acemoglu et al. (2012) holds in this more general case. Somewhat surprisingly, the circular regular economy and the completely connected regular economy will have the same stability in this environment with entry and exit (at least locally). This means that we can plot the exit-centralities for different regular economies, without worrying about the exact nature of the interconnections. These are plotted in figure 6, where we find that exit centrality falls with the number of industries, the elasticity of substitution within and across industries, and with the labor share parameter.

6 Conclusion

This paper highlights the importance of firm entry and exit in propagating and amplifying shocks in a production network. I show that in a model with entry and exit, notions of systemic influence are decoupled from size. The systemic importance of an industry depends on its role as a supplier and as a consumer of inputs, as well as on the market structure of the different industries. Two industries with the same demand-chains can have very different effects on the equilibrium if their supply-chains are different, and vice versa. I derive a new measure of systemic influence called exit centrality that ranks industries by how an exit or entry in that industry would affect output.

I also show that in an environment with non-unitary elasticity of substitution, entry and exit complicates the notions of strategic complementarity and substitutability. Two industries that are substitutes in their product market, can become complements if their supply chains are intertwined and sufficiently uncompetitive. This is because with a non-unitary elasticity of substitution, shocks can reverse directions along the network. In other words, a negative demand

Figure 6: Exit centralities for a regular example varying the different parameters. For each panel, one parameter is varies as the others are held constant. Non-changing parameters of the model are $N = 5$, $\alpha = 0.5$, $\varepsilon = 3$, and $\sigma = 0.5$.

shock to GM can become a negative supply shock to Ford through upstream exits in the auto-parts industry. Similarly, a negative supply shock to the steel mills in Michigan can become a negative demand shock to tire manufacturers through downstream exits in the automobile industry. In a future draft, the model will be calibrated to get a sense of how important these forces may be for the US economy.

A promising area for further research is to characterize how granularity, or minimum scale requirements on firms, may interact with these forces. Gabaix (2011) and Sutton and Trefler (2015) have drawn attention to the fact that certain industries contain very large firms. If firms in these industries are large due to minimum scale requirements, for instance implied by decreasing average cost, then a failure of one of these systemically important firms can snowball into an avalanche of failures. Such domino chain reactions are likely to amplify the results in this paper dramatically, and this is an interesting avenue for future work.

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Local Amplification

The following proposition shows that the equilibrium effect of a productivity shock z_i to industry i 's sales, in percentage change terms, is greatest for industry i . Furthermore, the effect on other industries decays geometrically according to the sum of geometric matrix series Ψ^s . Since sales are proportional to gross profits, the proposition also applies to gross profits.

Proposition .1. *For any economy E , if we fix the mass of firms in each industry, we have*

$$\frac{d \log(\text{sales}_i)}{dz_i} - \frac{d \log(\text{sales}_j)}{dz_i} = \frac{\Psi_{ii}^s \alpha_i}{\tilde{\alpha}_i} - \frac{\Psi_{ji}^s \alpha_i}{\tilde{\alpha}_j} \geq 0.$$

Proof of proposition .1. Using the same argument as in the proof of proposition E.1, we can show that

$$\frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = \frac{1}{\tilde{\alpha}_j} e'_j \Psi \alpha_i e_i + \frac{1}{P_c^\sigma C} \frac{d P_c^\sigma C}{dz_i^{\sigma-1}},$$

This means that

$$\frac{d \log(\text{sales}_i)}{dz_i^{\sigma-1}} - \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = \alpha_i \left(\frac{e'_i \Psi e_i}{\tilde{\alpha}_j} - \frac{e'_j \Psi e_i}{\tilde{\alpha}_j} \right),$$

as required. Lemma E.2 then implies that this is always greater than zero. ■

A Appendix: Proofs

Lemma A.1. *Demand for the output of firm i in industry k is*

$$\begin{aligned} y(k, i) &= c(k, i) + \sum_l \sum_j^{N_l} x(l, j, k, i) \Delta_l, \\ &= \beta_k \left(\frac{p(k, i)}{p_k} \right)^{-\varepsilon_k} M_k^{-\varphi_k \varepsilon_k} \left(\frac{p_k}{p_c} \right)^{-\sigma} C + \sum_l M_l \omega_{lk} \left(\frac{p(k, i)}{p_k} \right)^{-\varepsilon_k} M_k^{-\varphi_k \varepsilon_k} \left(\frac{p_k}{\lambda_l} \right)^{-\sigma} y(l, j), \end{aligned}$$

where λ_l is the marginal cost of the representative firm in industry l .

Proof. Cost minimization by each firm implies firm j in industry l 's demand for inputs from firm i in industry k is given by

$$x(l, j, k, i) = \omega_{lk} \left(\frac{p(k, i)}{p_k} \right)^{-\varepsilon_k} M_k^{-\varphi_k \varepsilon_k} \left(\frac{p_k}{\lambda_l} \right)^{-\sigma} y(l, j),$$

where λ_l is the marginal cost of firms in industry l ,

$$\lambda_k = \left(\alpha_k z_k^{\sigma-1} w^{1-\sigma} + \sum_l \omega_{kl} p_l^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

and p_k is the price index for industry k

$$p_k = \left(M_k^{-\varphi_k \varepsilon_k} \sum_i^{N_k} \Delta_k p(k, i)^{1-\varepsilon_k} \right)^{\frac{1}{1-\varepsilon_k}}.$$

Household demand for goods from firm i in industry k are

$$c(k, i) = \beta_k \left(\frac{p(k, i)}{p_k} \right)^{-\varepsilon_k} M_k^{-\varphi_k \varepsilon_k} \left(\frac{p_k}{P_c} \right)^{-\sigma} C,$$

where P_c is the consumer price index

$$P_c = \left(\sum_k \beta_k p_k^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Adding the household and firms' demands together gives the result, assuming the symmetric equilibrium. ■

Proof of lemma 3.1. By lemma A.1

$$y(k, i) = \beta_k M_k^{-\varphi_k \varepsilon_k} p(k, i)^{-\varepsilon_k} p_k^{\varepsilon_k - \sigma} p_c^\sigma C + \sum_l M_l \omega_{lk} M_k^{-\varphi_k \varepsilon_k} p(k, i)^{-\varepsilon_k} p_k^{\varepsilon_k - \sigma} \lambda_l^\sigma y(l, j).$$

In equilibrium, we can substitute $p_k = M_k^{\frac{1-\varphi_k \varepsilon_k}{1-\varepsilon_k}} p(k, i)$ to get

$$M_k^{\frac{1-\varphi_k \varepsilon_k}{\varepsilon_k - 1} \varepsilon_k - \varphi_k \varepsilon_k} p_k^\sigma y(k, i) = \beta_k p_c^\sigma C + \sum_l M_l \lambda_l^\sigma \omega_{lk} y(l, j). \quad (2)$$

Observe that, in equilibrium,

$$y_k = \left(M_k^{-\varphi_k} \sum_i^{N_k} \Delta_k y(k, i)^{\frac{\varepsilon_k - 1}{\varepsilon_k}} \right)^{\frac{\varepsilon_k}{\varepsilon_k - 1}} = M_k^{\frac{1-\varphi_k}{\varepsilon_k - 1} \varepsilon_k} y(k, i).$$

Furthermore

$$\frac{1 - \varphi_k \varepsilon_k}{\varepsilon_k - 1} \varepsilon + \varphi_k \varepsilon_k = \frac{1 - \varphi_k}{1 - \varepsilon_k} \varepsilon_k.$$

Therefore, we can rewrite (2) as

$$p^\sigma y_k = \beta_k P_c^\sigma C + \sum_l M_l \omega_{lk} \lambda_l^\sigma y(l, j).$$

Now substitute in

$$\lambda_l = \mu_l^{-1} p(l, i) = \mu_l^{-1} p_l M_l^{\frac{1-\varphi_l \varepsilon_l}{\varepsilon_l - 1}}$$

to get

$$p^\sigma y_k = \beta_k P_c^\sigma C + \sum_l \omega_{lk} M_l^{1 + \frac{1-\varphi_l \varepsilon_l}{\varepsilon_l - 1} \sigma - \frac{1-\varphi_l \varepsilon_l}{\varepsilon_l} \varepsilon_l} \mu_l^{-\sigma} p_l^\sigma y_l.$$

Finally, note that

$$1 + \frac{1 - \varphi_l \varepsilon_l}{\varepsilon_l - 1} \sigma - \frac{1 - \varphi_l \varepsilon_l}{\varepsilon_l} \varepsilon_l = \frac{\sigma - 1}{\varepsilon_l - 1} (1 - \varphi_l \varepsilon_l).$$

Recall that $\tilde{M}$ is defined to be the diagonal matrix whose k th diagonal element is $M_k^{\frac{1-\varphi_k \varepsilon_k}{\varepsilon_k - 1}}$, and μ is the diagonal matrix whose k th element is industry k 's markup. So, denote $s_k = p_k^\sigma y_k$. This means that we can write

$$s' = \beta' P_c^\sigma C + s' \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega.$$

Rewrite this to get

$$\begin{aligned} s' &= \beta' (I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1} P_c^\sigma C \\ &= \tilde{\beta}' P_c^\sigma C. \end{aligned}$$

■

Proof of proposition 3.2. Let $A = \mu^{-\sigma} \tilde{M}^{1-\sigma} \Omega$. Note that, by assumption, $A_{jk} = A_{jl}$ for every j . Let $A_{ji}^{(n)}$ denote the j th entry of A^n . We wish to show, by induction, that

$$A_{jk}^{(n)} = A_{jl}^{(n)}, \quad (n \in \{1, 2, \dots\}).$$

To that end, fix $n \in \mathbb{N}$ and assume that

$$A_{jk}^{(n-1)} = A_{jl}^{(n-1)}.$$

Then, it follows that

$$\begin{aligned} A_{jk}^{(n)} &= \sum_i A_{ji} A_{ik}^{(n-1)}, \\ &= \sum_i A_{ji} A_{il}^{(n-1)}, \\ &= A_{jl}^{(n)}. \end{aligned}$$

Since the induction assumption holds for $n = 1$ by assumption, it follows that

$$A_{jk}^{(n)} = A_{jl}^{(n)}, \quad (n \in \{1, 2, \dots\}).$$

Finally, observe that

$$\begin{aligned} \tilde{\beta}_k &= \sum_{n=0}^{\infty} \beta_j A_{jk}^{(n)}, \\ &= \beta_k + \sum_{n=1}^{\infty} \beta_j A_{jk}^{(n)}, \end{aligned}$$

which we have shown is equal to

$$= \beta_k + \sum_{n=1}^{\infty} \beta_j A_{jl}^{(n)},$$

which, since $\beta_k = \beta_l$, equals

$$\begin{aligned} &= \sum_{n=0}^{\infty} \beta_j A_{jl}^{(n)}, \\ &= \tilde{\beta}_l. \end{aligned}$$

■

Proof of proposition 3.3. Similar to the proof of proposition 3.2. ■

Proof of lemma 3.4. Since all firms have constant returns to scale on the margin, firm i 's problem in industry k , conditional on entry, can be written as By definition

$$p(k, i) = \mu_k \lambda_k.$$

Substituting this into the definition of λ_k we get

$$p(k, i) = \mu_k \left(\alpha_k z_k^{\sigma-1} w^{1-\sigma} + \sum_l \omega_{kl} p_l^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Note that, in the symmetric equilibrium,

$$p_k = \mu_k M_k^{\frac{1-\varphi_k \varepsilon_k}{1-\varepsilon_k}} \lambda_k,$$

so,

$$\mu_k^{\sigma-1} M_k^{\frac{1-\varphi_k \varepsilon_k}{\varepsilon_k-1} (1-\sigma)} p_k^{1-\sigma} = \alpha_k z_k^{\sigma-1} w^{1-\sigma} + \sum_l \omega_{kl} p_l^{1-\sigma}.$$

Let P be the vector p_k and let $P^{1-\sigma}$ represent elementwise exponentiation. Then

$$\mu^{\sigma-1} \tilde{M}^{1-\sigma} P^{1-\sigma} = (\alpha \circ z^{\sigma-1}) w + \Omega P.$$

Rearrange this to get

$$P^{1-\sigma} = (I - \mu^{1-\sigma} \tilde{M}^{\sigma-1} \Omega)^{-1} \mu^{1-\sigma} \tilde{M}^{\sigma-1} (\alpha \circ z^{\sigma-1}) w^{1-\sigma}.$$

■

Proof of theorem 3.5. Note that the profits of firm i in industry k are

$$\pi(k, i) = p(k, i) y(k, i) - \lambda_k y(k, i) - w f_k.$$

This is equivalent to

$$\begin{aligned} \pi(k, i) &= p(k, i) y(k, i) - \frac{1}{\mu_k} p(k, i) y(k, i) - w f_k, \\ &= \frac{\mu_k - 1}{\mu_k} p(k, i) y(k, i) - w f_k. \end{aligned}$$

Since all active firms in industry k are identical this is

$$\pi(k, i) = \frac{\mu_k - 1}{\mu_k} \frac{1}{M_k} p_k y_k - w f_k.$$

By lemmas 3.1 and 3.4,

$$p_k y_k = p_k^\sigma y_k p^{1-\sigma} = \tilde{\beta}_k \tilde{\alpha}_k P_c^\sigma C w^{1-\sigma},$$

and so

$$\pi(k, i) = \frac{\mu_k - 1}{\mu_k} \frac{1}{M_k} \tilde{\beta}_k \tilde{\alpha}_k P_c^\sigma C w^{1-\sigma} - w f_k.$$

■

Proof of proposition 4.2. Note that industry k 's sales are given by

$$p_k y_k = p_k^\sigma y_k p^{1-\sigma} = \tilde{\beta}_k \tilde{\alpha}_k P_c^\sigma C w^{1-\sigma}.$$

Observe that in equilibrium with no shocks,

$$\tilde{\alpha} = (I - \Omega)^{-1} \alpha = \mathbf{1}.$$

Therefore,

$$p_k y_k = \tilde{\beta}_k P_c^\sigma C w^{1-\sigma}.$$

So an industry's shares of sales relative to other industries is determined solely by $\tilde{\beta}_k$. ■

Proof of theorem 4.1. Note that in this special case $\Psi_s = \Psi_d = (I - \Omega)^{-1}$. To simplify notation, let $\Psi = (I - \Omega)^{-1}$. First, let us consider real GDP $C(z|E)$ as a function of productivity shocks z for a Cobb-Douglas economy E .

Lemma A.2 (Productivity shock). *Let the elasticity of substitution across industries be equal to one, then*

$$\log(C(z|E)) = \tilde{\beta}'(\alpha \circ \log(z)) = \sum_{k=1}^N \frac{w l_k}{GDP} \log(z_k).$$

That is, the network-structure Ω which has N^2 parameters is summarized by N sufficient statistics. These sufficient statistics are each industry's expenditures on labor as a share of GDP.

This lemma is a slight generalization of results in Acemoglu et al. (2012). If we assume that each industry's labor share is constant, so that $\alpha_i = \alpha$ for all i , and household expenditure shares are uniform so that $\beta_i = 1/N$, then we exactly recover the result in Acemoglu et al. (2012), which tells us that the effect of a productivity shock on real GDP depends on share of sales.

Proof of proposition A.2. Note that real GDP can be written as

$$C = \frac{P_c C}{P_c} = \frac{w l + \pi}{P_c},$$

where π is total profits. By free entry, profits are zero in equilibrium. Normalize $w = 1$. Then

$$\log(C) = -\log(P_c).$$

The marginal costs of firms in industry k are given by

$$\lambda_k = \left(\frac{z_k}{w}\right)^{-\alpha_k} \prod_l p_l^{\omega_{kl}}.$$

Since industries are perfectly competitive, firms set prices equal to their marginal costs. Let P denote the vector of industry prices. Then, in equilibrium,

$$\log(P) = (I - \Omega)^{-1} (\alpha \circ (\log(w) - \log(z))).$$

Therefore,

$$\begin{aligned} \log(C) &= -\log(P_c), \\ &= -\beta' \log(P), \\ &= -\beta' (I - \Omega)^{-1} (-\alpha \circ \log(z)), \\ &= \tilde{\beta}' \alpha \circ \log(z). \end{aligned}$$

Note that $\tilde{\beta}$ is sales as a share of GDP, and $\tilde{\beta}_k \alpha_k$ is therefore industry k 's wage bill as a share of GDP. ■

Corollary (Productivity shock). *Let the elasticity of substitution across industries be equal to one, and all industries have the same labor share α , then*

$$\log(C(z|E)) = \alpha \sum_{k=1}^N \frac{p_k y_k}{GDP} \log(z_k).$$

That is, the network-structure Ω which has N^2 parameters is summarized by N sufficient statistics. These sufficient statistics are each industry's share of sales.

These propositions imply that when $\sigma = 1$, the supplier centrality $\tilde{\beta}$ is a sufficient statistic for translating productivity shocks into real GDP. In particular, the aggregate impact of a vector of shocks depends on the supplier centrality weighted average of the productivity shocks. As lemma 3.1 shows, $\tilde{\beta}_k$ is equal to industry k 's share of sales. Therefore, the bigger the industry in equilibrium, the larger the impact of its productivity shocks on GDP. The exact nature of the network-structure is irrelevant since an industry i could be big because it sells a lot to the household (large β_i) or because it supplies many other firms (the i th column of Ψ is large).

These intuitions also carry over to the case where $\sigma \neq 1$.

Lemma A.3 (Productivity shock). *When the elasticity of substitution across industries is not equal to*

one, then

$$C(\alpha|E) = (\tilde{\beta}'\alpha)^{\frac{1}{\sigma-1}}.$$

That is, the network-structure Ω , which has N^2 parameters, is summarized by N sufficient statistics: $\tilde{\beta}$.

Outside of the Cobb-Douglas case, the supply-side centrality $\tilde{\beta}' = \beta'\Psi_s$ is no longer an industry's share of sales. Lemma 3.1 shows that it is still related to an industry's share of sales however. In fact, even though $\tilde{\beta}'$ is not equal to share of sales globally, it is still closely related to share of sales.

Real consumption is given by

$$\frac{wl + \pi}{P_c} = \frac{1}{P_c}.$$

We have shown that

$$P_c = (\beta'(I - \Omega)^{-1}\alpha)^{\frac{1}{1-\sigma}}.$$

Let $\tilde{\beta}' = \beta'(I - \Omega)^{-1}$ to get the desired result. ■

Proof of proposition ??. Under condition (a), the markups at the industry level are exogenous and determined by the industry-level elasticity of substitution $\mu_k = \varepsilon_k/(\varepsilon_k - 1)$. To derive markups under condition (b), note that firm i in industry k face the following demand function

$$\begin{aligned} y(i, k) &= \left(\frac{p(i, k)}{p_k} \right)^{-\varepsilon_k} y_k, \\ &= \left(\frac{p(i, k)}{p_k} \right)^{-\varepsilon_k} \left(\beta_k \left(\frac{p_k}{P_c} \right)^{-\sigma} C + \sum_l \omega_{lk} M_l \left(\frac{p_k}{\lambda_l} \right)^{-\sigma} y(l, j) \right), \end{aligned}$$

Under condition (b), the firm internalizes its effects on the industry-level aggregates p_k and y_k (but not on economy-wide aggregates like P_c and C) when it sets its prices. Then we can write the profit function of firm i in industry k as being proportional to

$$p(i, k)^{1-\varepsilon_k} p_k^{\varepsilon_k-\sigma} - \lambda_i p(i, k)^{-\varepsilon} p_k^{\varepsilon-\sigma}.$$

Maximizing this gives the first order condition

$$\begin{aligned} (1 - \varepsilon_k) p(i, k)^{-\varepsilon_k} p_k^{\varepsilon_k-\sigma} + (\varepsilon_k - \sigma) p(i, k)^{1-\varepsilon_k} p_k^{\varepsilon_k-\sigma-1} \Delta_k \left(\frac{p(i, k)}{p_k} \right)^{-\varepsilon_k} + \\ \lambda_i p(i, k)^{-1-\varepsilon_k} p_k^{\varepsilon_k-\sigma} - \lambda p(i, k)^{-\varepsilon_k} (\varepsilon_k - \sigma) p_k^{\varepsilon_k-\sigma-1} \Delta_k \left(\frac{p(i, k)}{p_k} \right)^{-\varepsilon_k} = 0. \end{aligned}$$

Rearrange this to get

$$(1 - \varepsilon_k) + (\varepsilon_k - \sigma)\Delta_k \left(\frac{p(i, k)}{p_k} \right)^{1-\varepsilon_k} + \lambda_i \frac{\varepsilon_k}{p(i, k)} - \lambda_i (\varepsilon_k - \sigma)\Delta_k \left(\frac{p(i, k)}{p_k} \right)^{1-\varepsilon_k} = 0.$$

In the symmetric equilibrium, this is

$$(1 - \varepsilon_k) + \frac{\varepsilon_k - \sigma}{N_k} + \frac{\lambda_i \varepsilon_k}{p(i, k)} - \lambda_i \frac{\varepsilon_k - \sigma}{N_k} \frac{1}{p(i, k)} = 0.$$

Rearrange this to get the optimal price as

$$p(i, k) = \mu_k \lambda_i = \frac{\varepsilon_k N_k - (\varepsilon_k - \sigma)}{(\varepsilon_k - 1)N_k - (\varepsilon_k - \sigma)} \lambda_i.$$

In the limit, as $N_k \rightarrow \infty$, this gives an exogenous markup of $\varepsilon_k/(\varepsilon_k - 1)$. In the case where the firm is a monopolist, and $N_k = 1$, this gives the markup $\sigma/(\sigma - 1)$. To make this sensible, we need that $\varepsilon_k > \sigma > 1$.

Finally, to derive markups under condition (c), let $\varepsilon_k \rightarrow \infty$, so that all firms in industry k are producing the same product. Now note that demand for industry k 's output is given by

$$y_k = \sum_j^{N_k} y(k, j) = p_k^{-\sigma} \text{const},$$

where const depends on economy-wide variables the firm treats as exogenous. The firm chooses its quantity to maximize profits

$$p_k y(k, i) - \lambda_k y_k,$$

which gives the first order condition

$$\text{const}^{\frac{1}{\sigma}} - \text{const}^{\frac{1}{\sigma}} \left(\frac{1}{\sigma} \right) \frac{y(k, i)}{y_k} = \lambda_k y_k^{\frac{1}{\sigma}}.$$

Rearrange this to get

$$p_k = \lambda_k \left(1 - \frac{1}{\sigma N_k} \right)^{-1}.$$

■

Proof of lemma 5.5. Recall that

$$\tilde{\beta}' = \beta' (I - S\Omega)^{-1}.$$

So

$$\begin{aligned}
\frac{d\tilde{\beta}'}{dM_i} &= -\beta' \Psi_s \frac{d(I - S\Omega)}{dM_i} \Psi_s, \\
&= \beta' \Psi_s \frac{dS}{dM_i} S^{-1} S\Omega \Psi_s, \\
&= \beta' \Psi_s \frac{d \log S}{dM_i} (\Psi_s - I), \\
&= \tilde{\beta}' \frac{d \log S}{dM_i} (\Psi_s - I).
\end{aligned}$$

So

$$\frac{d\tilde{\beta}_k}{dM_i} = \tilde{\beta}_i \frac{d \log S_i}{dM_i} (\Psi_{ik}^s - \mathbf{1}(i = k)).$$

Assuming the Dixit-Stiglitz structure, so that $S = \tilde{M}^{\sigma-1} \mu^{-\sigma}$ means,

$$\tilde{\beta}' = \beta' (I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1}.$$

So

$$\begin{aligned}
\frac{d\tilde{\beta}'}{d \log(M_i)} &= M_i \frac{d\tilde{\beta}'}{dM_i}, \\
&= -M_i \beta' \Psi_s \frac{d(I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)}{dM_i} \Psi_s, \\
&= M_i \beta' \Psi_s \frac{d\tilde{M}^{\sigma-1}}{dM_i} \mu^{-\sigma} \Omega \Psi_s, \\
&= M_i \beta' \Psi_s \frac{d\tilde{M}^{\sigma-1}}{dM_i} \tilde{M}^{\sigma-1} \tilde{M}^{1-\sigma} \mu^{-\sigma} \Omega \Psi_s, \\
&= M_i \beta' \Psi_s \frac{d\tilde{M}^{\sigma-1}}{dM_i} \tilde{M}^{\sigma-1} (\Psi_s - I),
\end{aligned}$$

The k th element of this vector is

$$\frac{d\tilde{\beta}_k}{d \log(M_i)} = \frac{\sigma - 1}{\varepsilon_i - 1} \tilde{\beta}_i (\Psi_s - I) e_k.$$

Putting this all into a matrix gives

$$\frac{d\tilde{\beta}'}{d \log(M\mathbf{1})} = \text{diag}(\tilde{\beta}) \text{diag}\left(\frac{\sigma - 1}{\varepsilon - 1}\right) (\Psi_s - I).$$

■

Proof of lemma 5.6. Recall that

$$\tilde{\alpha} = (I - D\Omega)^{-1}D\alpha.$$

So

$$\begin{aligned} \frac{d\tilde{\alpha}}{dM_i} &= (I - D\Omega)^{-1} \frac{dD}{dM_i} \Omega (I - D\Omega)^{-1} D\alpha + (I - D\Omega)^{-1} \frac{dD}{dM_i} \alpha, \\ &= (I - D\Omega)^{-1} D D^{-1} \frac{dD}{dM_i} D^{-1} D\Omega (I - D\Omega)^{-1} D\alpha + (I - D\Omega)^{-1} \frac{dD}{dM_i} \alpha, \\ &= \Psi_d D^{-1} \frac{dD}{dM_i} D^{-1} (\Psi_d D^{-1} I) D\alpha + \Psi_d D^{-1} \frac{dD}{dM_i} \alpha, \\ &= \Psi_d \frac{d \log D}{dM_i} D^{-1} \tilde{\alpha}. \end{aligned}$$

So

$$\frac{d\tilde{\alpha}_k}{dM_i} = \tilde{\alpha}_i \frac{d \log D_i}{dM_i} \Psi_{ki}^d \frac{1}{D_i}.$$

Assuming the Dixit-Stiglitz structure, so that $D = \tilde{M}^{\sigma-1} \mu^{-\sigma}$ means,

$$\tilde{\alpha} = (I - \tilde{M}^{\sigma-1} \mu^{1-\sigma} \Omega)^{-1} \tilde{M}^{\sigma-1} \mu^{1-\sigma} \alpha.$$

To simplify the notation, for this proof, let

$$B = (I - \tilde{M}^{\sigma-1} \mu^{1-\sigma} \Omega)^{-1}.$$

So

$$\begin{aligned} \frac{1}{M_i} \frac{d\tilde{\alpha}}{d \log(M_i)} &= \frac{d\tilde{\alpha}}{dM_i}, \\ &= B \frac{d\tilde{M}^{\sigma-1}}{dM_i} \mu^{1-\sigma} \alpha + B \frac{d(\tilde{M}^{\sigma-1})}{dM_i} (\tilde{M}^{\sigma-1})^{-1} (\tilde{M}^{\sigma-1}) \mu^{1-\sigma} \Omega B (\tilde{M}^{\sigma-1}) \mu^{1-\sigma} \alpha, \\ &= B \frac{d\tilde{M}^{\sigma-1}}{dM_i} \mu^{1-\sigma} \alpha + B \frac{d(\tilde{M}^{\sigma-1})}{dM_i} (\tilde{M}^{\sigma-1})^{-1} (B - I) (\tilde{M}^{\sigma-1}) \mu^{1-\sigma} \alpha, \\ &= B \frac{d(\tilde{M}^{\sigma-1})}{dM_i} (\tilde{M}^{\sigma-1})^{-1} B (\tilde{M}^{\sigma-1}) \mu^{1-\sigma} \alpha, \\ &= B \frac{d(\tilde{M}^{\sigma-1})}{dM_i} (\tilde{M}^{\sigma-1})^{-1} \tilde{\alpha}, \\ &= \Psi_d (\tilde{M})^{1-\sigma} \mu^{\sigma-1} \frac{d(\tilde{M}^{\sigma-1})}{dM_i} (\tilde{M}^{\sigma-1})^{-1} \tilde{\alpha}, \end{aligned}$$

The k th element of this vector is

$$\frac{d\tilde{\alpha}_k}{d\log(M_i)} = \left(\frac{\sigma-1}{\varepsilon_i-1}\right) \left(\frac{\varepsilon_i}{\varepsilon_i-1}\right)^{\sigma-1} e'_k \Psi_d e_i \tilde{\alpha}_i \frac{1}{\tilde{M}^{\sigma_i-1}}.$$

Putting this all into a matrix gives

$$\frac{d\tilde{\alpha}}{d\log(M\mathbf{1})} = \Psi_d \text{diag}(\tilde{\alpha}) \mu^{\sigma-1} \tilde{M}^{1-\sigma} \text{diag}\left(\frac{\sigma-1}{\varepsilon-1}\right).$$

■

Proof of proposition 5.7. Observe that, with the Dixit-Stiglitz structure,

$$\log(M\mathbf{1}) = \log(\tilde{\beta}) + \log(\tilde{\alpha}) + \log(P_c^\sigma C)\mathbf{1} - \log(f) - \log(\varepsilon).$$

Hence,

$$\frac{d\log(M\mathbf{1})}{d\log(f_i)} = \Lambda \frac{d\log(M\mathbf{1})}{d\log(f_i)} + \frac{d\log(P_c^\sigma C)}{d\log(f_i)} \mathbf{1} - e_i.$$

Rearrange this to get

$$\frac{d\log(M\mathbf{1})}{d\log(f_i)} = (I - \Lambda)^{-1} \left(\frac{d\log(P_c^\sigma C)}{d\log(f_i)} \mathbf{1} - e_i \right).$$

■

Proof of proposition 5.9. Recall that

$$\log(C) = (\beta' \tilde{\alpha})^{\frac{1}{\sigma-1}},$$

whence

$$\frac{d\log(C)}{d\log(f_i)} = \frac{1}{\sigma-1} \frac{1}{\beta' \tilde{\alpha}} \beta' \frac{d\tilde{\alpha}}{d\log(M\mathbf{1})} \frac{d\log(M\mathbf{1})}{d\log(f_i)}.$$

Since

$$\frac{d\log(P_c^\sigma C)}{d\log(f_i)} = -(\sigma-1) \frac{d\log(C)}{d\log(f_i)},$$

We can write

$$\begin{aligned} \frac{d\log(C)}{d\log(f_i)} &= \frac{1}{\sigma-1} \frac{1}{\beta' \tilde{\alpha}} \beta' \Psi_2 (I - \Lambda)^{-1} \left(\frac{d\log(P_c^\sigma C)}{d\log(f_i)} \mathbf{1} - e_i \right), \\ &= -\frac{1}{\sigma-1} \frac{1}{\beta' \tilde{\alpha}} \beta' \Psi_2 (I - \Lambda)^{-1} \left(e_i + (1-\sigma) \frac{d\log(C)}{d\log(f_i)} \mathbf{1} \right), \\ &= -\frac{1}{1 + \frac{1}{\beta' \tilde{\alpha}} \beta' \Psi_2 (I - \Lambda)^{-1} \mathbf{1}} \frac{\sigma-1}{\beta' \tilde{\alpha}} \beta' \Psi_2 (I - \Lambda)^{-1} e_i. \end{aligned}$$

Divide $d\log(C)/d\log(f_i)$ by $d\log(C)/d\log(f_j)$ to get the result.

■

Proof of proposition 5.1. Note that real GDP can be written as

$$C = \frac{P_c C}{P_c} = \frac{wl + \pi}{P_c},$$

where π is total profits. By free entry, profits are zero in equilibrium. Normalize $w = 1$. Then

$$\log(C) = -\log(P_c).$$

The marginal costs of firms in industry k are given by

$$\lambda_k = \left(\frac{\alpha_k z_k}{w} \right)^{-\alpha_k} \prod_l \left(\frac{\omega_{kl}}{p_l} \right)^{-\omega_{kl}}.$$

Substitute

$$\lambda_k = M_k^{\frac{1}{\varepsilon_k - 1}} \frac{\varepsilon_k - 1}{\varepsilon_k} p_k,$$

and let P denote the vector of industry prices. Then, in equilibrium,

$$\log(P) = (I - \Omega)^{-1} \left(-\alpha \circ \log(z) + \log(\mu \mathbf{1}) - \log(\tilde{M} \mathbf{1}) \right).$$

Free entry implies that

$$\tilde{M}_k = \left(\frac{\tilde{\beta}_k P_c C}{f_k \varepsilon_k} \right)^{\frac{1}{\varepsilon_k - 1}}.$$

Substituting this in to $\tilde{M}$ and combining with the fact that

$$\log(P_c) = \beta' \log(P),$$

gives

$$\log(C) = -\log(P_c) = \beta' (I - \Omega)^{-1} \alpha \circ \log(z) - \sum_k \frac{1}{\varepsilon_k - 1} \tilde{\beta}_k \log(f_k) + \text{const},$$

where

$$\text{const} = \tilde{\beta}' \left(\log(\mu \mathbf{1}) + \frac{1}{\varepsilon - 1} \circ \log(\tilde{\beta}) - \log(\varepsilon) \right).$$

In the above expression $1/(\varepsilon - 1)$ is the vector of $1/(\varepsilon_k - 1)$. ■

Proof of proposition E.5. By theorem 3.5, the profits of industry i are

$$\pi_i = \frac{\beta_i \alpha_i}{f_i \varepsilon_i} P_c^\sigma C,$$

when $\Omega = \mathbf{0}$. Therefore

$$\frac{d\pi_i}{dM_j} = \frac{\beta_i \tilde{\alpha}_i}{f_i \varepsilon_i} \frac{dP_c^\sigma C}{dM_j}.$$

We can write

$$P_c^\sigma C = \frac{1 - M'f}{\beta' \tilde{\alpha} - \text{diag}(\varepsilon)^{-1} \beta' \tilde{\alpha}}.$$

Hence,

$$\frac{dP_c^\sigma C}{dM_j} = \frac{-f_j}{\beta' \tilde{\alpha} - \text{diag}(\varepsilon)^{-1} \beta' \tilde{\alpha}} - \frac{1 - M'f}{\beta' \tilde{\alpha} - \text{diag}(\varepsilon)^{-1} \beta' \tilde{\alpha}} \beta_i (1 - 1/\varepsilon_i) \frac{\sigma - 1}{\varepsilon_i - 1} M_i^{\frac{\sigma - \varepsilon_i}{\varepsilon_i - 1}} \mu_i^{\sigma - 1} \alpha_i.$$

Since $\sigma > 1$, both of these terms are negative and industries can only be enemies. ■

Proof of theorem E.6. Let $\pi(M)$ be the vector industrial profits with mass of entrants M . Note that an equilibrium of the continuous limit corresponds to $M(0)$ such that $\pi(M(0)) = 0$. Let $M(\Delta)$ correspond to the equilibrium mass of entrants for some $\Delta \gg 0$. Then $\pi(M(\Delta)) \geq 0$. By the mean-value inequality

$$\|\pi(M(\Delta))\| = \|\pi(M(\Delta)) - \pi(M(0))\| \leq \|D\pi(M^*)\| \|M(\Delta) - M(0)\|,$$

where M^* is in the convex hull of $M(0)$ and $M(\Delta)$. Rearrange this expression to get the left hand side of the inequality. A similar computation on π^{-1} gives the right hand side, as long as π^{-1} exists over $\text{co}\{M(0), M(\Delta)\}$ ■

B Vertical and Horizontal Example

B.1 Vertical Example

Consider the example where $\beta = (1, 0, \dots, 0)'$, and $\alpha = (0, \dots, 0, 1)'$, and

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For this example σ should not matter. The formula then implies that

$$\tilde{\beta}_k = \beta'(I - \tilde{M}^{\sigma-1} \mu^{-\sigma} \Omega)^{-1} e_k = \prod_{i=1}^{k-1} M_i^{\frac{\sigma-1}{\varepsilon_i-1}} \mu_i^{-\sigma},$$

and

$$\tilde{\alpha}_k = e'_k (I - \tilde{M}^{\sigma-1} \mu^{1-\sigma} \Omega)^{-1} \tilde{M}^{\sigma-1} \mu^{1-\sigma} \alpha = \prod_{i=k}^N M_i^{\frac{\sigma-1}{\varepsilon_i-1}} \mu_i^{1-\sigma}.$$

Then

$$\begin{aligned} \frac{d\tilde{\beta}}{d \log(M\mathbf{1})} &= \left(\text{diag}(\tilde{\beta}) \text{diag} \left(\frac{\sigma-1}{\varepsilon-1} \right) (\Psi_s - I) \right)' \\ &= \left[\text{diag}(\tilde{\beta}) \text{diag} \left(\frac{\sigma-1}{\varepsilon-1} \right) \begin{pmatrix} 0 & M_1^{\frac{\sigma-1}{\varepsilon_1-1}} \mu_1^{-\sigma} & M_1^{\frac{\sigma-1}{\varepsilon_1-1}} \mu_1^{-\sigma} M_2^{\frac{\sigma-1}{\varepsilon_2-1}} \mu_2^{-\sigma} & \dots \\ 0 & 0 & M_2^{\frac{\sigma-1}{\varepsilon_2-1}} \mu_2^{-\sigma} & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \dots & M_{N-1}^{\frac{\sigma-1}{\varepsilon_{N-1}-1}} \mu_{N-1}^{-\sigma} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right]' \\ &= \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \frac{\sigma-1}{\varepsilon_1-1} \tilde{\beta}_2 & 0 & \dots & \dots & 0 \\ \frac{\sigma-1}{\varepsilon_1-1} \tilde{\beta}_3 & \frac{\sigma-1}{\varepsilon_2-1} \tilde{\beta}_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma-1}{\varepsilon_1-1} \tilde{\beta}_N & \frac{\sigma-1}{\varepsilon_2-1} \tilde{\beta}_N & \dots & \frac{\sigma-1}{\varepsilon_{N-1}-1} \tilde{\beta}_N & 0 \end{pmatrix} = \Psi_1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d\tilde{\alpha}}{d \log(M\mathbf{1})} &= \Psi_d \text{diag}(\tilde{\alpha}) \mu^{\sigma-1} \tilde{M}^{1-\sigma} \text{diag} \left(\frac{\sigma-1}{\varepsilon-1} \right), \\ &= \begin{pmatrix} M_1^{\frac{\sigma-1}{\varepsilon_1-1}} \mu_1^{1-\sigma} & M_1^{\frac{\sigma-1}{\varepsilon_1-1}} \mu_1^{1-\sigma} M_2^{\frac{\sigma-1}{\varepsilon_2-1}} \mu_2^{1-\sigma} & \dots & \dots & \prod_{i=1}^N (\tilde{M}_i / \mu_i)^{\sigma-1} \\ 0 & M_2^{\frac{\sigma-1}{\varepsilon_2-1}} \mu_2^{1-\sigma} & \dots & \dots & \prod_{i=2}^N (\tilde{M}_i / \mu_i)^{\sigma-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & M_N^{\frac{\sigma-1}{\varepsilon_N-1}} \mu_N^{1-\sigma} \end{pmatrix} \left[\text{diag}(\tilde{\beta}) \text{diag} \left(\frac{\sigma-1}{\varepsilon-1} \right) \right] \\ &= \begin{pmatrix} \tilde{\alpha}_1 \frac{\sigma-1}{\varepsilon_1-1} & \tilde{\alpha}_1 \frac{\sigma-1}{\varepsilon_2-1} & \dots & \tilde{\alpha}_1 \frac{\sigma-1}{\varepsilon_{N-1}-1} \\ 0 & \tilde{\alpha}_2 \frac{\sigma-1}{\varepsilon_2-1} & \dots & \tilde{\alpha}_2 \frac{\sigma-1}{\varepsilon_{N-1}-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\alpha}_N \frac{\sigma-1}{\varepsilon_{N-1}-1} \end{pmatrix} = \Psi_2. \end{aligned}$$

Now note that that

$$\begin{aligned} \Lambda &= \text{diag}(\tilde{\beta})^{-1} \Psi_1 + \text{diag}(\tilde{\alpha})^{-1} \Psi_2, \\ &= (\sigma-1) \left(\frac{1}{\varepsilon_1-1} \mathbf{1} \mid \dots \mid \frac{1}{\varepsilon_{N-1}-1} \mathbf{1} \right), \end{aligned}$$

where $\mathbf{1}$ denotes a column vector of all ones. This means that

$$\begin{aligned} (I - \Lambda)^{-1} &= I + \Lambda + \Lambda^2 + \dots, \\ &= I + \Lambda + (\sigma - 1) \sum_{i=1}^N \frac{1}{\varepsilon_i - 1} \Lambda + (\sigma - 1)^2 \left(\sum_{i=1}^N \frac{1}{\varepsilon_i - 1} \right)^2 \Lambda + \dots, \\ &= I + \Lambda \frac{\sigma - 1}{1 - (\sigma - 1) \sum_{i=1}^N \frac{1}{\varepsilon_i - 1}}. \end{aligned}$$

Exit centrality, or $d \log(C)/d \log(f_i)$ is then given by

$$\tilde{v} = \frac{1}{\sigma - 1} \frac{1}{\beta' \tilde{\alpha} + \beta' \Psi_2 (I - \Lambda)^{-1} \mathbf{1}} \beta' \Psi_2 (I - \Lambda)^{-1}$$

putting everything together, we get

$$\tilde{v} = \left(\frac{1}{\varepsilon_1 - 1}, \dots, \frac{1}{\varepsilon_N - 1} \right)'.$$

B.2 Horizontal Example

Now consider a standard multi-sector model without an input-output structure, so that $\Omega = \mathbf{0}$ and $\beta_i = 1/N$ for all i and $\alpha_i = 1$. Now note that demand for industry i 's output is given by

$$y_i = \frac{1}{N} \left(\frac{p_i}{p_c} \right)^{-\sigma} C,$$

or

$$p_i y_i = \frac{1}{N} \left(\frac{p_i}{p_c} \right)^{1-\sigma}.$$

Furthermore, we have that

$$p_i^{1-\sigma} = M_i^{\frac{1-\sigma}{1-\varepsilon_i}} \mu_i^{1-\sigma} w^{1-\sigma},$$

and

$$P_c^{1-\sigma} = \frac{1}{N} \sum_i M_i^{\frac{1-\sigma}{1-\varepsilon_i}} \mu_i^{1-\sigma} w^{1-\sigma}.$$

Free entry implies that

$$\begin{aligned}
M_i &= \frac{p_i y_i}{\varepsilon_i f_i} \\
&= \frac{M_i^{\frac{1-\sigma}{1-\varepsilon_i}} \mu_i^{1-\sigma}}{\sum_j M_j^{\frac{1-\sigma}{1-\varepsilon_j}} \mu_j^{1-\sigma}} \frac{1}{\varepsilon_i f_i}, \\
&= \mu_i^{1-\sigma} M_i^{\frac{1-\sigma}{1-\varepsilon_i}} \frac{C^{1-\sigma}}{\varepsilon_i f_i},
\end{aligned}$$

since $C = 1/P_c$. This then implies that

$$\frac{d \log(M_k)}{d \log f_i} = M_k^{\frac{1-\sigma}{1-\varepsilon_k}} \left(\frac{1-\sigma}{1-\varepsilon_k} \right) \frac{d \log M_k}{d \log f_i} - \mathbf{1}(i=k) + \frac{d \log C}{d \log f_i} (1-\sigma),$$

or

$$\frac{d \log(M_k)}{d \log f_i} = \left[-\mathbf{1}(i=k) + \frac{d \log C}{d \log f_i} (1-\sigma) \right] \frac{1}{1 - M_k^{\frac{1-\sigma}{1-\varepsilon_k}} \left(\frac{1-\sigma}{1-\varepsilon_k} \right)}, \quad (3)$$

We also have that

$$\log C = \frac{1}{\sigma-1} \left(\frac{1}{N} \sum_i \mu_i^{1-\sigma} M_i^{\frac{1-\sigma}{1-\varepsilon_i}} \right).$$

This means that

$$\begin{aligned}
\frac{d \log C}{d \log f_i} &= \frac{1}{\sigma-1} \frac{1}{\frac{1}{N} \sum_k \mu_k^{1-\sigma} M_k^{\frac{1-\sigma}{1-\varepsilon_k}}} \frac{1}{N} \sum_j \frac{1-\sigma}{1-\varepsilon_j} \mu_j^{1-\sigma} M_j^{\frac{1-\sigma}{1-\varepsilon_j}} \frac{d \log M_j}{d \log f_i}, \\
&= \frac{1}{\sum_k \mu_k^{1-\sigma} M_k^{\frac{1-\sigma}{1-\varepsilon_k}}} \sum_j \frac{1}{\varepsilon_j - 1} M_j^{\frac{1-\sigma}{1-\varepsilon_j}} \mu_j^{1-\sigma} \frac{d \log M_j}{d \log f_i},
\end{aligned}$$

substituting in (3) and rearranging gives

$$\frac{d \log C}{d \log f_i} = \frac{1}{1-\varepsilon_i} \frac{M_i^{1-\sigma} \mu_i^{1-\sigma}}{1 - M_i^{\frac{1-\sigma}{1-\varepsilon_i}} \frac{1-\sigma}{1-\varepsilon_i}} \left[\sum_j \mu_j^{1-\sigma} M_j^{\frac{1-\sigma}{1-\varepsilon_j}} - \sum \frac{1-\sigma}{1-\varepsilon_j} \frac{M_j^{\frac{1-\sigma}{1-\varepsilon_j}} \mu_j^{1-\sigma}}{1 - M_j^{\frac{1-\sigma}{1-\varepsilon_j}} \frac{1-\sigma}{1-\varepsilon_j}} \right]^{-1},$$

Figure 7: Example inspired by the auto-makers.

Now, evaluate this at $M_i = 1$ for all i to get

$$\begin{aligned}
&= -\frac{1}{\varepsilon_i - 1} \frac{\mu_i^{1-\sigma}}{1 - \frac{1-\sigma}{1-\varepsilon_i}} \left[\sum_j \mu_j^{1-\sigma} + \sum_j \frac{1-\sigma}{1-\varepsilon_j} \frac{\mu_j^{1-\sigma}}{1 - \frac{1-\sigma}{1-\varepsilon_j}} \right]^{-1}, \\
&= -\frac{1}{\varepsilon_i - 1} \frac{\mu_i^{1-\sigma}}{1 - \frac{1-\sigma}{1-\varepsilon_i}} \left[\sum_j \mu_j^{1-\sigma} \left(1 - \frac{1-\sigma}{\sigma - \varepsilon_j} \right) \right]^{-1}, \\
&= -\frac{1}{\varepsilon_i - 1} \frac{\mu_i^{1-\sigma} \left(\frac{\varepsilon_i - 1}{\varepsilon_i - \sigma} \right)}{\sum_j \mu_j^{1-\sigma} \left(\frac{\varepsilon_j - 1}{\varepsilon_j - \sigma} \right)}.
\end{aligned}$$

In the Cobb-Douglas case where $\sigma = 1$, we have

$$\frac{d \log C}{d \log f_i} = -\frac{1}{\varepsilon_i - 1} \frac{1}{N}.$$

C Example Inspired by the Automakers

Consider the economy depicted in figure 7. To simplify the algebra, consider the case where A_3 and A_2 are perfectly competitive industries, but A_1 and S are not. Let $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = 1$. Also, let $\beta = (1/3, 1/3, 0, 1/3)$.

Proposition C.1. *Let $\sigma > 1$. Then there exists a critical value ε_3^* such that $dsales_{A_2}/df_{A_1} < 0$ if and only if $\varepsilon_3 < \varepsilon_3^*$.*

If the supplying industry A_3 is sufficiently competitive, then A_2 benefits from the bad shock to A_1 , but otherwise, it is hurt by a shock. Intuitively, a negative shock to A_1 reduces the number of varieties in A_1 . This has two effects: (1) household demand substitutes away from A_1 towards A_2

and A_3 (beneficial to A_2), (2) household demand substitutes away from S reducing the number of upstream varieties (harmful to A_2). Which effect dominates depends on the size of ε_3 .

It is important to make three observations here. First, we need $\sigma > 1$, so that industry A_1, A_2, A_3 are symmetric substitutes. If the industries were complements, then a negative shock to one would negatively affect the others, since they would lose demand to the industry that experienced the negative shock. Furthermore, in the case where $\sigma = 1$, in other words Cobb-Douglas, there would be no effect on A_2, A_3 , and A_4 in response to a shock to A_1 , because there would be no substitution effects.

Second, not only do we need $\sigma > 1$, we need external economies to be sufficiently strong in the upstream industry A_3 , so that the negative supply effect dominates the positive demand effect for A_2 . Recall that in the general framework, external economies of scale occur due to the presence of variable markups and love-of-variety effects. In this specific example, where markups are set to the constant, it is only the love-of-variety effect that matters, and the strength of the variety effect is decreasing in ε_3 .

Finally, knowing everything about the nature of competition in the downstream industries A_1, A_2 , and A_4 is not sufficient to tell us how one industry is affected by a negative shock to the other. This example illustrates that in order to understand these effects, we need to have a more complete picture that takes into account the entire ecosystem that these firms operate in.

Proof of proposition C.1. Then, we have that

$$M_1 = \frac{1}{3} \frac{\tilde{M}_1^{\sigma-1} \mu_1^{1-\sigma} \tilde{M}_3^{\sigma-1} \mu_3^{1-\sigma}}{f_1 \varepsilon_1} P_c^\sigma C,$$

$$M_3 = \frac{1}{3} \frac{(\tilde{M}_1^{\sigma-1} \mu_1^{1-\sigma} + 1) \tilde{M}_3^{\sigma-1} \mu_3^{1-\sigma}}{f_3 \varepsilon_3} P_c^\sigma C.$$

Furthermore, the sales of industry 2 are given by

$$p_2 y_2 = \frac{1}{3} \tilde{M}_3^{\sigma-1} \mu_3^{1-\sigma} P_c^\sigma C = \frac{1}{3} \tilde{M}_3^{\sigma-1} \mu_3^{1-\sigma} C^{1-\sigma}.$$

This implies that

$$\frac{d \log(p_2 y_2)}{d \log(f_1)} = (\sigma - 1) \left(\frac{1}{1 - \varepsilon_3} \frac{d \log(M_3)}{d \log(f_1)} - \frac{d \log C}{d \log(f_1)} \right),$$

so the sales of A_2 are negatively affected, if

$$\frac{d \log(M_3)}{d \log(f_1)} < \frac{d \log C}{d \log(f_1)}.$$

Note that

$$\frac{d \log(M_1)}{d \log(f_1)} = \frac{\sigma - 1}{\varepsilon_1 - 1} \frac{d \log(M_1)}{d \log(f_1)} + \frac{\sigma - 1}{\varepsilon_3 - 1} \frac{d \log(M_3)}{d \log(f_1)} + (1 - \sigma) \frac{d \log(C)}{d \log(f_1)} - 1,$$

$$\frac{d \log(M_3)}{d \log(f_1)} = \frac{1}{3\tilde{\beta}_3} \mu^{-\sigma} \tilde{M}_1^{\sigma-1} \frac{\sigma - 1}{\varepsilon_1 - 1} \frac{d \log(M_1)}{d \log(f_1)} + \frac{\sigma - 1}{\varepsilon_3 - 1} \frac{d \log(M_3)}{d \log(f_1)} + (1 - \sigma) \frac{d \log(C)}{d \log(f_1)},$$

and

$$\frac{d \log(C)}{d \log(f_1)} = \frac{1}{\beta' \tilde{\alpha}} \frac{1}{3} \left[\frac{1}{\varepsilon_3 - 1} \tilde{M}_3^{\sigma-1} \mu_3^{1-\sigma} (1 + \tilde{M}_1^{\sigma-1} \mu_1^{1-\sigma}) \frac{d \log(M_3)}{d \log(f_1)} + \tilde{M}_1^{\sigma-1} \mu_1^{1-\sigma} \tilde{M}_3^{\sigma-1} \mu_3^{1-\sigma} \frac{d \log(M_1)}{d \log(f_1)} \right].$$

These three equations are collectively a linear system, so denote

$$X = \begin{bmatrix} \frac{d \log(M_1)}{d \log(f_1)} \\ \frac{d \log(M_3)}{d \log(f_1)} \\ \frac{d \log(C)}{d \log(f_1)} \end{bmatrix},$$

and write

$$X = AX - e_1,$$

where e_1 is the first column of the identity matrix and

$$A = \begin{bmatrix} \frac{\sigma-1}{\varepsilon_1-1} & \frac{\sigma-1}{\varepsilon_3-1} & 1-\sigma \\ \frac{\sigma-1}{\varepsilon_1-1} \frac{1}{3\tilde{\beta}_3} \mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1} & \frac{\sigma-1}{\varepsilon_3-1} & 1-\sigma \\ \frac{\tilde{\alpha}_1}{3\beta' \tilde{\alpha}(\varepsilon_1-1)} & \frac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta' \tilde{\alpha}(\varepsilon_3-1)} & 1 \end{bmatrix}.$$

Then we can write

$$X = -(I - A)^{-1} e_1.$$

In order to solve this, we only need to know the first column of $(I - A)^{-1}$, which can be solved for by hand. So,

$$X = \frac{1}{\det A} \begin{bmatrix} \frac{\varepsilon_3 - \sigma}{\varepsilon_3 - 1} + (\sigma - 1) \frac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta' \tilde{\alpha}(\varepsilon_3 - 1)} \\ \frac{\sigma - 1}{\varepsilon_1 - 1} \left(\frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3} - \frac{\tilde{\alpha}_1}{3\beta \tilde{\alpha}} \right) \\ \frac{\sigma - 1}{\varepsilon_1 - 1} \frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3} \frac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta \tilde{\alpha}(\varepsilon_3 - 1)} + \frac{\varepsilon_3 - \sigma}{\varepsilon_3 - 1} \frac{\tilde{\alpha}_1}{3\beta' \tilde{\alpha}(\varepsilon_1 - 1)} \end{bmatrix},$$

where we know that $\det A < 0$, because $d \log(M_1)/d \log(f_1) < 0$. Finally, note that

$$\frac{d \log(p_2 y_2)}{d \log(f_1)} = (\sigma - 1) \left(\frac{1}{1 - \varepsilon_3} \frac{d \log(M_3)}{d \log(f_1)} - \frac{d \log C}{d \log(f_1)} \right),$$

which is a linear combination of the values of X . So, $\frac{d \log(p_2 y_2)}{d \log(f_1)} < 0$ if and only if

$$\frac{1}{\varepsilon_3 - 1} \frac{d \log(M_3)}{d \log(f_1)} < \frac{d \log(C)}{d \log(f_1)}.$$

In other words, we need

$$\frac{\sigma - 1}{\varepsilon_1 - 1} \left(\frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3} - \frac{\tilde{\alpha}_1}{3\beta\tilde{\alpha}} \right) > \frac{\sigma - 1}{\varepsilon_1 - 1} \frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3} \frac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta\tilde{\alpha}} + (\varepsilon_3 - \sigma) \frac{\tilde{\alpha}_1}{3\beta'\tilde{\alpha}(\varepsilon_1 - 1)}.$$

Rearrange this to get

$$\varepsilon_3 < (\sigma - 1) \left(\frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1} \beta' \tilde{\alpha}}{\tilde{\beta}_3 \tilde{\alpha}_1} - 1 \right) - \frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1} + 1}{\tilde{\beta}_3} (\sigma - 1) + \sigma.$$

Simplify this to get

$$\varepsilon_3 < (\sigma - 1) \left(\frac{\mu_1^{1-\sigma} \tilde{M}_1^{\sigma-1}}{\tilde{\beta}_3} \left(\frac{\beta' \tilde{\alpha}}{\tilde{\alpha}_1} - 1 \right) - 1 \right) + \sigma = \varepsilon_3^*.$$

■

D Appendix: Labor Productivity Shocks and the Extensive Margin

Proposition D.1. *The derivative of the equilibrium mass of firms in each industry M with respect to a labor productivity shock in industry k is given by*

$$\left(\frac{d \log(M)}{dz_k} \right) = (I - \Lambda)^{-1} \left(\frac{\mathbf{1} dP_c^\sigma C / dz_k}{P_c^\sigma C} + \text{diag}(\tilde{\alpha})^{-1} \Psi_d e_k \right).$$

This expression is very intuitive. We can rewrite the expression in proposition D.1 as

$$\begin{aligned} \left(\frac{d \log(M)}{dz_k} \right) &= \sum_{t=0}^{\infty} \Lambda^t \left(\frac{\mathbf{1} dP_c^\sigma C / dz_k}{P_c^\sigma C} + \text{diag}(\tilde{\alpha})^{-1} \Psi_s e_k \right) \\ &= \underbrace{\sum_{t=0}^{\infty} \Lambda^t \left(\frac{\mathbf{1} dP_c^\sigma C / dz_k}{P_c^\sigma C} \right)}_{\text{GE effect}} + \underbrace{\sum_{t=0}^{\infty} \Lambda^t \left(\text{diag}(\tilde{\alpha})^{-1} \Psi_s e_k \right)}_{\text{network-structure effect}}. \end{aligned}$$

First, consider the “network-structure” term. The initial impact of a productivity shock to k , holding fixed the mass of firms in each industry, is given by $\text{diag}(\tilde{\alpha})^{-1} \Psi_s e_k$ – this is the traditional input-output effect which captures the total intensity with which each industry uses inputs from industry k . In proposition .1, I show this term captures the extent to which industry sales respond

to shocks when there is no entry/exit. However, when we have entry/exit, the traditional input-output effect must be multiplied by a new term $(I - \Lambda)^{-1}$. This captures how the industry's sizes change in response to the intensive margin shock. Recall that

$$\Lambda = \frac{\partial \log(\tilde{\beta}) + \log(\tilde{\alpha})}{\partial \log(M)} \propto \frac{\partial \log(\text{share of sales})}{\partial \log(M)},$$

where the proportionality sign follows from lemmas 3.1 and 3.4. Therefore, $(I - \Lambda)^{-1}$ captures the cumulative effect of the shock on the size of the industries.

So we see that in equilibrium, a productivity shock to industry k will first affect the masses in all other industries through its effect on the marginal costs of anyone who buys from k . However, the initial change in masses causes the supplier and consumer centralities of all industries to change (captured by Ψ_1 and Ψ_2). This change in centrality measures, in turn, causes the masses to adjust again, and this changes the centralities again, and so on ad infinitum. This gives rise to a new amplification channel captured by the geometric series of Λ . The cumulative effect on the equilibrium of these changes is captured by the “network-structure effect” term. The shock also has an effect on the general price level and real GDP, and the second “GE effect” captures this general equilibrium effect. It is a simple matter to show that in equilibrium, $dP_c^\sigma C/dz_k$ is just a weighted average of the network-structure effects.

Proof of proposition D.1. Note that

$$M_i = \frac{\tilde{\beta}_i \tilde{\alpha}_i}{\varepsilon_i f_i} P_c^\sigma C.$$

Therefore,

$$\frac{d \log(M)}{d\alpha_i} = M \mathbf{1} \frac{dP_c^\sigma C/d\alpha_i}{P_c^\sigma C} + M \text{diag}(\tilde{\beta})^{-1} \frac{d\tilde{\beta}}{dM} \frac{dM}{d\alpha_i} + M \text{diag}(\tilde{\alpha})^{-1} \frac{d\tilde{\alpha}}{dM} \frac{dM}{d\alpha_i} + M \text{diag}(\tilde{\alpha})^{-1} \Psi_d e_i.$$

Rewrite this as

$$\frac{d \log(M)}{d\alpha_i} = M \mathbf{1} \frac{dP_c^\sigma C/d\alpha_i}{P_c^\sigma C} + M \left(\text{diag}(\tilde{\beta})^{-1} \Psi_1 + \text{diag}(\tilde{\alpha})^{-1} \Psi_2 \right) \frac{dM}{d\alpha_i} + M \text{diag}(\tilde{\alpha})^{-1} \Psi_d e_i.$$

Rearrange this to get the desired result. ■

Now, let us analyze how real GDP responds to productivity shocks in this massless limit. This result should be compared to proposition ??, which gave the response when there was no extensive margin of entry and exit. Finally, it helps to compare the output responses of this model to the canonical input-output model.

Proposition D.2. *With free entry,*

$$\frac{d \log(C)/d \log(\alpha_k)}{d \log(C)/d \log(\alpha_j)} = \frac{\tilde{v} (\text{diag}(\tilde{\alpha})^{-1} + I) \Psi_d e_k}{\tilde{v} (\text{diag}(\tilde{\alpha})^{-1} + I) \Psi_d e_j}.$$

The presence of exit centrality $\tilde{v}$, which depends on Ψ_1 and Ψ_2 , and therefore depends on in-degrees and out-degrees, shows that the most influential industry is not simply the industry who supplies the most or consumes the most. It also provides an additional channel of amplification to the purely intensive margin result. Furthermore, output is no longer linearly related to productivity shocks, due to the nonlinear propagation mechanisms of entry and exit. Lastly, since $\tilde{v}$ and Ψ_d are now endogenous, and depend on how many firms are in each industry, the sufficient statistics approach no longer works.

Proof of proposition D.2. Note that

$$C = \frac{P_c C}{P_c} = \frac{1 + \sum_k \pi_k}{P_c} = \left(\frac{1}{\beta' \Psi_d (\alpha \circ z^{\sigma-1})} \right)^{\frac{1}{1-\sigma}}.$$

Therefore,

$$\begin{aligned} \frac{d \log(C)}{dz_i} &= \frac{1}{\sigma-1} \frac{d \left(\log \left(\beta' \Psi_d (\alpha \circ z^{\sigma-1}) \right) \right)}{dz_i}, \\ &= \frac{1}{\sigma-1} \frac{1}{\beta' \tilde{\alpha}} \beta' \frac{\partial \tilde{\alpha}}{\partial M} \frac{\partial M}{\partial z_i} + \beta' \Psi_d \frac{d(\alpha \circ z^{\sigma-1})}{dz_i}, \end{aligned}$$

by lemma 5.6 and proposition D.1,

$$= \frac{1}{\sigma-1} \frac{1}{\beta' \tilde{\alpha}} \beta' \Psi_2 M M^{-1} (I - \Lambda)^{-1} \left(\mathbf{1} \frac{dP_c^\sigma C / dz_i}{P_c^\sigma C} + \text{diag}(\tilde{\alpha})^{-1} \Psi_d e_i \right) + \frac{1}{\beta' \tilde{\alpha}} \beta' \Psi_d e_i \alpha_i z_i^{\sigma-2}.$$

Note that

$$\frac{dP_c^\sigma C / dz_i}{P_c^\sigma C} = \frac{d \log(P_c^\sigma C)}{dz_i} = -(\sigma-1) \frac{d \log(C)}{dz_i}.$$

Substituting this into the previous expression and rearranging gives

$$\left(1 + \frac{\beta' \Psi_2 (I - \Lambda)^{-1} \mathbf{1}}{\beta' \tilde{\alpha}} \right) \frac{d \log(C)}{dz_i} = \frac{1}{\beta' \tilde{\alpha}} \left(\beta' \Psi_2 (I - \Lambda)^{-1} (\text{diag}(\tilde{\alpha})^{-1} + I) \Psi_d e_i \right) \alpha_i z_i^{\sigma-2}.$$

Rearranging this gives the desired result. ■

E Appendix: Local Amplification

Let us briefly investigate the nature of shock propagation in the version of the model without entry. The following proposition shows that the equilibrium effect of a productivity shock z_i to industry i 's sales, in percentage change terms, is greatest for industry i . Furthermore, the effect on other industries decays geometrically according to the sum of geometric matrix series Ψ .

Proposition E.1. *The semi-elasticity of firm i 's sales relative to firm j 's sales with respect to a shock to firm i 's productivity around the steady state $z = \mathbf{1}$ is given by*

$$\frac{d \log(\text{sales}_i)}{dz_i^{\sigma-1}} - \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = \alpha_i (\Psi_{ii} - \Psi_{ji}) > 0.$$

Proof of proposition E.1. To show this, first we need the following technical lemma.

Lemma E.2. *Let A be a non-negative $N \times N$ matrix whose rows sum to one. Let a, b, c be $N \times 1$ vectors in the unit cube with $c = a + b$. Let $B = (I - \text{diag}(1 - c)A)^{-1}$. Then*

$$\frac{e'_i B e_i}{e'_i B a} \geq \frac{e'_j B e_i}{e'_j B a} \quad (i \neq j),$$

where e_i is the i th standard basis vector. When a is strictly positive, then the inequality is strict.

The sales of industry j are given by

$$\tilde{\beta}_j e'_j \Psi (\alpha \circ z^{\sigma-1}) P_c^\sigma C.$$

Therefore

$$\begin{aligned} \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} &= \frac{1}{\tilde{\beta}_j} \frac{d \tilde{\beta}_j}{dz_i^{\sigma-1}} + \frac{1}{\tilde{\alpha}_j} \frac{d e'_j \Psi (\alpha \circ z^{\sigma-1})}{dz_i^{\sigma-1}} + \frac{1}{P_c^\sigma C} \frac{d P_c^\sigma C}{dz_i^{\sigma-1}}, \\ &= 0 + \frac{1}{\tilde{\alpha}_j} \frac{d e'_j \Psi (\alpha \circ z^{\sigma-1})}{dz_i^{\sigma-1}} + \frac{1}{P_c^\sigma C} \frac{d P_c^\sigma C}{dz_i^{\sigma-1}}. \end{aligned}$$

Note that

$$\left. \frac{d \Psi (\alpha \circ z^{\sigma-1})}{dz_i^{\sigma-1}} \right|_{z=1} = \Psi \left. \frac{d (\alpha \circ z^{\sigma-1})}{dz_i^{\sigma-1}} \right|_{z=1} = \Psi \alpha_i e_i.$$

Substitute this in to get

$$\frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = e'_j \Psi \alpha_i e_i + \frac{1}{P_c^\sigma C} \frac{d P_c^\sigma C}{dz_i^{\sigma-1}},$$

where we use the fact that $\tilde{\alpha} = \mathbf{1}$. This means that

$$\frac{d \log(\text{sales}_i)}{dz_i^{\sigma-1}} - \frac{d \log(\text{sales}_j)}{dz_i^{\sigma-1}} = \alpha_i (e_i' \Psi e_i - e_j' \Psi e_i),$$

as required. Lemma E.2 then implies that this is always greater than zero. ■

This proposition formalizes the intuition that a shock has to decay as it travels, and the input-output connections Ω control the rate of decay. The rate of decay can be slow enough to overturn the law of large numbers, as shown by Acemoglu et al. (2012). However, even though the network can slow the decay, proposition E.1 shows that shocks cannot be amplified as they travel out from their source. In order for shocks to industry k to have a big impact, industry k must have strong connections to the household, or strong connections to other industries that have strong connections to the household, and so on. But having strong connections to the household implies that the industry has to be large in equilibrium. The lack of local amplification means that size and influence are closely linked in this class of models. This, and the earlier results in this section, show that as long as firms are small, and they do not supply large fractions of the economy, then shocks to them cannot have significant aggregate effects.

Local Amplification with Entry

The equilibrium impact of a change in an industry's productivity at the firm level is very stark.

Proposition E.3. *In equilibrium, the effect of a productivity shock z_k on the sales of firm i in industry k is the same as its effect on firm j in industry l .*

$$\frac{d \log(\text{sales}_{ki})}{dz_k} - \frac{d \log(\text{sales}_{lj})}{dz_k} = 0.$$

At the firm level, the mass of firms in each industry adjusts to ensure that all firms are equally exposed to productivity shocks. At the industry-level, the conclusion is less stark.

Proof of proposition E.3. Let $\bar{s}$ be the sales of the representative firm in each industry. Then

$$\frac{d \log(\bar{s})}{d\alpha} = \frac{d}{d\alpha} (\log(\tilde{\beta}) + \log(\tilde{\alpha}) - \log(M)) + \frac{d}{d\alpha} \log(P_c^\sigma C).$$

By the chain rule,

$$\begin{aligned}
\frac{d \log(\bar{s})}{d\alpha} &= \left(\text{diag}(\tilde{\beta})^{-1} \frac{d\tilde{\beta}}{dM} + \text{diag}(\tilde{\alpha})^{-1} \frac{d\tilde{\alpha}}{dM} - \text{diag}(M)^{-1} \right) \frac{dM}{d\alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C), \\
&= \left(\text{diag}(\tilde{\beta})^{-1} \Psi'_1 \text{diag}(M)^{-1} + \text{diag}(\tilde{\alpha})^{-1} \Psi_2 \text{diag}(M)^{-1} - \text{diag}(M)^{-1} \right) \frac{dM}{d\alpha} \\
&\quad + \text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C), \\
&= \left(\text{diag}(\tilde{\beta})^{-1} \Psi'_1 + \text{diag}(\tilde{\alpha})^{-1} \Psi_2 - I \right) \text{diag}(M)^{-1} \frac{dM}{d\alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C), \\
&= \left[\left(\text{diag}(\tilde{\beta})^{-1} \Psi'_1 + \text{diag}(\tilde{\alpha})^{-1} \Psi_2 - I \right) \text{diag}(M)^{-1} \text{diag}(M) \left(I - \text{diag}(\tilde{\beta})^{-1} \Psi'_1 - \text{diag}(\tilde{\alpha})^{-1} \Psi_2 \right)^{-1} + I \right] \\
&\quad \text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C), \\
&= 0,
\end{aligned}$$

where the second to last line uses proposition D.1. ■

Proposition E.4. *In equilibrium, the difference in the response of the profits of industry k relative to industry j to a productivity shock to industry k is given by*

$$\frac{d \log(s_k)}{dz_k} - \frac{d \log(s_j)}{dz_k} = (e_k - e_j)' (I - \Lambda)^{-1} \left(\text{diag}(\tilde{\alpha})^{-1} \Psi_s e_k \right). \quad (4)$$

Proof of proposition E.4. Let s be the sales of the industries. Then

$$\frac{d \log(s)}{d\alpha} = \frac{d}{d\alpha} \left(\log(\tilde{\beta}) + \log(\tilde{\alpha}) + \log(P_c^\sigma C) \right).$$

By the chain rule,

$$\begin{aligned}
\frac{d \log(s)}{d\alpha} &= \left(\text{diag}(\tilde{\beta})^{-1} \frac{d\tilde{\beta}}{dM} + \text{diag}(\tilde{\alpha})^{-1} \frac{d\tilde{\alpha}}{dM} \right) \frac{dM}{d\alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C), \\
&= \left(\text{diag}(\tilde{\beta})^{-1} \Psi'_1 \text{diag}(M)^{-1} + \text{diag}(\tilde{\alpha})^{-1} \Psi_2 \text{diag}(M)^{-1} \right) \frac{dM}{d\alpha} + \text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C), \\
&= \left[\Lambda \text{diag}(M)^{-1} \text{diag}(M) (I - \Lambda)^{-1} + I \right] \left[\text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C) \right], \\
&= (I - \Lambda)^{-1} \left[\text{diag}(\tilde{\alpha})^{-1} \Psi_{de_k} + \frac{d}{d\alpha} \log(P_c^\sigma C) \right],
\end{aligned}$$

which gives the desired result. Note that the second to last line of the derivation uses proposition D.1, and the last line uses the fact that

$$\Lambda(I - \Lambda)^{-1} + I = (I - \Lambda)^{-1}.$$

E.1 Lumpy Firms

In this section, we consider the equilibrium where $\|\Delta\| > 0$. As argued by Gabaix (2011) and Sutton and Trefler (2015), certain industries are dominated by very large firms. It turns out that the nonlinear propagation mechanism in this model can behave very differently when firms are not infinitesimal. An equilibrium will now feature as many firms in each industry as possible, in the sense that adding a firm to any industry would drive that industry's profits to be negative.

Let $\Pi : \mathbb{R}_+^N \times \mathbb{R}_+^N \rightarrow \mathbb{R}^N$ be the function mapping the masses of entrants M and vector of productivity shocks z to industrial profits assuming general equilibrium in period 2. In theorem 3.5, I analytically characterize this function.

Definition E.1. A vector of integers $\{N_k\}_{k=1}^N$ is an equilibrium number of entrants if

$$\Pi_i(M_i, M_{-i}, z) \geq 0 > \Pi_i(M_i + \Delta_i, M_{-i}, z) \quad (i \in \{1, 2, \dots, N\}),$$

where $M_j = N_j \Delta_j$, for all j .

Intuitively, a vector of integers is an equilibrium number of entrants if all firms make non-negative profits, and the entry of an additional firm in any industry results in firms in that industry making negative profits. Note that if we let $\Delta_i \rightarrow 0$, the entry inequalities become a zero-profit equality condition.

The model with finitely many firms is a formidable combinatorial problem. Pure-strategy equilibria need not always exist. Technically, the non-existence of a pure-strategy equilibrium means that this model of the economy can feature non-fundamental or non-exogenous randomness. Intuitively, pure-strategy equilibria can fail to exist due to cycling, where the entry of a firm causes the profits of another industry to go negative. Once a firm from that industry exists, another firm can enter that causes the profits of the first entrant to be negative, and so on.

An equilibrium when $\|\Delta\| > 0$ is a solution to a nonlinear integer programming problem. Results from computational complexity theory suggest that we cannot hope to fully characterize the set of equilibria. A naive brute-force computation would become infeasible very rapidly, since with even a few hundred firms, we may need to consider more cases than there are atoms in the observable universe! Therefore, a full-fledged theoretical analysis of the model with lumpy firms is left for future work. In this paper, I simply highlight the importance these granularities can have by way of illustrative examples. Our approach here will be to compare the discontinuous model's equilibria to the equilibria of the continuous model, with the hopes of gleaning some intuition. To demonstrate some of the subtleties, consider the following example illustrated in figure 8.

Figure 8: Two production lines selling to the household HH. Industries 2 – 6 form on production line and industry 1 forms a second (degenerate) production line. The direction of arrows represent the flow of goods and services. To keep the numerical example simple, I assume the Dixit-Stiglitz market structure.

$$\begin{aligned}
 \beta &= (3/5, 2/5, 0, 0, 0, 0)' , \\
 f &= (0.01, 0.01, 0.01, 0.01, 0.01, f_6), \\
 \sigma &= 1.1, \\
 \varepsilon &= (3, 3, 3, 3, 3, 1.2). \Delta = (1, 1, 1, 1, 1, 0.5).
 \end{aligned}$$

Let us consider the equilibrium mass of entrants $M(\Delta, f_6)$ for different values of f_6 and mass vector Δ . First, let us consider the massless limit,

$$M(\mathbf{0}, 0.04) = \begin{pmatrix} 15.7 \\ 17.6 \\ 10.1 \\ 5.8 \\ 3.3 \\ 1.2 \end{pmatrix}, \quad M(\mathbf{0}, 0.05) = \begin{pmatrix} 16.9 \\ 16.2 \\ 9.4 \\ 5.3 \\ 3.0 \\ 0.8 \end{pmatrix}, \quad M(\mathbf{0}, 0.06) = \begin{pmatrix} 17.8 \\ 15.4 \\ 8.7 \\ 4.9 \\ 2.7 \\ 0.6 \end{pmatrix}, \quad M(\mathbf{0}, 0.07) = \begin{pmatrix} 18.7 \\ 14.6 \\ 8.2 \\ 4.6 \\ 2.5 \\ 0.5 \end{pmatrix}.$$

We see how increasing the fixed costs of the final supplier in the long chain reduces the mass of firms active in the long chain and increases the mass of firms in the competing short chain in a continuous and intuitive way.

Now, let us consider this same equilibrium away from the massless limit.

$$M(\Delta, 0.04) = \begin{pmatrix} 16 \\ 17 \\ 10 \\ 5 \\ 3 \\ 1 \end{pmatrix}, \quad M(\Delta, 0.05) = \begin{pmatrix} 19 \\ 15 \\ 8 \\ 4 \\ 2 \\ 0.5 \end{pmatrix}, \quad M(\Delta, 0.06) = \begin{pmatrix} 19 \\ 14 \\ 8 \\ 4 \\ 2 \\ 0.5 \end{pmatrix}, \quad M(\Delta, 0.07) = \begin{pmatrix} 33 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In going from $f_6 = 0.04$ to $f_6 = 0.05$, the discontinuous model amplifies the impact of the shock because the shock is large enough to force a discontinuous change. From $f_6 = 0.05$ to $f_6 = 0.06$ however, the discontinuous model attenuates the impact of the shock since the firms are large enough that they can absorb the losses without exiting. This represents a phase transition since shocks below a threshold are attenuated, and above that threshold, they are amplified. In going from $f_6 = 0.06$ to $f_6 = 0.07$, the interior equilibrium disappears, and the long supply chain collapses. Furthermore, this is the unique equilibrium, so this is *not* a coordination failure resulting from equilibrium switching. The situation is illustrated in figure 9.

This example suggests that under some conditions, we can use the continuous model to approximate the lumpy model. But it also warns that this approximation can break down dramatically. These differences are not only large, but they can also be very complex. Figure 10 illustrates that when dealing with firm entry and exit, even in an example as simple as this one, the size, number, sales and prices of firms in an industry provide a poor guide to how important those industries are to the equilibrium. As it happens, despite having the lowest share of sales, industry 6 has the highest exit centrality. On the other hand, industry 1 is by far the largest industry, features the highest sales, and the lowest costs, but it has the smallest exit centrality.

E.2 Approximation Error

A combinatorial analysis of the lumpy model is left for future work, however, we can gain some intuition for the behavior of this model by comparing its behavior to the limiting case $\Delta \rightarrow 0$. We can ask when should we expect the continuous model, which is tractable, to provide a good approximation to the lumpy model, which is intractable. The lumpy model can behave very differently to the continuous model for reasons that are related to numerical instability in finding the roots of certain systems of equations. In numerical analysis, such problems are called ill-conditioned. In this section, I show that a similar intuition applies to this problem, where the difference between the lumpy and continuous model explodes under economically interpretable conditions.

To formalize the approximation error, we need a few definitions. These definitions have some resemblance to the concepts of *natural friends* and *natural enemies* in international trade theory.

Figure 9: Masses of industries with different levels of f_6 . Note that the discrete model amplifies the shock when going from the first panel to the second panel, attenuates it in going from the second to the third, and experiences a chain reaction in going from the third to the fourth.

Figure 10: Various measures of influence plotted for the mass of entrants $M(\Delta, 0.04)$ as an example. We see that the exit centrality measure is not at all related to any of the other measures of centrality. Specifically, the final industry is the most exit-central, despite having the lowest overall sales. Recall that supplier centrality $\tilde{\beta}$ measures the importance of the industry as a supplier to the household, and the consumer centrality $\tilde{\alpha}$ measures the importance of the industry as a consumer of labor inputs from the household.

Motivated by the Stolper and Samuelson (1941) result, Deardorff (2006) defines an industry to be a natural friend (enemy) of a factor when an increase in its prices will increase (decrease) the returns to that factor. Inspired by these terms, I define the following.

Definition E.2. Let $\pi_i(M_1, \dots, M_N)$ be the profit function of industry i . Industry j is an *enemy* of industry i when

$$\frac{\partial \pi_i}{\partial M_j} < 0,$$

and industry j is a *friend* of industry i when

$$\frac{\partial \pi_i}{\partial M_j} > 0.$$

Industry j is a *frenemy* of industry i when

$$\frac{\partial \pi_i}{\partial M_j}$$

takes both positive and negative values.

Although the definition is reminiscent of the one in Deardorff (2006), these definitions are *out-of-equilibrium*. In other words, we change the mass of firms in one industry without changing the masses in other industries.

When industries are frenemies, the entry decisions of firms in one industry are strategic substitutes for some regions and strategic complements for other regions. When industries are substitutes, frenemies can only occur with non-degenerate (non-diagonal) input-output connections.

Proposition E.5. Let $\sigma > 1$. When the network structure Ω is degenerate (diagonal), all industries are enemies. When the network is non-degenerate, industries can be frenemies.

It turns out that we can guarantee that approximating the discontinuous model with a continuous limit will be bad when firms are frenemies.

Theorem E.6. Let $M(\Delta)$ correspond to the mass of firms in each industry in an equilibrium where firm-level masses are given by Δ . Let $M(0)$ denote equilibrium masses in each industry when $\Delta \rightarrow 0$. Let

$$\|D\pi(\tilde{M})\| = \sup \{\|D\pi(M)\| : M \in \text{co}\{M(0), M(\Delta)\}\}, \quad \|D\pi(\hat{M})^{-1}\| = \sup \{\|D\pi(M)^{-1}\| : M \in \text{co}\{M(0), M(\Delta)\}\}.$$

As long as these exist, then

$$\|\pi(M(\Delta))\| \|D\pi(\tilde{M})\|^{-1} \leq \|M(\Delta) - M(0)\| \leq \|\pi(M(\Delta))\| \|D\pi(\hat{M})^{-1}\|.$$

In particular, the error gets larger when the profit functions have flat slopes, which can only occur if industries are frenemies. In other words, the approximation errors explode when the

entry decisions of firms in two industries switch from being strategic substitutes to being strategic complements. On the other hand, we see that when profits are small, then the discrete equilibrium will be close to the continuous equilibrium. For the discrete model and continuous model to give similar answers, the profit function, and inverse profit function, must not change dramatically when there are small changes in the mass of players. Or in the language of Trefethen and Bau (1997), these conditions ensure that we get “nearly right answers to nearly right questions.”

E.3 Bail-outs and the Nature of Externalities

The example in figure 8 suggests that bail-outs and government interventions may be desirable in the lumpy model, since the failure of an important firm or industry can take down entire parts of a network, and cause a jump from one equilibrium to another.⁶ However, figuring out which failures are efficient and which ones are not, in general, is very difficult. To implement the optimum for the discrete model, a social planner would not only need access to an infeasible amount of information, but it would also have to solve an intractable computational and coordination problem.

Instead, inspired by the example where the president of Ford testified in favor of GM, we can imagine a scenario where the policy-maker can, in response to a shock, survey firms if other firms should be rescued. The firms will truthfully tell the policy-maker whether a rescue would increase their profits or decrease their profits. Is it the case that we can trust the testimony of other firms, particularly if those firms compete with the party that received the shock? Unfortunately, the answer to this question is, in general, negative. To see this consider the example in figure 11, and suppose that GM is hit with a negative fixed cost shock. If GM's exit causes D to also exit, then it can be in F's interest to lobby for a bailout even if it is more efficient for the entire left-hand side of the figure to disappear.

To see this, recall that the profit function of a firm in industry k is given by

$$g(N_k)P_c^\sigma \tilde{\beta}_k \times \tilde{\alpha}_k - f_k,$$

where g is a decreasing function of the number of competitors N_k in industry k . Denote the equilibrium with no bail-out as NB and the equilibrium with a bail-out as B. Ideally, we would want to find a firm i such that

$$\pi_i^B > \pi_i^{NB} \Rightarrow C^B > C^{NB}.$$

However, the presence of the network terms $\tilde{\beta}$, $\tilde{\alpha}$, the price-level term P_c^σ , and the competition term $g(N_k)$ mean that profits may not move in the same direction as real GDP. So, F may want GM rescued to protect D, even if it results in lower consumption C . In general, it is impossible

⁶In fact, even in the continuous limit, as long as the network-structure is not degenerate, the entry margin is inefficient in this model. Loosely speaking, double-marginalization means that upstream industries, appropriately defined, receive too little demand and are too small in equilibrium. A full analysis of optimal industrial policy is beyond the scope of this paper.

Figure 11: A stylized example of a supply chain. HH is household; GM is General Motors; F is Ford Motor Company; D is Dunlop Tires; M is Mitsubishi Motors; T is Toyota; and, B is Bridgestone Tires, a Japanese tire manufacturer. The arrows represent the flow of goods and services.

to guarantee that another firm's testimony is reliable, even if that firm is completely disconnected from origin of the shock. So that, in figure 11, even pure competitors like T or M may not give reliable feedback on whether or not GM should be bailed out.