

# On Interest Rate Policy and Asset Bubbles\*

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## Abstract

In a provocative paper, Gali (2014), showed that a policymaker who raises interest rates because of concerns about a bubble will paradoxically make the bubble bigger. In this paper, we argue Gali's framework abstracts from the possibility that a policymaker who raises rates might crowd out resources that would have otherwise been spent on the bubble. We show that when we modify Gali's model to allow for this possibility, interventions that lead to higher interest rates can dampen bubbles. However, even if raising rates effectively dampens bubbles, such an intervention is not Pareto improving in the modified version of Gali's model we analyze. We then show that if we modify the model so that it can generate the type of credit-driven bubbles policymakers worry about, raising rates may still be effective against bubbles, and that there may be scope for such interventions to make society better off.

## Preliminary and Incomplete

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# Introduction

In a provocative paper, Gali (2014) argued that a policymaker who raises interest rates because she is concerned about the possibility of an asset bubble – a policy known as “leaning against the wind” – will paradoxically make the bubble larger if one is present. This result stands in sharp contrast to conventional wisdom on how interest rate policy can be used as a tool for financial stability. Although central banks have long been reluctant to use interest rates to combat asset bubbles, this was not out of concern that raising rates would exacerbate bubbles. To the contrary, policymakers viewed interest rates as an effective but blunt tool that would not only mitigate bubbles but also affect economic activity and inflation in ways that may not be desirable, a view formalized in the work of Bernanke and Gertler (1999). Gali’s result forces the question of whether conventional wisdom on interest rate policy and bubbles is flawed, or whether conventional wisdom is correct but relies on features absent in his model. Gali was aware of the latter possibility, and cautioned as much in his paper. For example, he concludes his paper by noting that his model assumes agents are rational, while in practice policymakers may be concerned about bubbles driven by “irrational exuberance” – a phrase notably coined by a central banker.<sup>1</sup>

In this paper, we argue that Gali’s model precludes a more elementary reason for why raising rates can mitigate bubbles, even when agents are rational. In particular, we show that Gali’s framework rules out the possibility that raising interest rates might crowd out resources that would otherwise have gone to the bubble. Essentially, Gali considers an economy in which a higher interest rate is associated with higher savings in equilibrium. Since Gali assumes the bubble is the only asset agents can use to save, and since this asset is available in fixed supply, then the price of the asset must rise when agents attempt to save more. A higher interest rate draws more resources towards the bubble asset rather than crowd them out.

We demonstrate that under modest modifications to an economy that is qualitatively similar to the one Gali analyzed, policy interventions that raise interest rates can crowd out resources from the bubble asset and so dampen an existing bubble, just as conventional wisdom dictates. For example, if we allow agents to save using either the bubble asset or government bonds as opposed to just the bubble asset, policy interventions that lead the public to hold more government debt raise interest rates and dampen the bubble. These interventions succeed in dampening the bubble because they leave agents with fewer resources to buy the bubble asset. As another example, if higher interest rates induce a temporary fall in output, as is true of many models that feature nominal price rigidity, agents may reduce their savings as interest rates rise. In this case, agents would spend less on the bubble asset as their income temporarily contracts, and the extent to which the asset is overvalued would once again fall.

One of the contributions of this paper, then, is to identify a channel through which leaning-against-the-wind policies can be effective against bubbles. This not only offers a contrast to Gali’s result that raising interest rates is counterproductive, but provides microfoundations for the view common among policymakers

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<sup>1</sup>See Greenspan (1996).

that raising rates helps mitigate bubbles.<sup>2</sup> Developing such a framework offers several insights on the use of interest rate policy to combat bubbles. For example, the channel we identify suggests policymakers seeking to use interest rate policy to combat bubbles should not simply be satisfied if their intervention succeeds in raising real interest rates, but should also seek to gauge whether the rise in interest rates is accompanied by lower savings or a change in the portfolio held by the public that would be consistent with crowding out. Our results also suggest that it may be difficult to dampen asset bubbles without inducing a recession, since a reduction in the resources agents earn and can use to buy assets may be essential for dampening the bubble. In one of our examples, the only reason the bubble is mitigated is because economic activity falls.

While our analysis confirms that policymakers *can* dampen asset bubbles by raising rates, as is widely believed in policy circles, a separate question is whether they *should* do so. In our benchmark model that draws on Gali's setup, interventions that raise rates and dampen bubbles do not constitute Pareto improvements. This benchmark model also fails to speak to the concerns policymakers express about bubbles, namely that letting a bubble grow and then collapse could trigger a wave of defaults and a possible financial crisis. This view is expressed, for example, Mishkin (2011). In the second part of the paper, then, we modify our benchmark model to incorporate credit and the possibility of bursting bubbles. These modifications give rise to a different type of asset bubble, one which we argue comes closer to capturing the type of credit-driven bubbles policymakers identify as a key source of concern. We show that even with these bubbles, there can be interventions that raise interest rates and crowd out resources that would have otherwise been spent on asset bubbles. Such interventions still fail to produce a Pareto improvement if the bubble asset is available in fixed supply. However, when the supply of the bubble asset is variable, there may be scope for combating bubbles before they burst in order to discourage the creation of more bubble assets and thus reducing the amount of aggregate risk in the economy. This logic would suggest that the case for intervention is weaker for bubbles on assets like land whose stock cannot be easily augmented than for bubbles on assets like housing or commercial real estate whose supply can respond to changes in prices.

The paper is organized as follows. In Section 1, we lay out our model and use it to demonstrate Gali's original result that in a model where agents can only save using the bubble asset, a higher real interest rate will be associated with a larger bubble. In Section 2, we allow agents who wish to save to buy both a bubble asset and government bonds, and show that in this environment there exists a policy intervention that drives up interest rates and reduces the size of the bubble. In Section 3 we allow agents to hold the bubble asset, bonds, and money, and show that there exist monetary interventions that dampen the bubble as well. Since neither of these settings involve borrowing against bubble assets, we modify the model in Section 4 to allow for credit-driven bubbles, i.e. bubbles that arise when agents borrow with the intent of profiting if the asset rises in value enough and defaulting otherwise. We argue that in this case, interventions that raise rates can still serve to dampen bubbles, and that these models more closely correspond to the type of bubbles policymakers are worried about. We conclude in Section 5.

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<sup>2</sup>In independent work, Dong, Miao, and Wang (2017) offer a different model in which a move to raise rates can dampen bubbles. Their analysis uses a model with credit market frictions where agents can hold bubble assets as collateral and thus relax the constraints they face. Monetary policy impacts credit market frictions, which in turn affects bubble assets.

# 1 Reproducing Gali's Result

We begin by replicating Gali's result. The environment we present differs along some dimensions from the one Gali described, although both are essentially variants of Samuelson (1958). The framework we use is analytically more convenient to work with, and we will argue below that our framework captures the key features behind Gali's results.

Time is discrete and indexed  $t = 0, 1, 2, \dots$ . There is a single consumption good available at each date. This good can be stored at no cost, allowing agents to convert a unit of consumption goods at date  $t$  into a unit of consumption goods at date  $t + 1$ .

We assume overlapping generations of agents with two-period lives. Agents are risk neutral. They derive utility only from the goods they consume in the second period of life. Specifically, the utility of agents born at date  $t$  from consumption  $c_t$  and  $c_{t+1}$  at dates  $t$  and  $t + 1$ , respectively, is given by

$$u(c_t, c_{t+1}) = c_{t+1} \tag{1}$$

The cohort born at date  $t$  is endowed with  $e_t > 0$  units of consumption goods when young and none when old. The amount of resources successive cohorts are endowed with grows over time according to

$$e_t = (1 + g)^t e_0 \tag{2}$$

The key problem agents in this environment face is that they need to convert the goods they are endowed with when young into goods they can consume when old. They can do this on a one-for-one basis by storing their endowment. However, they might be able to do even better by exchanging their endowment for assets when young and then back into goods when old.

Gali essentially assumes there is only one asset that agents can trade.<sup>3</sup> We begin with this case as well, although we eventually relax this assumption. The asset is available in fixed supply, normalized to 1, and yields a constant dividend flow of  $d \geq 0$  consumption goods per period. Gali assumed  $d = 0$ , rendering the asset intrinsically worthless. For now, we also assume  $d = 0$ , although we will eventually allow  $d > 0$ . Assuming  $d > 0$  is not only more realistic given that the historical episodes often described as bubbles involve dividend-bearing assets, but it turns out to matter for the determinacy of equilibria.

All shares in the asset are initially endowed to those who are old at date 0. We use  $p_t$  to denote the price of a share of the asset at date  $t$ , measured in units of consumption goods. Given our normalization, this is also the total market value of the asset.

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<sup>3</sup>More precisely, Gali also allows agents to borrow and lend to one another, so in principle agents can hold privately issued debt as well. But since only the young would ever borrow or lend and all young agents are assumed to be identical, there is no borrowing or lending between agents in equilibrium. The bubble asset is thus the only asset agents hold. We return to this point in Section 4, where we introduce within-cohort heterogeneity to allow private debt to circulate in equilibrium.

Asset market clearing in our overlapping generations economy requires that at each date  $t$ , the old agree to sell all of their asset holdings to the young. This ensures all assets will be owned by someone each period. Hence, an equilibrium is a path of prices  $\{p_t\}_{t=0}^{\infty}$  such that for each date  $t$ , the old are willing to sell their asset holdings for a price  $p_t$  and the young are willing to buy them at this price. In principle,  $\{p_t\}_{t=0}^{\infty}$  can be a sequence of random variables. However, we restrict attention to deterministic price paths.

Let  $r_t$  denote the rate of return that those who buy the asset at date  $t$  anticipate to earn from it in equilibrium. Since we are assuming the asset yields no dividends, the return to buying the asset at date  $t$  is just the rate at which the price of the asset grows between dates  $t$  and  $t + 1$ , i.e.

$$1 + r_t = \frac{p_{t+1}}{p_t} \quad (3)$$

We will refer to  $r_t$  as the real interest rate. The presence of a storage technology implies that  $r_t \geq 0$  for all  $t$  in equilibrium. Otherwise, the asset market would fail to clear, since the young at date  $t$  would not want to hold the asset and would prefer storage.

To characterize the equilibrium paths  $\{p_t\}_{t=0}^{\infty}$ , consider first the possibility that the real interest rate  $r_t$  at date  $t$  implied by a candidate equilibrium path of asset prices is positive. In this case, the cohort born at date  $t$  would strictly prefer buying the asset to storage. Since we normalized the supply of the asset to 1, and since agents will trade all of their endowment for the asset, it follows that

$$p_t = e_t$$

Hence, in any period  $t$  in which the interest rate  $r_t > 0$ , the price of the asset is uniquely determined.

If the real interest rate were instead equal to zero at date  $t$ , i.e.  $r_t = 0$ , the young would be indifferent between storing their endowment and using it to buy the asset. While the equilibrium price  $p_t$  would no longer be unique, there is still something we can say about the path of prices in this case. Specifically, if  $r_t = 0$ , then (3) implies  $p_{t+1} = p_t$ . Since  $p_t \leq e_t$ , then the fact that  $r_t = 0$  implies the price in the next period  $p_{t+1}$  will necessarily be less than the endowment of agents, since

$$p_{t+1} = p_t \leq e_t < e_{t+1}$$

But this implies  $r_{t+1} = 0$ , since if agents at date  $t + 1$  do not use all their endowment to buy the asset, storage must not be dominated. Hence, a zero real interest rate is absorbing: Once the real interest rate falls to 0, it will remain there indefinitely. Moreover, since the price of the asset grows at the rate of interest, it follows that once  $r_t = 0$ , from then on the price of the asset will remain equal to its value at date  $t$ .

The above observations allow us to fully characterize all deterministic equilibrium price paths in our economy. In particular, any equilibrium path for asset prices can be characterized in terms of a cutoff date  $t^*$ . Before the cutoff date  $t^*$ , the interest rate  $r_t$  is positive and the asset price is equal to  $e_t$ , while after cutoff date  $t^*$  the interest rate is zero and the asset price remains constant. At the cutoff date  $t^*$ , the price of the asset is indeterminate, although it must be at least  $e_{t^*-1}$  and at most  $e_{t^*}$ . Formally, we have

**Proposition 1:** Suppose  $d = 0$ . A deterministic path  $\{p_t\}_{t=0}^{\infty}$  is an equilibrium iff there exists a cutoff date  $t^*$  with  $0 \leq t^* \leq \infty$  such that

$$p_t = \begin{cases} e_t & \text{if } t < t^* \\ p_{t^*} & \text{if } t > t^* \end{cases} \quad (4)$$

and  $p_{t^*}$  can assume any value in  $[e_{t^*-1}, e_{t^*}]$ .

As in Gali's economy, the model admits multiple equilibrium paths. This indeterminacy implies that for date  $t$ , any value  $\rho \in [0, e_t]$  can be observed in equilibrium. However, the set of equilibria exhibits a particular structure, in the sense that if  $p_t$  assumes a particular value  $\rho$  at some date  $t$ , the asset price either before or after date  $t$  can only assume certain values.

In order to determine whether the asset in this economy should be viewed as a bubble, we first need to define a fundamental value for the asset to which we can compare its price. We follow the common practice of defining the fundamental value of the asset  $f_t$  as the value of the dividends the asset is expected to generate, properly discounted to date  $t$ . When  $d = 0$ , what discount rate we use is irrelevant as  $f_t$  would equal 0 for any discount rate. We will refer to an asset as a bubble whenever  $p_t \neq f_t$ , i.e. when the asset trades at a price that deviates from its fundamental value, and we define the size of the bubble as

$$\Delta_t \equiv p_t - f_t \quad (5)$$

Since the fundamental value  $f_t$  of an intrinsically worthless asset is 0, then in the special case where  $d = 0$ , the size of the bubble  $\Delta_t = p_t$ , i.e. the bubble and the price of the asset are one and the same.

We now turn to the question of how changing the path of interest rates  $\{r_t\}_{t=0}^{\infty}$  would affect the size of the bubble  $\Delta_t$ . At first, it may not be clear whether this is a well-posed question. First, interest rates are endogenous in our setup: the path of interest rates is directly implied by the path of asset prices  $\{p_t\}_{t=0}^{\infty}$  from (3), and the latter are equilibrium objects. How then can we talk about a policymaker changing interest rates? And even if we could settle this question, recall that our model exhibits multiple equilibria. Can we make any clear predictions about the size of the bubble for a given interest rate path?

Gali faced these same questions in his paper. He began his discussion with a partial equilibrium model in which he took the interest-rate path  $\{r_t\}_{t=0}^{\infty}$  as exogenous. Condition (3) provides a first-order difference equation that the path of asset prices  $\{p_t\}_{t=0}^{\infty}$  must satisfy. Given a path for interest rates, we can use this equation to determine the path of equilibrium  $p_t$  up to an initial price  $p_0$ . But since the model exhibits multiple equilibria, the initial price  $p_0$  is indeterminate; it can assume any value between 0 and  $e_0$ . To compare asset prices under two interest rate paths, we need a way to assign an initial price  $p_0$  to each path. At this point, Gali suggested requiring that  $p_0$  be the same for the two paths. He argued this rules out the case where policy affects the bubble indirectly through some "indeterminacy" channel, i.e. where asset prices change because we jump from one equilibrium to the other as we switch from one exogenous interest rate path to another. If we restrict  $p_0$  to be the same for both interest rate paths, then the fact that  $p_0$  starts at the same point and  $p_t$  grows at the rate of interest implies  $p_t$  will be higher after date 0 when interest rates are higher. Higher interest rates are associated with larger asset bubbles.

The above argument might give the impression that Gali’s result rests on an arbitrary (even if plausible) assumption about initial conditions for a given interest rate path, and that alternative assumptions would lead to different conclusions. However, Gali offers his partial equilibrium analysis only as motivation, and ultimately turns to a general equilibrium model to explore how interest rates and bubbles are related. As we now argue, a general equilibrium model provides additional structure that governs  $p_0$ . This restriction renders assumptions about  $p_0$  redundant and potentially in conflict with the model. Notwithstanding the intuition that emerges from Gali’s partial equilibrium analysis, we show that the full equilibrium model yields unambiguous predictions on the relationship between interest rates and bubbles.

We develop the argument using our model and then relate it to Gali’s setup. Recall that in Proposition 1 we showed that the model exhibits multiple equilibrium price paths. Each equilibrium path for asset prices is associated with a corresponding equilibrium path for interest rates. That is, we can interpret Proposition 1 to mean that there exists a set of deterministic equilibrium interest rate paths  $\{r_t\}_{t=0}^{\infty}$  that can arise in our model. These interest rate paths can be similarly characterized by the cutoff date  $t^*$ , since an interest rate path  $\{r_t\}_{t=0}^{\infty}$  corresponds to an equilibrium if and only if

$$r_t = \begin{cases} g & \text{if } t < t^* - 1 \\ 0 & \text{if } t > t^* - 1 \end{cases} \quad (6)$$

for some date  $t^* = 1, 2, \dots$ , and at date  $t^* - 1$  the interest rate can assume any value between 0 and  $g$ . This structure implies that the set of equilibrium interest rate paths can be ordered: Given any two paths, one path of interest rates will be weakly higher than the interest rates in the other path at each and every date. A policymaker can set interest rates by selecting a particular path for interest rates from the set of all equilibrium interest rate paths as given by (6). Indeed, this is essentially what Gali assumes when he analyzes a general equilibrium model later in his paper.<sup>4</sup> The effect of policy thus reduces to how asset prices and interest rates vary together across the equilibria in Proposition 1: If a policymaker selects an equilibrium with higher interest rates, what asset prices would she face?

As we noted above, the deterministic equilibrium interest rate paths in our model can be ordered. Without loss of generality, we assume that the interest rate between the two paths we consider differs from date 0; before the interest rate paths diverge, both would have to equal  $g$  to allow interest rates to diverge later, and so whatever happens before rates diverge is inconsequential and can be ignored. Figure 1 shows how the initial asset price  $p_0$  and the initial interest rate  $r_0$  vary across equilibria. As evident from the figure, any equilibrium in which the initial asset price  $p_0$  is below  $e_0$  will be associated with an interest rate of  $r_0 = 0$ . Likewise, any equilibrium in which the interest rate  $r_0$  is positive will be associated with an equilibrium

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<sup>4</sup>Gali assumes the central bank can select the nominal interest rate. Since he assumes prices are sticky, this means it also selects the real interest rate. To be sure, Gali’s model is much richer, including stochastic shocks that are only immediately observable to the central bank. This allows him to deal with the fact that price setters might anticipate central bank policy. But this shouldn’t obscure from the fact that Gali effectively assumes the central bank selects the interest rate without intervening in asset markets, presumably by promising to take certain actions if the nominal rate it wants does not prevail. One can similarly assume a policymaker in our model promises to intervene if the real interest rate differs from its desired level.

asset price of  $p_0 = e_0$ . Note that in this case, the initial price  $p_0$  is uniquely determined, and so it will be inappropriate to impose restrictions on  $p_0$  as Gali does in his partial equilibrium analysis. Figure 1 shows that if the policymaker selects an interest rate path with a higher initial interest rate  $r_0$ , the associated equilibrium price  $p_0$  must be the same or higher. Since a higher initial interest rate  $r_0$  implies all subsequent interest rates will be weakly higher as well, choosing a higher initial rate will result in a bubble that is at least as large at date 0, grows more between dates 0 and 1, and then grows at least as fast thereafter. Raising rates will force the policymaker to tolerate a strictly larger bubble from date 1 on.

Gali's insight can thus be understood as an observation that in a certain class of models, the equilibria that feature higher interest rates also feature larger bubbles. The intuition for this result can be understood as follows. A higher interest rate in our economy leads agents to shift from storage to buying assets. Thus, a higher interest rate will be associated with weakly higher desired savings. In Gali's model there is no storage, but agents value consumption both when young and old. Agents in his model therefore choose between consumption and saving as opposed to between storage and saving. A higher equilibrium interest rate will induce young agents to allocate more resources to buying assets in his model as well. Since there is only one asset agents can buy, which is available in fixed supply, this asset necessarily becomes more valuable when desired savings rises. In Gali's setup, in fact, the analog of Figure 1 is a strictly upward-sloping curve, so a higher interest rate path in his model will be associated with a strictly larger bubble starting at date 0, contrary to the restriction he imposes in his partial equilibrium analysis.<sup>5</sup>

Our model can thus elucidate why Gali's model implies a higher real interest rate is associated with a larger bubble. His model, like ours, feature multiple equilibria, and those equilibria with higher interest rates also feature larger bubbles. If policymakers intervene by selecting those equilibria with higher interest rates, they will end up with larger bubbles. The fact that policymakers select equilibria rather than intervene directly in markets plays an important role in this result, as does the fact that there is only one asset agents can use to save. In the next section, we modify the model so that it admits a unique equilibrium. Policymakers will only be able to affect interest rates by intervening in financial markets, e.g. by changing the amount of government debt the public must hold. In this case, we will show that policymakers can intervene in ways that both raise interest rates and dampen asset bubbles.

## 2 Overturning Gali's Result

Our first modification to the model in Section 1 is to replace  $d = 0$  with  $d > 0$ . As we show below, this seemingly trivial change reduces the continuum of possible equilibria when the asset is intrinsically

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<sup>5</sup>While agents in Gali's model and ours can only buy a bubble asset, Diamond (1965) and Tirole (1985) let agents buy a bubble asset or invest in physical capital. Their models also exhibit multiple equilibria, and still imply that equilibria with higher interest rates feature larger bubbles. Intuitively, if a policymaker merely selected an equilibrium with a high interest rate, she would have to choose an equilibrium with less capital and a higher marginal product of capital. Agents who hold less capital but want to save more given higher interest rates must spend more on the bubble asset.

worthless to a single equilibrium. Moreover, this unique equilibrium corresponds to a bubble, i.e. the uniquely determined price of the asset will exceed the present discounted value of its dividends.<sup>6</sup>

To show that  $d > 0$  implies a unique equilibrium in the asset market, we now argue that  $r_t = 0$  cannot occur in equilibrium. For suppose  $r_t$  did equal 0. Then (3) would imply

$$p_{t+1} = p_t - d < p_t$$

Since  $p_t \leq e_t$ , it follows that  $p_{t+1} < e_t < e_{t+1}$ , which implies that  $r_{t+1} = 0$ . By this logic, the price would continue to decline by  $d$  at dates  $t + 1$ ,  $t + 2$ , and so on, until the price would eventually turn negative. But this cannot be an equilibrium, since at a negative price the cohort that owns them would prefer not to sell them. The only candidate equilibrium price path is one with strictly positive interest rates at all dates. But in that case, buying the asset dominates storage and young agents will exchange all of their endowment for the asset. The price of the asset will thus be uniquely determined in every period. We confirm this in the next proposition. The proof of this and other propositions are contained in an appendix.

**Proposition 2:** Suppose  $d > 0$ . Then the unique equilibrium path  $\{p_t\}_{t=0}^{\infty}$  is given by

$$p_t = e_t \tag{7}$$

and the unique equilibrium interest rate is given by

$$r_t = \frac{d}{e_t} + g \tag{8}$$

Note that while Proposition 1 allowed for the existence of additional stochastic equilibria in which  $\{p_t\}_{t=0}^{\infty}$  are random variables, in the case where  $d > 0$  the equilibrium is unique, meaning there are no stochastic equilibria. This result is noteworthy, since previous work on bubbles with intrinsically worthless assets has at times explored the possibility of stochastically bursting bubbles, i.e. equilibria in which the bubble can collapse to zero at a random date. Such phenomena are not possible when  $d > 0$  without modifying the model: The bubble in our model persists indefinitely. We return to this point in Section 4.

We now argue that the equilibrium in Proposition 2 indeed constitutes a bubble. To show this, we must first compute the fundamental value of the asset  $f_t$ . The rate at which we discount dividends is the market interest rate  $r_t$ , i.e.

$$f_t = \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} \frac{1}{1 + r_{t+i}} \right) d \tag{9}$$

The reason for using the market interest rate is that if we take resources from a young agent at date  $t$ , he would demand  $1 + r_t$  units at date  $t + 1$  to remain equally well off. The interest rate thus faithfully captures the way society trades off resources between adjacent dates.

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<sup>6</sup>Our results do not hinge on dividends being constant. If we allowed for time-varying dividends  $d_t$ , the equilibrium would continue to be unique as long as  $\lim_{t \rightarrow \infty} d_t > 0$ . At the same time, the asset would continue to be a bubble so long as  $\lim_{t \rightarrow \infty} d_t/e_t = 0$ , since we know from Tirole (1985) and Rhee (1991) that no bubble can arise when  $\lim_{t \rightarrow \infty} d_t/e_t > 0$ .

From Proposition 2, we know that  $r_t > g > 0$  for all  $t$ . Hence, (9) is finite, specifically

$$f_t \leq \sum_{j=1}^{\infty} \left( \frac{1}{1+g} \right)^j d = \frac{d}{g}$$

and, moreover,  $\lim_{t \rightarrow \infty} f_t = d/g < \infty$ . Thus, the fundamental value of the asset is bounded. At the same time, the price of the asset grows without bound, since  $\lim_{t \rightarrow \infty} p_t = \lim_{t \rightarrow \infty} (1+g)^t e_0 = \infty$ . It follows that the asset price will eventually exceed its fundamental value. Thus, our model still gives rise to a bubble, at least asymptotically. But we can show that the equilibrium price of the asset exceeds the fundamental value at all dates rather than just asymptotically. To see this, note that the equilibrium interest rate  $r_t$  in Proposition 2 implies that

$$p_t = \frac{d + p_{t+1}}{1 + r_t}$$

At the same time, the fundamental value  $f_t$  in (9) satisfies

$$f_t = \frac{d + f_{t+1}}{1 + r_t}$$

Subtracting the latter expression from the former reveals that the difference  $\Delta_t \equiv p_t - f_t$  must satisfy

$$\Delta_t = \frac{\Delta_{t+1}}{1 + r_t}$$

By repeated substitution, it follows that  $\Delta_0 = \left( \prod_{t=0}^{T-1} \frac{1}{1+r_t} \right) \Delta_T$  for any  $T > 0$ . But since we just argued  $\Delta_T > 0$  as  $T \rightarrow \infty$ , it follows that  $\Delta_0 > 0$ . Hence, the price of the asset exceeds its fundamental value at all dates, just as in the case where  $d = 0$ . The asset is thus still a bubble, and, in contrast to the case where  $d = 0$ , the scenario where the asset is not a bubble cannot be an equilibrium.<sup>7</sup>

Given that our model admits a unique equilibrium, we can no longer model policy interventions as selecting equilibria. Instead, we need to allow policy to affect the unique equilibrium interest rate directly. Towards this end, we introduce government debt into the model. Not only is debt a natural channel through which policy can affect interest rates, but it also allows agents to save using more than one asset, and we previously argued that the restriction to one asset played an important role behind Gali's result. Arguably the most relevant policy interventions that use government debt to alter interest rates are open-market operations, i.e. when a monetary authority sells bonds if it wants to raise rates. But modelling this would require us to introduce money as well as bonds. For simplicity, we at first ignore money and focus on the consequences of an intervention that involves issuing more debt. While this should more naturally be interpreted as a fiscal intervention, it still helps to demonstrate why interventions that raise interest rates can dampen bubbles. We introduce money and turn to monetary policy in the next section.<sup>8</sup>

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<sup>7</sup>The fact that the equilibrium must feature a bubble is anticipated by Tirole (1985). He shows that if the limiting interest rate without a bubble is zero or negative, the introduction of an asset that yields positive dividends ensures an asymptotic bubble must be present in equilibrium. Intuitively, the fundamental value of an asset bearing dividends would be infinite if the bubble ever vanished. Since the return to storage in our economy is zero, we satisfy Tirole's condition.

<sup>8</sup>Although policy interventions against bubbles are often framed in terms of monetary policy, issuing debt has been proposed as a policy in economies that are vulnerable to bubbles. For example, Diamond (1965) interprets his model to mean that issuing public debt can potentially restore dynamic efficiency to an economy that would otherwise be vulnerable to bubbles.

Consider a government that issues real one-period debt obligations. For each unit of consumption the government collects at date  $t$ , it promises to pay  $1 + r_t$  units date  $t + 1$ . The government can finance these promises using lump sum taxes, which we assume are levied only on the young. Let  $b_t$  denote the amount of resources collected from debt issuance at date  $t$  and  $\tau_t$  denote the lump sum tax collected at date  $t$ . A lump-sum tax  $\tau_t$  is feasible only if  $\tau_t \leq e_t$ ; with no storage, the government cannot collect more resources than agents are endowed with that period. The flow government budget constraint is given by

$$b_{t+1} = (1 + r_t) b_t - \tau_{t+1} \quad (10)$$

That is, any part of the government's debt obligation not covered by tax revenue must be financed by new debt. More generally, we could have allowed the government to purchase goods with the revenue it collects, but we ignore this possibility in what follows. The initial obligation of the government to the old at date 0 is denoted  $(1 + r_{-1}) b_{-1}$ . Feasibility requires that the obligation  $(1 + r_{-1}) b_{-1}$  not exceed  $e_0$ , i.e. the government cannot transfer to the old more than it can collect in the same period.

Market clearing requires that at each date  $t$ , the young are willing to buy all of the debt  $b_t$  issued by the government and all shares of the asset held by the old. The old must be willing to sell all of their shares of the asset. In equilibrium the interest rate on government bonds and the asset be equal, i.e.

$$1 + r_t = \frac{d + p_{t+1}}{p_t} \quad (11)$$

Using the same argument as before, we can conclude that  $r_t > 0$  for  $t = 0, 1, 2, \dots$ . This implies storage is dominated, and so the equilibrium price of the asset is once again uniquely determined:

$$p_t = e_t - \tau_t - b_t$$

To study the effects of changes in debt issuance, we begin by defining fiscal policies in our economy. A path for fiscal policy can be summarized by two objects: an initial obligation of the government  $(1 + r_{-1}) b_{-1}$  to the old at date 0, and the path for lump-sum taxes  $\{\tau_t\}_{t=0}^{\infty}$  it collects from the young from date 0 on. As we now show, these two objects determine a unique path for both the interest rate  $\{r_t\}_{t=0}^{\infty}$  and debt issuance  $\{b_t\}_{t=0}^{\infty}$  that satisfy both the intertemporal budget constraint (10) and the equilibrium condition (11).

**Lemma 1:** Suppose  $d > 0$ . Given an initial debt obligation  $(1 + r_{-1}) b_{-1} \leq e_0$  and tax path  $\{\tau_t\}_{t=0}^{\infty}$  where  $\tau_t \leq e_t$  for all  $t$ , there is a unique path  $\{b_t, r_t\}_{t=0}^{\infty}$  that satisfies both the intertemporal budget constraint (10) and the equilibrium interest rate condition (11).

In what follows, we further restrict fiscal policy to ensure the government sector doesn't eventually crowd out all economic activity. That is, we wish to avoid the case where young agents ultimately hand over all of their endowment to the government, either as tax payments or as purchases of government bonds. Formally, we assume the initial debt obligation  $(1 + r_{-1}) b_{-1}$  and subsequent tax payments  $\{\tau_t\}_{t=0}^{\infty}$  satisfy

$$\lim_{t \rightarrow \infty} \frac{\tau_t + b_t}{e_t} < 1 \quad (12)$$

where  $b_t$  is the debt issuance implied by the path of fiscal policy that is summarized in Lemma 1. Our next result provides expressions for the unique equilibrium asset price  $p_t$  and interest rate  $r_t$ .

**Proposition 3:** Suppose  $d > 0$ . Given an initial debt obligation  $(1 + r_{-1})b_{-1}$  and a path  $\{\tau_t\}_{t=0}^\infty$ , the equilibrium price of the asset and the interest rate are uniquely determined each period and given by

$$\begin{aligned} p_t &= e_t - \tau_t - b_t \\ 1 + r_t &= \frac{d}{e_t - \tau_t - b_t} + \frac{(1 + g)e_t - \tau_{t+1} - b_{t+1}}{e_t - \tau_t - b_t} \end{aligned}$$

where  $\{b_t\}_{t=0}^\infty$  is the unique path for debt issuance in Lemma 1. If (12) is satisfied,  $\lim_{t \rightarrow \infty} p_t = \infty$ .

In gauging the effect of policy interventions, it will prove convenient to work with policies that can be summarized with a single parameter. Suppose that there is a single parameter  $b$  such that

$$\begin{aligned} (1 + r_{-1})b_{-1} &= (1 + r_{-1})b \\ b_t &= b \text{ for } t = 0, 1, 2, \dots \end{aligned} \tag{13}$$

A higher value of  $b$  means the government increases the amount of outstanding obligations to all generations, including the initial old who we can view as earning an implicit return  $r_{-1}$  on outstanding debt they would have obtained as young. To ensure the amount of debt issued at all dates is equal to  $b$ , the fiscal authority would need to set taxes  $\tau_t$  equal to the interest obligation  $r_{t-1}b$ , leaving the principle amount borrowed unchanged. Our next result shows that an intervention that increases government liabilities at all dates would result in a higher interest rate  $r_t$  and a lower price for the asset  $p_t$ .

**Proposition 4:** Suppose fiscal policy is given by (13) where  $r_{-1} > 0$ . Then the equilibrium interest rate  $r_t$  is increasing increasing in  $b$  for all  $t$  and the equilibrium asset price  $p_t$  is decreasing in  $b$  for all  $t$ .

Intuitively, a higher  $b$  requires lump sum taxes  $\tau_t$  to be higher at all dates.<sup>9</sup> With more taxes to pay and more bonds to hold, each cohort will have fewer available resources with which to buy the asset. As a result, the price of the asset will fall. Simply put, issuing more bonds crowds out savings in the bubble asset. This crowding out will also lead to higher rates of return. Intuitively, issuing more debt implies agents invest fewer resources in the asset but continue to earn the same dividends. Hence, the dividend yield  $d/p_t$  on the asset rises. Of course, the return on the asset also depends on how the intervention affects the price of the asset in the future. An intervention that increases the amount of debt issued each period by the same amount will ensure that the capital gain term does not offset the increase in the dividend yield.<sup>10</sup>

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<sup>9</sup>Here is where  $r_{-1} > 0$  plays an important role, since it ensures the government's initial obligation increases by more than the obligation the government takes on at date 0 increases, forcing  $\tau_0$  to rise.

<sup>10</sup>If agents were to value bonds differently from other assets as in Krishnamurthy and Vissing-Jorgensen (2012) so that assets were imperfect substitutes, issuing more government debt would drive the return on those bonds up to compel agents to hold more of these assets. Our model ignores this possibility by assuming agents only care about rates of return.

For a more precise explanation of why issuing more debt at all dates leads to higher interest rates, recall that in equilibrium no goods will be stored and the entire endowment each period will be consumed by that period's old. That is, the cohort born at date  $t$  will consume the same amount  $e_{t+1} + d$  when they turn old regardless of what the fiscal authority does. They finance this consumption from the return to their savings, which corresponds to all of their disposable income when young, i.e. their endowment  $e_t$  net of the taxes they pay  $\tau_t$ . The return on private savings for the cohort born at date  $t$  is thus

$$1 + r_t = \frac{e_{t+1} + d}{e_t - \tau_t} \quad (14)$$

The higher the tax  $\tau_t$ , then, the higher the rate of return: Agents earn the same amount on their savings but save less. Interventions only lead to higher interest rates if they increase taxes. This is true of the intervention we considered, namely increasing the amount of debt issued by the same amount at all dates. That policy requires setting  $\tau_t = r_{t-1}b$ , and from Proposition 4 we know that interest rates rise as  $b$  rises. More generally, issuing more debt can be compatible with a fall in tax rates in certain dates; for example, issuing a little more debt at date  $t$  but a lot more at date  $t + 1$  will lower taxes at date  $t + 1$ . However, the intertemporal government budget constraint implies higher debt obligations must be financed by higher taxes eventually, so issuing more debt will lead to higher interest rates eventually even if not immediately.

Proposition 4 establishes that there exists a policy intervention that increases interest rates and drives down the price of the asset. However, with  $d > 0$ , a lower price no longer necessarily corresponds to a smaller bubble. Given interest rates rise, the fundamental value  $f_t$  as defined in (9) will fall as interest rates rise, and in principle the fundamental could fall by the same amount or even more than the price. Our next result confirms that under assumption (12), an increase in  $b$  will reduce the size of the bubble  $\Delta_t = p_t - f_t$  at all dates. Thus, the policy intervention we consider increases the real interest rate and reduces the bubble, mitigating the extent to which the asset is overvalued.

**Proposition 5:** If fiscal policy is given by (13), and (12) is satisfied, then  $\Delta_t \equiv p_t - f_t$  is decreasing in  $b$  for all  $t$ .

Thus, we have an example in which, contrary to Gali's result, a policy intervention that leads to higher interest rates also reduces the extent to which assets are overvalued. At the core of our example is the fact that agents can save using more than one financial instrument, so it is possible to raise rates while crowding out the amount agents spend on the bubble asset. A policymaker intent on shrinking a bubble would effectively induce the public to hold other assets instead of the bubble and thus depress its price.

While our example accords with conventional wisdom that raising rates can help mitigate bubbles, in the example there is also no welfare-based reason to intervene in order to dampen the bubble. In fact, such interventions have no effect on welfare in our model: Regardless of policy, each cohort consumes the entire endowment of the next cohort together  $e_{t+1}$  as well as the dividend  $d$ . Issuing more debt only affects how resources are transferred between generations. If the government issues no debt, the young transfer their resources to the old directly by paying them for the asset. If the government does issue debt, the young

will transfer resources to the government when they pay taxes and buy bonds, which the government then transfers to the old when they repay previous debt. Intervention may affect market interest rates and prices, but it does not affect what any cohort consumes.

Our irrelevance result is admittedly special. It arises because only the old value consumption, and so they will consume all resources regardless of prices. If we assumed agents value consumption in both periods of their life, as Gali does, issuing debt would generally change how agents allocate consumption over the life cycle. In that case, intervening to dampen a bubble would matter for welfare. However, as Tirole (1985) and others have emphasized, asymptotic bubbles in models like the ones we analyze are Pareto efficient. This implies that any intervention which changes allocations must make at least some agents worse off. More generally, in models where bubbles arise because of dynamic inefficiency such as the one Gali considered and we sought to replicate, acting against a bubble cannot make society as a whole better off.

This is not to say that it is impossible to modify such models to allow for welfare-improving interventions. For example, if agents could create additional assets at a cost, a bubble could lead to excessive creation of assets: Society would be better off having the government issue debt than wasting resources to create assets that achieve the same thing. While this offers a reason for intervening against bubbles, it is not the one policymakers typically invoke to argue for leaning against potential bubbles. They tend to worry about bubbles collapsing and the consequences thereof. Recall that in our economy where  $d > 0$ , the bubble persists indefinitely and does not burst, so introducing variable asset supply would not touch on these concerns even if it provided a reason for intervention. We return to these considerations in Section 4. For now, we simply note that while we have shown is that policymakers can successfully fight bubbles by raising interest rates, we have yet to offer a compelling reason why they should.

### 3 Monetary Policy

In the previous section, we showed that by issuing more debt, a fiscal authority could raise interest rates and dampen bubbles. It might be tempting to infer from this that contractionary monetary policy should work similarly, since one example of contractionary monetary policy is an open market operation that also seeks to get the public to hold more bonds. However, open market operations replace one asset held by the public with another, i.e. money with bonds. It isn't obvious, then, that such a policy would still crowd out spending on the bubble asset. To understand the effects of monetary interventions, we need to explicitly add money as another asset agents can hold. In this section, we sketch out how to add money into our model and the consequences of doing so. The full details of the analysis are described in Appendix B. To make the analysis transparent, we continue to use the model in Section 2 in which there is no welfare gain from deflating a bubble. We turn to the question of why intervention might be beneficial in the next section.

To anticipate our main findings, we show that when goods prices are flexible and respond instantly to changes in the money stock, a monetary intervention can result in higher interest rates and a smaller

bubble, but only if it forces a fiscal intervention of the type we discussed in Section 2. In particular, a monetary intervention that lowers the price of goods will increase the real value of the government's outstanding nominal obligations. Increasing the government's obligations will crowd out resources from the asset and reduce the bubble. But for the real interest rate to rise, the monetary intervention must induce the government to collect more taxes; if the government finances its obligations using seigniorage revenue from offsetting monetary policy it implements later, real rates will be unchanged. With flexible prices, then, monetary policy shapes interest rates not directly but indirectly through fiscal policy. We then show that with nominal price rigidity, a monetary intervention can raise rates and dampen bubbles even if fiscal policy is held fixed. This is because when prices are sticky, a monetary contraction temporarily reduces economy activity, which in turn lowers the income of the agents who buy the bubble.<sup>11</sup>

To establish these results, we need to modify the model from Section 2 in two key ways. First, we introduce money as an additional financial instrument agents can hold. To allow money injections or withdrawals to affect interest rates, we further need agents to treat money differently than other assets; otherwise, money would have to carry the same return as other assets agents can hold. Second, we assume agents are endowed with labor rather than with a fixed amount of goods. This allows monetary policy to affect the quantity of goods produced, which is the same as the amount young agents earn.

We begin with incorporating money. Young agents who wish to save can still store their endowment or exchange it with either the dividend-bearing asset or government debt. But now they can also exchange goods for money, i.e. a non-interest bearing asset issued by the government. Let  $M_t$  denote the amount of money circulating at date  $t$ , so that  $M_t - M_{t-1}$  corresponds to the amount of money issued or removed from circulation at date  $t$ . We model these injections or withdrawals as lump sum to the old rather than as open market operations. Let  $P_t$  denote the price of one unit of consumption goods in terms of money. Conversely,  $1/P_t$  represents the real price of one unit of money, and  $m_t = M_t/P_t$  denotes real money balances. Let  $\Pi_t$  denote the gross inflation rate between dates  $t$  and  $t + 1$ , i.e.  $\Pi_t = P_{t+1}/P_t$ . It will be convenient to define

$$x_{t+1} = \frac{M_{t+1} - M_t}{P_{t+1}}$$

as the injection of money between dates  $t$  and  $t + 1$  measured in terms of how much this amount could buy at date  $t + 1$ . If young individuals opt to set aside  $m_t$  of their original wealth to exchange for money, their real money balances at date  $t + 1$  will equal

$$m_{t+1} = \Pi_t^{-1} m_t + x_{t+1}$$

where agents view  $x_{t+1}$  as fixed and unaffected by their actions. We continue to denote the real rate of return on government debt by  $1 + r_t$ , and we use  $1 + i_t$  to denote the nominal interest rate  $(1 + r_t) \Pi_t$ .

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<sup>11</sup>The reason Gali fails to find such an effect is because he only contemplated contractionary monetary policy in response to a shock that increases the wealth of young agents. In that case, even if the income of young agents does decline, the amount they can spend on bubble assets can still rise. Dong, Miao, and Wang (2017) make a similar point.

As we noted above, if money were just another asset that can be used as a store of value, there would be limited scope for monetary policy to influence inflation or interest rates. Thus, we need agents to value money beyond its store of value. We follow Gali's approach (in Appendix 3 of his paper) of assuming that real balances enter preferences directly.<sup>12</sup> This specification is meant to be a stand-in for the liquidity value of holding money, since it makes agents willing to hold money even when it offers a lower return. In particular, we assume agents derive log utility from the real balances they set aside while young,  $m_t = M_t/P_t$ . As we discuss below, the log utility specification is convenient in that it implies money is superneutral, i.e. changing the growth rate of money will have no effect on economic activity in equilibrium.

The second way we modify our model is to assume young agents are endowed not with goods but with labor. Agents born at date  $t$  are endowed with one unit of labor. Production is linear, so if an agent devotes  $n_t \in [0, 1]$  units of labor he will produce  $A_t n_t$  units of output, where productivity  $A_t$  evolves as

$$A_t = (1 + g)^t A_0$$

Gali also assumed young agents are endowed with labor rather than goods, but he assumed they supply their labor inelastically. We instead follow Adam (2003) in assuming agents value time spent not working, and so might vary  $n_t$  in response to policy. In particular, we assume that the cohort of agents born at date  $t$  incur disutility  $v_t(n_t)$  from supplying  $n_t$  units of time, where  $v_t(\cdot)$  is increasing and convex. To simplify the exposition, we will proceed here as if agents are yeoman farmers who operate their own technology. In Appendix B, we assume some young agents are workers and others are entrepreneurs who know how to deploy labor, allowing a market for labor services at a market-clearing wage.

These assumptions imply that we replace the preferences in (1) with

$$u(c_t, c_{t+1}, m_t, n_t) = c_{t+1} + \theta \ln(m_t) - v_t(n_t) \quad (15)$$

It will be convenient to assume that  $v_t(n_t) = A_t v(n)$  for some common function  $v(\cdot)$ . This implies later cohorts will not necessarily work more even though they are more productive, and employment can be stable over time. We assume  $\lim_{n \rightarrow 0} v'(n) = 0$  and  $\lim_{n \rightarrow 1} v'(n) = \infty$  to ensure an interior solution.

Agents maximizing the utility in (15) are subject to the budget constraint

$$c_{t+1} = (1 + r_t)(A_t n_t - m_t - \tau_t) + \Pi_t^{-1} m_t + x_{t+1} \quad (16)$$

In equilibrium, both the government bond and the asset yield the same return  $r_t$ , and so the agent's portfolio decision boils down to choosing between how much to save in interest-bearing instruments and how much to save in money. The first order conditions of the problem above with respect to real balances and labor effort are given by

$$m_t = \frac{\theta \Pi_t}{(1 + r_t) \Pi_t - 1} = \frac{\theta \Pi_t}{i_t} \quad (17)$$

$$v'(n_t) = 1 + r_t \quad (18)$$

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<sup>12</sup>Waldo (1985) offers another example of a monetary OLG economy where agents value real balances directly.

Finally, we need to modify the intertemporal government budget constraint in (10) to take into account money injections and withdrawals. This constraint is now given by

$$\begin{aligned} b_t &= (1 + r_{t-1}) b_{t-1} - \tau_t - \frac{M_t - M_{t-1}}{P_t} \\ &= (1 + r_{t-1}) b_{t-1} - \tau_t - m_t + \frac{m_{t-1}}{\Pi_{t-1}} \end{aligned} \quad (19)$$

An equilibrium in this economy is a path  $\{P_t, p_t, r_t\}_{t=0}^{\infty}$  for the price of goods relative to money, the real price of the asset relative to goods, and the real interest rate, respectively, together with a path  $\{n_t\}_{t=0}^{\infty}$  for hours, which satisfy the first order conditions (17) and (18), the intertemporal government budget constraint (19), and the equilibrium condition that the return to buying the asset  $(d + p_{t+1})/p_t$  must be the same as the real interest rate  $r_t$ .<sup>13</sup> Since there is nothing to impede the price of goods  $P_t$  from changing when the amount of money in circulation  $\{M_t\}_{t=0}^{\infty}$  changes, we will refer to this setup as a flexible price economy.

As alluded to earlier, in Appendix B we show that log preferences ensure money is superneutral, in the sense that changes in  $\{M_t\}_{t=0}^{\infty}$  have no effect on the real interest rate  $r_t$  or the employment decision  $n_t$ . This is because demand for money under log preferences as given by (17) implies the real return agents earn on money will not vary with inflation, and so the real rate  $r_t$  will not depend on the stance of monetary policy. In particular, the real interest rate  $r_t$  will be given by

$$1 + r_t = \frac{A_{t+1}n_{t+1} + \theta + d}{A_t n_t - \tau_t} \quad (20)$$

Note the similarity between (20) and the expression for  $r_t$  in (14) in the absence of money. Although the argument is a bit more subtle given  $n_t$  is endogenous, in Appendix B we show that it is still the case that changing the interest rate requires a change in the path of taxes  $\tau_t$ . A change in the path of money  $\{M_t\}_{t=0}^{\infty}$  on its own will have no effect on the real interest rate given the preferences we assume.

While the path of  $\{M_t\}_{t=0}^{\infty}$  does not matter for the real interest rate, employment, or output, it will matter in general for the price of goods  $P_t$ , the level of real balances  $m_t = M_t/P_t$  agents hold, and government revenues. But it also matters for the real price of the asset  $p_t$ . To see this, consider the effect of a change in  $\{M_t\}_{t=0}^{\infty}$  that leads to a reduction in the initial price level  $P_0$ . Since young agents will prefer savings to storage, they will exchange all of the  $A_t n_t$  goods they produce for financial assets. Given the asset is in fixed supply of 1, total spending on the asset at date  $t$  is equal to  $p_t$ . Hence,  $p_t$  is equal to the income  $A_t n_t$  agents earn net of taxes  $\tau_t$ , government debt  $b_t$ , and real balances  $M_t/P_t$ . Applied to a date 0, this implies

$$p_0 = A_0 n_0 - \tau_0 - b_0 - \frac{M_0}{P_0} \quad (21)$$

From the intertemporal government budget constraint in (19), we know that

$$b_0 + \tau_0 + \frac{M_0}{P_0} = (1 + r_{-1}) b_{-1} + \frac{M_{-1}}{P_0}$$

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<sup>13</sup>The nominal rate is not included as an equilibrium object because this rate can be deduced from  $i_t = (1 + r_t) P_{t+1}/P_t - 1$ .

Substituting this into (21) yields

$$p_0 = A_0 n_0 - (1 + r_{-1}) b_{-1} - \frac{M_{-1}}{P_0} \quad (22)$$

Both  $(1 + r_{-1}) b_{-1}$  and  $M_{-1}$  are predetermined, and above we argued that  $n_0$  is independent of  $\{M_t\}_{t=0}^{\infty}$ . Hence, if changing  $\{M_t\}_{t=0}^{\infty}$  reduces  $P_0$ , it will also decrease  $p_0$ . Intuitively, a higher price level  $P_0$  reduces the seniorage revenue of the government, forcing it to either issue more debt or raise taxes. Either would leave agents with fewer resources to buy the asset, so its price will fall. The same intuition would apply if the initial debt obligation  $(1 + r_{-1}) b_{-1}$  were issued as a nominal obligation.

In short, when prices are flexible, pure monetary interventions can affect the real asset price  $p_t$ . However, since we argued that  $n_t$  will not change as we vary  $\{M_t\}_{t=0}^{\infty}$ , monetary policy on its own will not affect the real interest rate  $r_0$ , as evident from (20). As that expression shows, only if monetary policy forces a change in  $\tau_0$  will the real rate  $r_0$  change. Recall that this is also what we found in our analysis of fiscal policy, i.e. the interest rate  $r_t$  only changes when  $\tau_t$  does. There, we argued that issuing more debt would force the government to raise taxes eventually, so interest rates would rise eventually even if not immediately. But once we add money, the government can potentially meet its higher obligation using offsetting monetary policy later on to raise seniorage revenue. A monetary intervention will only affect the real interest rate  $r_0$  if it is not later undone by subsequent monetary policy, effectively forcing the type of fiscal policy intervention that we showed in Section 2 leads to higher rates and smaller bubbles.

Since a pure monetary policy intervention will not change real interest rates  $r_t$ , such interventions will have no effect on the fundamental value of the bubble asset, i.e. the value of its dividends discounted at the real interest rate  $r_t$ . Hence, a deflationary monetary policy that lowers  $p_0$  but leaves the fundamental value unchanged would reduce the bubble. But just as before, there is nothing inherently desirable about this outcome. Since deflationary monetary policy has no effect on output, agents will consume the same amount when the monetary authority changes the path of money. Monetary interventions will matter for welfare, since they affect real balances  $m_t$ , which agents care about. But the effect is independent of the price of the asset  $p_t$  or the extent to which the asset is overvalued. To the extent that monetary policy forces the fiscal authority to raise taxes, our earlier analysis applies and consumption would be unchanged.

The fact that monetary policy will not influence interest rates except through changes in fiscal policy relies crucially on flexible goods prices. However, much of the literature on monetary policy has assumed prices are sticky, in line with empirical evidence on the sluggish response of inflation to monetary policy. In Appendix B, we describe a version of the model above that allows for price-setting that borrows both from Gali (2014) and Adam (2012). In that version, agents produce not a final good but different intermediate goods that can be combined into a final good. Each intermediate good producer is a monopolist supplier of his own good. This formulation allows us to analyze what happens if producers set their prices before the monetary authority moves. We show in Appendix B that if the monetary authority unexpectedly withdraws money at date 0 and then pursues a predictable policy thereafter, there exists an intervention that raises the real interest rate  $r_0$  and lowers equilibrium employment  $n_0$ . Since young agents earn less, they will have

less to spend on the asset. As evident from (22), since  $P_0$  is fixed and both  $(1 + r_{-1})b_{-1}$  and  $M_{-1}$  are predetermined, a fall in  $n_0$  would cause the price of the asset  $p_0$  to fall. With a bit more work, we show that the bubble term  $\Delta_0 = p_0 - f_0$  also falls. Thus, with nominal price rigidity, there exist purely monetary interventions that raise the real interest rate and mitigate the bubble, even if we hold fiscal policy fixed.

Interestingly, the expression for  $p_0$  in (22) implies that if  $P_0$  cannot respond to changes in the stock of money, the only way a surprise monetary contraction at date 0 would affect  $p_0$  is if  $n_0$  fell. In other words, the only way to dampen bubbles is to generate a recession. Intuitively, dampening the bubble requires the crowding out of resources that agents spend to buy assets. In our sticky-price economy, the only way to achieve this is if agents earn less income. This suggests it might be difficult for a policymaker intent on dampening an asset bubble to deflate the bubble without also curtailing economic activity; the logic of crowding out dictates that income must fall for spending on the asset to fall. More generally, our analysis suggests that a policymaker seeking to deflate a bubble should not simply be content with raising interest rates, but should look to see if the increase in rates was associated with crowding out, i.e. whether it was associated with a decline in saving by agents or a shift in the composition of the portfolio of assets held by the public. In other words, since our theory emphasizes a particular channel through which higher rates depress bubbles, policymakers can verify whether there is any evidence that this channel is in fact operative.

## 4 Credit-Driven Bubbles

At this point, our analysis offers only mixed news for advocates of leaning against the wind. On the one hand, we show that intervening to raise rates can depress bubbles, in line with their view about appropriate policy in the face of bubbles. At the same time, our examples rely on models in which there is no reason for a policymaker to intervene and dampen bubbles. Of course, advocates of leaning against the wind would be quick to point out that the types of bubbles we have discussed so far do not coincide with the scenarios they worry about. Policymakers do not seem especially alarmed when a strong desire to save leads agents to bid up the price of assets above their fundamental value, especially if, as is true in our model, such a bubble will persist indefinitely. What worries policymakers is the prospect of a bubble collapsing, especially when its purchase was financed with debt. For example, Mishkin (2011) singles out what he calls credit-driven bubbles as a particularly acute concern for policymakers:

[N]ot all asset price bubbles are alike. Financial history and the financial crisis of 2007-2009 indicates that one type of bubble, which is best referred to as a credit-driven bubble, can be highly dangerous. With this type of bubble, there is the following typical chain of events: Because of either exuberant expectations about economic prospects or structural changes in financial markets, a credit boom begins, increasing the demand for some assets and thereby raising their prices... At some point, however, the bubble bursts. The collapse in asset prices then leads to a reversal of the feedback loop in which loans go sour, lenders cut back on credit supply, the demand for the assets declines further, and prices drop even more.

In this section, we show how our model can give rise to credit-driven bubbles along these lines. Obviously, this requires us to introduce both credit and the possibility of a bubble bursting. We now do so using our setup from Section 2, i.e. we return to ignoring money to keep the analysis simple.

## 4.1 Stochastic Shocks

We begin with features that allow a bursting bubble. As we noted earlier, when  $d = 0$ , there exist equilibria in which the asset initially trades at a positive price but its price can collapse to zero with positive probability in any given period and will stay at zero thereafter. Blanchard and Watson (1982) were among the first to discuss such stochastically bursting bubbles, and later work by Weil (1987) showed that such price paths can arise as equilibria in dynamically inefficient economies with intrinsically worthless assets. Various authors have since exploited this observation for models featuring intrinsically worthless assets to explore stochastically bursting bubbles. But since we assume the asset pays a positive dividend  $d > 0$ , our model features a unique equilibrium in which the bubble persists indefinitely. To get a bubble that bursts, we need to introduce a shock that can trigger its collapse.

One way to try to get a bubble to collapse is to introduce a shock that would render a bubble non-viable. For a bubble to be sustainable, the endowment agents use to buy it must grow as fast as the interest rate. Suppose, then, that we introduce the possibility that the endowment might stop growing from some random date  $T$  on. Since the return on buying the asset is positive given it yields a positive dividend, a bubble should not be possible beyond date  $T$ . Formally, we replace the endowment process in (2) with

$$e_t = \begin{cases} (1+g)^t e_0 & \text{if } t < T \\ (1+g)^{T-1} e_0 & \text{if } t \geq T \end{cases} \quad (23)$$

where  $T$  is some random variable. We can interpret this specification as a risk of the economy slipping into a permanent state of secular stagnation.<sup>14</sup> We now verify that indeed no bubble can exist from date  $T$  on once the endowment stops growing.

For expositional ease, let us consider the case where  $(1+r_{-1})b_{-1} = 0$  and  $\tau_t = 0$  for all  $t$ . This implies the government issues no debt and agents can only hold the dividend-bearing asset. Let us use a hat to denote the value of a variable once the endowment stopped growing. That is,  $\hat{p}_t$  denotes the price of the asset for  $t \geq T$  and  $\hat{f}_t$  denotes its fundamental value for  $t \geq T$ . Since the interest rate must be positive given  $d > 0$ , storage is dominated and young agents trade all of their endowment for the asset. This implies

$$\hat{p}_t = \hat{e}_t = e_{T-1}$$

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<sup>14</sup>To the extent that the bubble collapses when growth stops, a period of secular stagnation would further coincide with the collapse of a bubble and the onset of low real interest rates. These features seem to characterize the empirical pattern Summers (2016) associated when he rechristened the term secular stagnation.

The real interest rate for any date  $t \geq T$  will be given by

$$1 + \widehat{r}_t = \frac{d + \widehat{p}_{t+1}}{\widehat{p}_t} = 1 + \frac{d}{e_{T-1}}$$

We can use this interest rate to compute the fundamental value of the asset for any date  $t \geq T$ :

$$\widehat{f}_t = \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} \frac{1}{1 + \widehat{r}_{t+i}} \right) d = \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} \frac{1}{1 + d/e_{T-1}} \right) d = e_{T-1}$$

From date  $T$  on, then, both  $\widehat{p}_t$  and  $\widehat{f}_t$  are equal to  $e_{T-1}$ . Hence, there can be no bubble beyond date  $T$ . Note that the fundamental value  $\widehat{f}_t$  depends on  $T$ . This is because even though dividends are constant, the equilibrium interest rate  $\widehat{r}_t$  is a function of  $T$ .

Our specification thus ensures that if a bubble existed before date  $T$ , it would burst once the endowment stops growing. As for what happens while the endowment grows, we need to specify a distribution for  $T$ . We assume  $T$  is geometrically distributed, meaning there will be constant probability per period that the endowment stops growing as the endowment keeps rising. Formally,

$$\Pr(T = k) = (1 - \pi)^{k-1} \pi \text{ for } k = 1, 2, 3, \dots$$

We will now argue that the shock which renders a bubble unsustainable from date  $T$  on in fact rules out the possibility of a bubble altogether. That is, in our attempt to construct a bursting bubble, we have actually ruled out bubbles altogether. We hasten to add that once we allow private credit to circulate, our model will indeed admit a bubble while the endowment grows that will burst at date  $T$ . In this sense, we can refer to these bubbles as credit-driven, since they crucially rely on the existence of private debt. The fact that the desire to save on its own cannot generate a bubble highlights that, in line with Mishkin's comment, credit-driven bubbles are fundamentally different types of bubbles.

To show that a bubble cannot arise before date  $T$  when there is no private debt, we must first define the fundamental value of the asset when the endowment process is stochastic. In this case, it is natural to define the fundamental value of the asset as the *expected* present discounted value of dividends, i.e.

$$f_t = E \left[ \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} \frac{1}{1 + r_{t+i}} \right) d \right] \quad (24)$$

where  $(1 + r_t)^{-1}$  denotes the value as of date  $t$  of one unit of resources to be delivered with certainty at date  $t + 1$  and which generally depends on  $T$ . For dates  $t > T$ , we already argued that  $\widehat{r}_t = d/e_{T-1}$ . For  $t \leq T$ , the fact that agents are risk neutral implies they are indifferent between a unit of resources at date  $t + 1$  with certainty and a random amount of resources at date  $t + 1$  that equals 1 in expectation. The return on a risk-free instrument is thus the same as the expected return on the asset, i.e.

$$1 + r_t = \frac{d + E[p_{t+1}]}{p_t}$$

Substituting in  $p_t = e_t$  and  $\widehat{p}_t = \widehat{e}_t$  reveals that for  $t < T$ , the risk-free real interest rate at date  $t$  equals

$$r_t = (1 - \pi)g + \frac{d}{e_t}$$

With this definition for the fundamentals, we can establish the following result:

**Proposition 6:** Suppose  $d > 0$ . If  $e_t$  is given by (23) and there is no private debt that circulates in equilibrium, a bubble will not arise, i.e.  $p_t = f_t$  for  $t < T$  and  $\widehat{p}_t = \widehat{f}_t$  for  $t \geq T$ .

Note that the absence of a bubble beyond date  $T$  does not automatically rule out bubbles before date  $T$ . When  $d = 0$ , we can construct equilibria with a bubble that is positive before date  $T$  and 0 thereafter. It is only when  $d > 0$  that we can rule out a bubble while the endowment is still growing.<sup>15</sup>

The argument for the non-existence of a bubble when the endowment might stop growing goes through when we explicitly allow the government to issue debt. The only reason for ignoring government debt so far is because it led to simple expressions for  $p_t$  and  $\widehat{p}_t$ . From hereon, we will allow the government to issue debt. To ensure the government will never be forced to default, we further restrict  $(1 + r_t)b_t \leq e_t$  for all  $t < T$ . To simplify matters, we assume taxes  $\tau_t$  are not contingent on the realization of  $T$  and only vary with calendar time. To ensure that the government does not crowd out all economic activity, then, we need to impose  $\lim_{t \rightarrow \infty} b_t + \tau_t < e_0$ .

## 4.2 Credit and Bubbles

We are now ready to introduce credit into the model and show, eventually, that this will allow a bubble to emerge that persists while the endowment grows and collapses when it stops. As we noted in an earlier footnote, Gali allowed for private debt in his model, but since only the young are willing to lend or can credibly promise to repay, no private debt will circulate when all young agents are the same. For private debt to circulate, we need heterogeneity among the young so that some want to save while others want to borrow. Suppose, then, that in each period  $t$  there is still the same group of young agents with aggregate endowment is  $e_t$  as before whose objective is to save for period  $t + 1$ , but there is also a group of young entrepreneurs endowed with no resources but with the technical expertise to convert a unit of goods at date  $t$  into  $1 + y$  units of good at date  $t + 1$ . We assume  $y$  is large, in the sense that

$$1 + y > \sup_t \left\{ \frac{p_{t+1} + d}{p_t} \right\} \tag{25}$$

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<sup>15</sup>Technically, the set of equilibrium prices is upper hemicontinuous in  $d$ , since  $p_t = e_t$  is the unique equilibrium price when  $d > 0$  and continues to be an equilibrium – just not the unique one – when  $d = 0$ . By contrast, the fundamental value of the asset is discontinuous in  $d$  at 0, since  $f_t = e_t > 0$  when  $d > 0$ , but is equal to 0 when  $d = 0$ . Essentially, as  $d$  tends to 0, the interest rate  $\widehat{r}_t$  tends to 0. Thus, for small values of  $d$ , we have an infinite stream of vanishingly small dividends discounted at a vanishingly small rate. This allows the fundamental value to remain bounded strictly away from 0 when  $d$  is arbitrarily small but positive even as the fundamental equals 0 when  $d = 0$ .

That is, entrepreneurs can earn a higher return than the asset can deliver. Although this restriction concerns an endogenous object,  $p_t$ , the price is driven by  $e_t$  and fiscal policy, and so (25) amounts to a restriction on paths for fiscal policy. We further assume that there is a mass  $\kappa$  of entrepreneurs, where  $\kappa < e_0$ , and each entrepreneur faces a capacity constraint and can convert at most one unit of goods into  $1 + y$  units of goods one period later. These assumptions ensure the young who want to save and the entrepreneurs who want to produce can gain from trading with each other. For now, we assume this trade is carried out through debt contracts, i.e. young entrepreneurs promise to pay  $1 + R_t$  units of resources at date  $t + 1$  for each unit they receive at date  $t$ . Later on discuss whether agents would opt to use such contracts.

In the absence of credit market frictions, entrepreneurs receive the full  $\kappa$  units of resources they need and the wealthy lend to entrepreneurs in addition to buying the asset and government debt. However, the mere existence of credit will not give rise to a bubble. Our economy with entrepreneurs is essentially equivalent to an economy with no entrepreneurs where wealthy agents receive an endowment of  $e_t - \kappa$  they must save to consume one period later. In that case, we know from Proposition 6 that no bubble is possible. It is only when credit markets are subject to frictions that a bubble might emerge.

One credit market friction that has attracted considerable attention is borrowing constraints. In our context, this implies entrepreneurs collectively raise less than  $\kappa$  units of resources in the absence of bubbles. Several papers have demonstrated that this situation allows intrinsically worthless assets to trade at a positive price. Intuitively, trading assets this way can substitute for credit as a way of transferring resources to otherwise constrained agents: Agents buy intrinsically worthless assets with the intent of selling them when they need inputs for production; in turn, agents looking to save for the future buy these assets. This channel is explored in Farhi and Tirole (2012), Martin and Ventura (2012), and Hirano and Yanagawa (2017). Other papers have emphasized that purchasing intrinsically worthless assets that trade at a positive price can help agents relax the borrowing constraints they face; see, for example, Kocherlakota (2009), Miao and Wang (2015), and Martin and Ventura (2016).<sup>16</sup>

By contrast, we consider an economy where entrepreneurs receive all  $\kappa$  units they need so agents are not borrowing constrained. We do this not because borrowing constraints are not important, but because we wish to highlight a different credit market friction that can also give rise to bubbles, namely informational frictions that allow non-entrepreneurs to pass for entrepreneurs and borrow, not to produce but to buy risky assets with the hope that their value appreciates. This theory of bubbles is explored in Allen and Gorton (1993), Allen and Gale (2000), and Barlevy (2014). Informational frictions and borrowing constraints can certainly coexist. We focus on information frictions because they speak more to the concern about credit-driven bubbles in Mishkin's remarks. In particular, bubbles that arise because entrepreneurs are borrowing constrained tend to be socially beneficial, precisely because they facilitate trade. As such, it will typically not be optimal to lean against such bubbles as much as to intervene and prevent their collapse. Bubbles that arise because of informational frictions, by contrast, provide no comparable social benefit.

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<sup>16</sup>Kocherlakota (1992), Santos and Woodford (1997) and Kocherlakota (2008) also argued that the existence of borrowing constraints can give rise to bubbles. However, they considered endowment economies rather than production economies.

To capture these information frictions, we need three types of young agents. That is, in addition to the group of agents with a total endowment of  $e_t$  that they want to save for date  $t + 1$  and the  $\kappa$  entrepreneurs who have no endowments but the skills to produce, we add a third group of agents, whom we shall call speculators, who have no endowments nor any technical expertise. For simplicity, we assume all three groups have linear utility in the amount they consume at date  $t + 1$ . Wealthy agents cannot tell entrepreneurs and speculators apart nor monitor their actions. As a result, even though speculators bring neither resources nor skills to the table, they can still potentially interfere in the trade between wealthy agents and entrepreneurs by joining the latter borrowing to buy assets.<sup>17</sup> We assume that there are infinitely many speculators born each period. This amounts to allowing for free entry into speculation.

Why would speculators borrow to buy assets? Buying riskless government bonds is pointless: Wealthy agents can buy bonds themselves, and would require at least as much in compensation for lending as the return  $r_t$  on government bonds. There would similarly be no point in buying the dividend-bearing asset after date  $T$ : The return to buying the asset is deterministically equal to  $(\hat{p}_{t+1} + d)/\hat{p}_t$ , which must equal the interest rate on government bonds  $\hat{r}_t$ . But there might be benefit to buying the asset before date  $T$ . At this point, the return to buying the asset is stochastic: It will equal  $(p_{t+1} + d)/p_t$  if the endowment continues to grow and  $(\hat{p}_{t+1} + d)/p_t$  if the endowment stops growing. If the interest rate on loans  $1 + R_t$  falls between these two values, speculators would earn profits if the endowment grew, and would have to default and be left with nothing – no worse than what they start with – if the endowment stopped growing.

To model the way those who wish to save and those interested in obtaining resources interact, we assume any agent looking to save can offer a menu of contracts for financing at date  $t$ . Each contract stipulates an amount  $w_t$  to be transferred from the creditor at date  $t$  together with an amount  $z_{t+1}(e_{t+1})$  the creditor will be paid at date  $t + 1$  for each realization of  $e_{t+1}$ . Entrepreneurs and speculators then select which contract if any to enter. From date  $T$  on, when there is no uncertainty about  $e_{t+1}$ , we can always interpret such contracts as debt contracts in which the creditor lends  $w_t$  at an interest rate of  $1 + R_t \equiv z_{t+1}/w_t$ .

Before date  $T$ , while there is still uncertainty, creditors must set  $z_{t+1}(e_{t+1})$  high enough for each realization of  $e_{t+1}$  to leave speculators with zero profits. Otherwise, given free entry into speculation, demand for resources would exceed the finite amount  $e_t$  agents can lend out. Hence,

$$z_{t+1}(e_{t+1}) \geq \frac{d + p_{t+1}(e_{t+1})}{p_t} w_t \quad (26)$$

Since the lower bound in (26) corresponds to the total resources speculators have if they purchase risky assets, their contract will satisfy (26) with equality. We can interpret this as a debt contract that requires

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<sup>17</sup>The assumption that lenders cannot tell entrepreneurs and speculators apart seems plausible for new, hard-to-evaluate technologies, where it may be difficult to distinguish those who adept in the new technology from those who buy assets merely in the hope their price appreciates. The assumption seems less plausible for housing, where lenders certainly scrutinize borrowers and the assets they purchase. But in that case, the difference between speculators and good borrowers is not the assets they buy but their willingness to default if prices fall. See Barlevy and Fisher (2014) for an example of a model of risk-shifting bubbles where information frictions concern the preferences of borrowers rather than the assets they buy.

stipulates an interest rate  $1 + R_t$  which ensures the speculator will default if the endowment fails to grow. As for entrepreneurs, any contract offered to them must also satisfy (26), but it can hold as a weak inequality. Unlike the case with speculators, such contracts cannot in general be interpreted as debt contracts. In particular, entrepreneurs earn a fixed amount from their production regardless of the realization of  $e_{t+1}$ . Hence, the amount they can afford to pay does not depend on the realization of  $e_{t+1}$ . But to the extent that the contract stipulates state-contingent payments, i.e.  $z_{t+1}(e_{t+1}) \neq z_{t+1}(\widehat{e}_{t+1})$ , we can no longer interpret the contract as debt contract in which the lower payment corresponds to default. It is only when the lower payment exhausts the available resources of the borrower that this interpretation of a debt contract applies. Assuming that agents only trade using debt contracts thus amounts to an implicit assumption that writing contingencies into a contract is very costly, e.g. it is too costly to show a third party which realization of  $e_{t+1}$  materialized. Even without such costs, though, there exist equilibria in the credit market in which all creditors offer debt contracts given this is what other creditors offer.<sup>18</sup>

Proceeding with the assumption that agents use debt contracts, an equilibrium corresponds to a set of paths  $\{p_t, r_t, R_t\}_{t=0}^{\infty}$  and  $\{\widehat{p}_t, \widehat{r}_t, \widehat{R}_t\}_{t=0}^{\infty}$  for prices and interest rates that are consistent with market clearing. As before, the equilibrium price of the asset is easy to pin down. Since there is no storage, the endowment  $e_t$  at each date  $t$  will be used to pay the tax obligations of the young, to buy the bonds issued by government, to finance entrepreneurs, and to buy the asset. The equilibrium price for the asset is thus

$$\widehat{p}_t = p_t = e_t - b_t - \tau_t - \kappa \quad (27)$$

From date  $T$  on, there is no uncertainty and hence no way for speculators to profit. In that case,

$$\widehat{R}_t = \widehat{r}_t = \frac{\widehat{p}_{t+1} + d}{\widehat{p}_t} \quad (28)$$

Substituting in for  $\widehat{p}_t$  from (27), we can express  $\widehat{R}_t$  and  $\widehat{r}_t$  in terms of  $e_t$  and  $\tau_t$ .

Finally, we turn to the interest rates  $R_t$  and  $r_t$  while the economy is growing. From (26), we know that  $R_t$  must be set to deny speculators any profits, i.e.

$$1 + R_t \geq \frac{p_{t+1} + d}{p_t} \quad (29)$$

We will focus on the equilibrium where (29) holds with equality, i.e. where creditors charge no more than necessary to keep away the infinite number of potential speculators. There are other equilibria where (29) holds as a strict equality, but these will also give rise to bubbles just as in the case where (29) holds with equality. To solve for the risk-free rate  $r_t$  on government debt, note that in equilibrium creditors must be indifferent between making loans and buying government debt. Lenders expect entrepreneurs to pay them

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<sup>18</sup>If all agents offer debt contracts with  $1 + R_t = (d + p_{t+1})/p_t$ , a creditor might consider deviating and offer a contract that charges entrepreneurs less if  $e_{t+1} = \widehat{e}_{t+1}$ , if it was sure only entrepreneurs would take that contract. But speculators would be willing to take those contract as well, so there are beliefs that deter such deviations. Our result on the existence of a bubble does not hinge on agents using debt contracts, but that creditors expect to earn more from entrepreneurs than from speculators. As long as  $z_{t+1}(\widehat{e}_{t+1})$  exceeds the lower bound in (26), bubbles will emerge even if  $z_{t+1}(\widehat{e}_{t+1}) \neq z_{t+1}(e_{t+1})$ .

back in full given our restriction on  $y$  in (25). creditors also expect that speculators will pay them back in full if the endowment keeps growing and default and pay at a rate of  $(d + \widehat{p}_{t+1})/p_t$  if the endowment stops growing. The expected return to lending depends on the relative share of borrowing by entrepreneurs and speculators. We can show that the only possible equilibrium is one in which all assets are purchased by speculators. Intuitively, speculators value assets more than agents who purchase them outright, since the former have an option to default if the endowment stops growing. This means that in any equilibrium, wealthy agents lend  $\kappa + p_t$  in total, enough to finance all entrepreneurs and all asset purchases by speculators. The return on loans will yield an expected return of  $r_t$  if and only if

$$1 + r_t = (1 + R_t) \left[ \frac{\kappa}{\kappa + p_t} + (1 - \pi) \frac{p_t}{\kappa + p_t} \right] + \pi \left( \frac{d + \widehat{p}_{t+1}}{\kappa + p_t} \right)$$

Rearranging this equation and substituting in from (27), we obtain

$$\begin{aligned} R_t - r_t &= \frac{\pi (p_{t+1} - \widehat{p}_{t+1})}{\kappa + p_t} \\ &= \frac{\pi g e_t}{e_t - b_t - \tau_t} > 0 \end{aligned} \tag{30}$$

Thus, there will be a spread between the rate agents charge on loans  $R_t$  and the risk-free rate paid on government bonds  $r_t$ . This spread compensates lenders for losses they would incur on their loans to speculators if the endowment failed to grow between dates  $t$  and  $t + 1$ . We can rearrange (30) to arrive at an expression for  $r_t$ . This completes the description of equilibrium.

To sum up, in this economy wealthy agents will lend entrepreneurs the  $\kappa$  resources they need. While the endowment is growing, speculators borrow to buy the asset in the hope that their price appreciates. If the endowment stops growing, they will default. From that point on, there is no opportunity for speculators to gain from borrowing. The economy thus begins in a period of speculative growth where agents borrow in order to speculate and continues until a period of secular stagnation sets in.<sup>19</sup> We now argue the period of speculative growth will feature an asset bubble, i.e. that  $\Delta_t = p_t - f_t > 0$  for  $t < T$ .

If we combine (30) with the fact that  $1 + R_t = (d + p_{t+1})/p_t$ , we can express the risk-free rate  $r_t$  for dates  $t < T$  as follows:

$$1 + r_t = \frac{(1 - \pi) p_{t+1} + \pi \widehat{p}_{t+1} + d}{p_t} + \frac{\pi \kappa g e_t}{p_t (p_t + \kappa)} \tag{31}$$

This implies that if  $\kappa > 0$ , the risk-free rate  $r_t$  will strictly exceed the expected return on buying the asset. This is consistent with our earlier claim that lenders will not buy the asset themselves and only lend to speculators who buy them, since the asset will be dominated by anything that offers lenders the risk-free

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<sup>19</sup>Previous work on speculative growth has argued that bubbles encourage capital accumulation and hence growth. For example, Caballero, Farhi, and Hammour (2006) argue bubbles can lead to more savings that translate into capital accumulation, while Martin and Ventura (2012) and Hirano and Yanagawa (2016) argue bubbles improve the allocation of resources in economies with borrowing constraints. Our analysis highlights the opposite direction: growth can foster speculation, since it feeds into growth in asset prices that make it profitable to speculate on how long growth will continue.

rate of return  $r_t$ . Let us define  $1 + r_t^A$  as the expected return to buying the asset at date  $t$ , i.e.

$$1 + r_t^A \equiv E \left[ \frac{p_{t+1} + d}{p_t} \right]$$

In Proposition 6, we showed that if we discount dividends at the rate  $1 + r_t^A$  above, the expected present discounted value of dividends must equal the price, i.e.

$$p_t = E \left[ \sum_{t=1}^{\infty} \left( \prod_{i=0}^s \frac{1}{1 + r_t^A} \right) d \right]$$

By contrast, the equilibrium fundamental value of the asset  $f_t$  is defined as the expected present discounted value of dividends discounted at the riskless rate  $1 + r_t$ , i.e.

$$f_t = E \left[ \sum_{t=1}^{\infty} \left( \prod_{i=0}^s \frac{1}{1 + r_t} \right) d \right]$$

Since (31) implies  $r_t > r_t^A$  for  $t < T$ , it follows that the price of the asset will strictly exceed the fundamental value for as long as  $t < T$ . This can be summarized in the next Proposition:

**Proposition 7:** If  $\kappa > 0$ , then  $\Delta_t > 0$  for  $t < T$  and  $\Delta_t = 0$  for  $t \geq T$ . This is, the economy will exhibit a stochastically bursting bubble.

The reason a bubble emerges before  $T$  is that entrepreneurs cross-subsidize speculators by lowering the interest charged to all borrowers. Interestingly, the bubble arises not because speculators drive up the price of the asset, but because speculation leads to a higher risk-free rate, which in turn drives down the fundamental value of the asset. This is because in our formulation, agents spend all of their endowment on the asset, and so the price of the asset will be fixed by the endowment regardless of what speculators do. With more general preferences, the fact that speculation changes the risk-free rate might affect how much individuals choose to save and thus the price of the asset as well.

### 4.3 Policy Interventions

Now that we have generated a credit-driven bubble, we can look to see whether in this case it is still possible for a government intervention to drive up the interest rate and dampen the bubble. Our proof of Proposition 4 carries through as before. That is, if we restrict fiscal policies to paths as in (13) that can be parameterized by a single parameter  $b$  that governs the size of debt obligations, then increasing debt will drive down the price of the asset and drive up the risk-free rate.

**Proposition 8:** Suppose fiscal policy is given by (13), where  $r_{-1} > 0$ . Then the equilibrium interest rate  $r_t$  and loan rate  $R_t$  are both increasing in  $b$  and the equilibrium asset price  $p_t$  is decreasing in  $b$  for all  $t$ .

Recall that since  $d > 0$ , a decline in the price of the asset  $p_t$  need not imply a smaller bubble, since higher rates driven down both the price of the asset and its fundamental value. In fact, for credit-driven

bubbles, issuing more debt will not always reduce the bubble. This highlights an important difference between credit-driven bubbles and the bubbles in our original framework borrowed from Gali's setup. In the case of credit-driven bubbles, we can think of both the price of the asset and the fundamental value of the asset as present discounted expected value of dividends evaluated at different discount rates – the price corresponds to discounting at rate  $1 + r_t^A$ , the expected return on the asset, and the fundamental corresponds to discounting at rate  $1 + r_t$ , the return on risk-free assets. The effect of intervention depends on how the intervention affects these two rates, and in principle can make the bubble smaller or larger. By contrast, in the bubbles we originally studied, there was no uncertainty and thus only a single interest rate: The expected return to the asset and the return on risk-free assets were one and the same. The reason a bubble emerged could be interpreted to mean that the price of an asset corresponds to the present discounted value of a shadow dividend that exceeds  $d$  and which stems from the value the bubble provides as a superior savings instrument. In that case, there was no ambiguity about how an intervention affected the bubble: Raising rates lowers the discounted value of both dividends and shadow dividends. But since the latter is larger, the effect of higher rates will be larger on the price than on fundamentals. This is quite different from the effect of interventions in the environment we use to model credit-driven bubbles.

Calculating the fundamental value  $f_t$  in our model to determine the effect of increasing the interest rate  $r_t$  on  $\Delta_t$  turns out to be unwieldy. However, we can obtain analytical insights if we assumed that rather than assume a geometric distribution,  $T$  could only take on two values, i.e.

$$T = \begin{cases} 1 & \text{with probability } \pi \\ 2 & \text{with probability } 1 - \pi \end{cases}$$

In other words, the endowment will either grow for one period or not at all. Speculators at date 0 can gamble on whether this growth materializes or not, but there is no scope for gambling beyond date 0. In this case, we have the following result:

**Proposition 9:** Suppose fiscal policy is given by (13), where  $r_{-1} > 0$ . Then there exists a value  $\kappa^*$  where  $0 < \kappa^* < e_0$  such that an increase in  $b$  will lead to a larger bubble at date 0 if  $0 < \kappa < \kappa^*$  and a smaller bubble at date 0 if  $\kappa^* < \kappa < e_0$ .

In words, as long as there are enough safe borrowers to cross-subsidize speculation, a policy intervention that induces agents to hold more government debt will increase rates, drive down the price of the asset, and reduce the extent to which the asset is overvalued. Intuitively, a higher value of  $\kappa$  not only implies more cross-subsidization, but it also implies fewer resources will be used to purchase the asset, since by assumption production is assumed to carry a high return. This will drive down both the price and the fundamental value of the asset. When the fundamental value of the asset is low, a policy that increases the interest rate will have a small effect on the fundamental value given it is already low. But the effects of crowding out resources from the asset are not blunted when the price is low. In this case an intervention that induces agents to hold more assets will depress the bubble.

## 4.4 Welfare

We have thus generated an example of a bubble that resembles the type of credit-driven bubbles Mishin describes: These bubbles occur precisely because credit creates more demand for the asset, and agents who borrow against the asset default when the bubble bursts. An intervention that raises rates can, at least under certain conditions, dampen the bubble before it bursts. Thus, we have a setup that captures the precise scenario that motivates advocates of leaning against the wind. And yet, even in this case, it turns out that at least as the model is specified thus far, intervention still cannot make society as a whole better off. This is because when the supply of the asset is fixed, as we have assumed so far, any intervention would simply redistribute resources from entrepreneurs to lenders. Some agents benefit from intervention, but others lose. To see this, recall that speculators earn zero profits in equilibrium, by virtue of (29), and are therefore unaffected by the intervention. Next, observe that the wealthy agents who seek to convert their endowment when young into consumption when old always spend  $e_t - \tau_t$  to buy bonds and make loans, regardless of how much government debt circulates. Lenders earn  $(1 + r_t) b_t$  on the government bonds they hold,  $(1 + R_t) \kappa$  from their loans to entrepreneurs, and expect to earn  $(1 + r_t^A) p_t$  from lending to speculators to buy the asset, i.e. their expected return is

$$\frac{(1 + r_t) b_t + (1 + R_t) \kappa + (1 + r_t^A) p_t}{e_t - \tau_t}$$

Using the expressions for  $p_{t+1}$  and  $\hat{p}_{t+1}$ , we can deduce that the return for these young agents is equal to

$$\frac{(1 + (1 - \pi) g) e_t + d + R_t \kappa}{e_t - \tau_t}$$

Proposition 8 tells us that intervening to raise the risk-free rate will also increase the interest rate  $R_t$  on loans. But an increase in  $R_t$  makes young agents who wish to save better off at the expense of those young agents who undertake production and must hand over more of the  $y$  units of output they produce to creditors. Beyond this redistribution, an intervention that raises  $r_t$  and drives down  $p_t$  will have no effects on welfare. In particular, the risk from the uncertain growth rate for the economy is always fully borne by young agents who wish to save, regardless of  $b$ . Reducing the bubble cannot eliminate this risk which is inherent to the endowment of agents. Dampening the bubble in our model will therefore not be inherently desirable.

To break this logic, we need to allow the supply of assets to be variable. In that case, society might be worse off if a larger bubble encouraged the creation of too many bubble assets. Developing a full-fledged example of this is beyond the scope of this paper. However, in Appendix C we illustrate the potential for a model with variable asset supply to admit a role for policy intervention. In that section, we consider a different setup in which the uncertainty that speculators gamble on does not concern the aggregate endowment but to the dividend paid by the asset. The key distinction is that risk to the aggregate endowment is fixed and unaffected by the amount of risky assets, but when assets themselves are risky, the amount of assets will determine the extent of aggregate risk the economy faces. Our example in Appendix C shows that the type of credit-driven bubbles we have analyzed here can emerge in an environment where the supply of assets

is variable, and so the amount of aggregate risk is endogenous. What we do not do here is show that this implies intervention is desirable if and when a bubble arises. But our example suggests how one might go about establishing this. First, if we assumed agents were risk-averse rather than risk-neutral, any effects on the endogenous amount of aggregate risk would certainly matter for welfare. Since the agents who create new assets are responding to its price rather than its fundamental, there is no reason the economy on its own would be associated with the socially optimal amount of risky assets. As a separate argument, more aggregate risk magnifies the losses lenders would realize if and when the bubble bursts would be greater. Lenders take this into account and set higher rates on loans. But the realized losses can still be greater the larger the bubble. Developing this would require us to model the consequences of losses to financial intermediaries, which again is beyond the scope of our paper. Hence, our analysis only suggests at how intervention might be desirable. But it identifies the key components that would be needed to justify leaning-against-the-wind type policies. This in itself is informative. As we noted in the Introduction, for example, our results suggest there is probably less scope for intervening against bubbles on assets available in fixed supply, e.g. land, than on assets that can be accumulated. And it also suggests that the reason to move against an incipient bubble is not so much to align its price with fundamentals but to move against the excessive risk that high prices might foster.

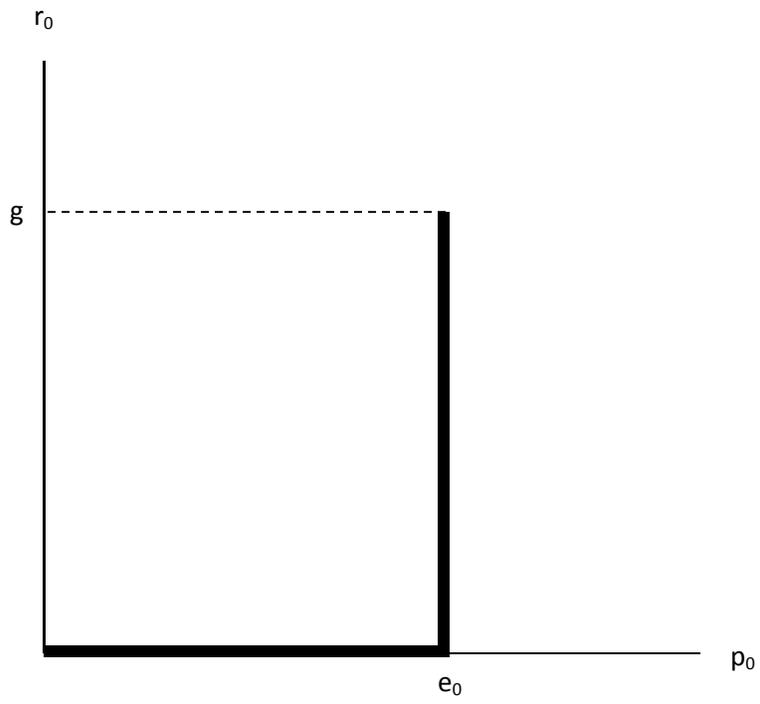
## 5 Conclusion

In this paper, we argue that moving to raise interest rates can be an effective tool against asset bubbles. This result stands in contract to recent work by Gali (2014) which showed that moving to raise interest rates only served to increase the bubble if one were in fact present. We began by replicating Gali's result in a slightly different setting. We argued that the intervention Gali analyzed amounted to selecting equilibria, and across the equilibria that arise in his model, equilibria that feature higher interest rates also feature larger bubbles. We then showed that introducing a seemingly inconsequential modification to our model – assuming the bubble asset is not intrinsically worthless but yields actual dividends – can eliminate the multiplicity of equilibria, forcing us to think of policy interventions not in terms of selecting an equilibrium but in terms of a direct intervention in financial markets such as issuing or selling government bonds. We construct a few examples of interventions by a policymaker that can simultaneously raise the interest rate and reduce the extent to which the bubble asset is overvalued. Thus, the notion that a policymaker intent on mitigating a bubble would act in a way that raised interest rates can be plausible, notwithstanding the findings laid out by Gali. Given this, the natural next question is whether policymakers should raise rates if they believe a bubble is present in asset markets. We argued that the model Gali used, a variant of the Samuelson (1958) model in which bubbles arise when agents are eager to save, does not on its own suggest such interventions are desirable. We then showed how the model we use can be modified to incorporate credit-driven bubbles. That setup seems to better capture the reasons policymakers are concerned about bubbles. We argue that in such an environment, intervening to dampen a bubble can affect the extent of macroeconomic risk agents face, which may create a role for policy.

We conclude our analysis with a few observations. First, one of the points Gali emphasizes in his paper is that theoretically, higher interest rates ought to be associated with more rapid asset price appreciation. In subsequent work, Gali and Gambetti (2015) provide empirical evidence in support of this proposition. Our model, which also features rational agents, is consistent with this implication. However, whether a policymaker who intervenes to raise rates succeeds in depressing a bubble depends not on how interest rates affect the growth rate of asset prices, but how they affect asset prices as compared to fundamentals. Unfortunately, since this requires measuring the fundamental value of assets, the latter is difficult if not impossible to resolve empirically. But our analysis nevertheless shows that the case in which higher rates affect asset prices more than they do fundamentals is a plausible theoretical possibility.

Even if empirical evidence on how interest rates impact prices and fundamental is hard to come by, our analysis does suggest empirical measures that policymakers can use to assess whether contractionary policies are likely to have their intended effect on bubbles, at least in line with the channels we emphasize here. The reason contractionary policy mitigates bubbles in our model is that it crowds out resources from overvalued assets. This is more essential for mitigating bubbles than raising rates per se, and in some of our examples a policymaker can crowd out resources from the asset market without raising interest rates, at least not immediately. This suggests that for an interest rate hike to depress bubbles through the channel we describe in this paper, it should be accompanied by a decline in savings or in the share of the bubble asset in the aggregate wealth portfolio of investors. Absent these, our model would suggest an interest rate hike would not be successful in reining in bubbles. Of course, there may be other channels through which a higher interest rate might depress bubbles that our model fails to capture. For example, one view that is sometimes articulated in policy circles is that low interest rates trigger a search for yield that induces agents to take on more risks. Our model does not admit such a channel. We also ignore any role for interest rate policy to affect bubbles by affecting credit markets. Although we do describe a model in Section 4 where agents borrow to speculate on assets, the amount speculators borrow in our model is unaffected by the interest rate. In principle, though, a higher interest rate might make speculation unprofitable, even if a higher rate increases the expected rate of appreciation. Formalizing this reasoning would presumably require a richer model of credit spreads than we develop here.

The last point we wish to make is that while our analysis here can be viewed as a first step towards a model that can be used to justify leaning against the wind policies, further work is needed to determine whether such policies represent the best way to combat asset bubbles. Even if one can show that dampening credit-driven bubbles is welfare improving by reducing macroeconomic volatility, there is more than one way to attempt to reduce a bubble. For example, a central bank could impose regulatory restrictions on credit markets that discourage or mitigate credit-driven bubbles. In the model we develop, for example, a restriction on the amount agents can borrow against an asset would have implications on the size of the bubble. Such restrictions would also discourage socially valuable trade between creditors and entrepreneurs. Determining what is the most cost effective way to dampen bubbles remains a challenge for future work.



**Figure 1:** Set of possible equilibrium values  $(p_0, r_0)$  at date 0

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## Appendix A: Proofs of Propositions

**Proof of Proposition 1:** In the text, we argued that  $r_t = 0$  is an absorbing state. That is, there exists a  $t^*$  where  $0 \leq t^* \leq \infty$  such that  $r_t > 0$  for  $t < t^*$  and  $r_t = 0$  for  $t \geq t^*$ . For  $t < t^*$ , we know that  $r_t > 0$  implies storage is dominated, and so  $p_t = e_t$  for  $t < t^*$ . Since  $r_t = 0$  for  $t \geq t^*$  we can use (3) to conclude that  $p_{t+1} = p_t$  for  $t \geq t^*$ , and by induction we can infer  $p_t = p_{t^*}$  for all  $t \geq t^*$ .

The last step is to show that at date  $t^*$ , any  $\rho \in [e_{t^*-1}, e_{t^*}]$  can be an equilibrium, and only these values can be an equilibrium. Since  $r_t \geq 0$ ,  $p_{t^*} \geq p_{t^*-1} = e_{t^*-1}$ . Since  $p_t \leq e_t$  at all dates  $t$ , this is also true for date  $t^*$ . For any  $\rho \in (e_{t^*-1}, e_{t^*}]$ , the rate of return on the asset to those purchasing the asset at date  $t^* - 1$  will be positive, so they would buy the asset at this price at date  $t^* - 1$  and sell all their holdings at date  $t^*$ . Since  $r_{t^*} = 0$ , the young at date  $t^*$  are indifferent between storage and buying the asset, so they would be willing to buy any amount the old sell. Hence, we can construct an equilibrium where  $p_{t^*} = \rho$  for any  $\rho \in (e_{t^*-1}, e_{t^*}]$ . ■

**Proof of Proposition 2:** In the text, we argued that  $r_t > 0$ . Since storage is dominated, the price  $p_t = e_t$ , since at any other price the young would be using storage which is inferior or saving more resources than they are endowed with. The interest rate is then pinned down by the requirement that

$$1 + r_t = \frac{d + p_{t+1}}{p_t}$$

and substituting in  $p_t = e_t$  and  $p_{t+1} = (1 + g)e_t$ . ■

**Proof of Lemma 1:** The proof is by induction. Suppose we are given histories  $\{b_{-1}, \dots, b_{t-1}\}$  and  $\{r_{-1}, \dots, r_{t-1}\}$  up to but not including date  $t$ . We want to show that these histories imply a unique candidate pair  $b_t$  and  $r_t$  that satisfy both the government budget constraint and the equilibrium interest rate condition. From the government's flow budget constraint, we have

$$b_t = (1 + r_{t-1})b_{t-1} - \tau_t$$

Since  $\tau_t$ ,  $b_{t-1}$ , and  $r_{t-1}$  are all given to us,  $b_t$  is uniquely determined. Next, consider the equilibrium condition for interest rates (11). We already argued that when  $d > 0$ , the equilibrium interest rate  $r_t$  must be positive or else the price of the asset will eventually turn negative. Hence, storage will be dominated, and agents will exchange all of their endowment net of taxes for bonds and the asset, i.e.

$$p_t = e_t - \tau_t - b_t$$

Substituting in for  $p_t$  and  $p_{t+1}$  into (11), we get

$$(1 + r_{t-1})b_{t-1} - b_t = (1 + r_{t-1})(e_{t-1} - \tau_{t-1}) - (e_t - \tau_t) - d$$

But from (10), we know that

$$(1 + r_{t-1})b_{t-1} - b_t = \tau_t$$

and so we have

$$\tau_t = (1 + r_{t-1})(e_{t-1} - \tau_{t-1}) - (e_t - \tau_t) - d$$

which reduces to

$$d = (1 + r_t)(e_t - \tau_t) - e_{t+1} \quad (32)$$

Since  $e_t$ ,  $e_{t+1}$ , and  $\tau_t$  are given,  $r_t$  is uniquely determined. The claim then follows by induction. ■

**Proof of Proposition 3:** We know from Lemma 1 that the path  $\{r_t, b_t\}_{t=0}^{\infty}$  is uniquely determined in equilibrium. Since  $r_t > 0$ , we know that storage is dominated, so the young will use all of their available endowment to purchase assets. This is equal to  $e_t - \tau_t$ . Since they must allocate  $b_t$  to government debt, the amount spent on the asset is equal to  $p_t = e_t - \tau_t - b_t$ . The expression for  $r_t$  then follows from the fact that in equilibrium,  $(1 + r_t)p_t = d + p_{t+1}$ . Finally, (12) implies that

$$\lim_{t \rightarrow \infty} \frac{\tau_t + b_t}{e_t} = \theta$$

for some  $\theta < 1$ . Hence,

$$\begin{aligned} \lim_{t \rightarrow \infty} p_t &= \lim_{t \rightarrow \infty} e_t - \tau_t - b_t \\ &= \lim_{t \rightarrow \infty} e_t - \theta e_t \\ &= (1 - \theta) \lim_{t \rightarrow \infty} e_t = \infty \end{aligned}$$

where the last step uses the fact that  $\theta < 1$ . ■

**Proof of Proposition 4:** At  $t = 0$ , we have  $\tau_0 = r_{-1}b$  which is increasing in  $b$ . From the intertemporal budget constraint (10) and the equilibrium condition (11), we know that

$$d = (1 + r_t)(e_t - \tau_t) - (1 + g)e_t$$

Since  $\tau_0$  is increasing in  $b$ , it follows that  $r_0$  is increasing in  $b$  as well. Next, suppose we know  $r_0, \dots, r_{t-1}$  are increasing in  $b$ . Then we have  $\tau_t = r_{t-1}b$  and by the same logic as above, we can conclude that  $\tau_t$  and  $r_t$  are increasing in  $b$ . To finish the claim, we need to prove that  $p_t$  is decreasing in  $b$ . But from Proposition 4, we know that

$$p_t = e_t - b - \tau_t$$

which establishes the result. ■

Before proving Proposition 5, we establish the following lemma:

**Lemma A1:** Suppose fiscal policy is given by (13) and satisfies (12). Then  $\lim_{t \rightarrow \infty} r_t = g$ .

**Proof of Lemma A1:** Recall from (32) that

$$d = (1 + r_t)(e_t - \tau_t) - e_{t+1}$$

Using the fact that  $e_{t+1} = (1 + g)e_t$  and the fact that when  $b_t = b$  for all  $t$ ,

$$\begin{aligned}\tau_t &= (1 + r_{t-1})b_{t-1} - b_t \\ &= (1 + r_{t-1})b - b = r_{t-1}b\end{aligned}$$

we can rewrite (32) as

$$d = (r_t - g)e_t - (1 + r_t)r_{t-1}b$$

which, upon rearranging, yields a difference equation that relates  $r_t$  to  $r_{t-1}$ :

$$r_t = \frac{d + ge_t + r_{t-1}b}{e_t - r_{t-1}b} \equiv h_t(r_{t-1}) \quad (33)$$

Since  $r_{t-1}b = b_t + \tau_t$ , we can rewrite this as

$$r_t = \frac{d + ge_t + b_t + \tau_t}{e_t - b_t - \tau_t}$$

From (12), we know that

$$\lim_{t \rightarrow \infty} b_t + \tau_t = \lim_{t \rightarrow \infty} \theta e_t$$

for some  $\theta < 1$ . Hence, taking limits as  $t$  tends to infinity, we get

$$\begin{aligned}\lim_{t \rightarrow \infty} r_t &= \lim_{t \rightarrow \infty} \frac{d + ge_t + b_t + \tau_t}{e_t - b_t - \tau_t} \\ &= \lim_{t \rightarrow \infty} \frac{d + (g + \theta)e_t}{(1 - \theta)e_t} \\ &= \frac{g + \theta}{1 - \theta}\end{aligned}$$

where the last expression is finite. But if  $\lim_{t \rightarrow \infty} r_t$  is finite, then

$$\begin{aligned}\lim_{t \rightarrow \infty} r_t &= \lim_{t \rightarrow \infty} \frac{d + ge_t + r_{t-1}b}{e_t - r_{t-1}b} \\ &= \lim_{t \rightarrow \infty} \frac{g + \frac{d + r_{t-1}b}{e_t}}{1 - \frac{r_{t-1}b}{e_t}} \\ &= g\end{aligned}$$

where the last equality follows from the fact that if  $\lim_{t \rightarrow \infty} r_t$  is finite, so is  $\lim_{t \rightarrow \infty} r_t b$ . Hence, (12) together with (13) imply that  $\lim_{t \rightarrow \infty} \frac{b_t + \tau_t}{e_t} = \theta = 0$  and  $\lim_{t \rightarrow \infty} r_t = g$ , as claimed. ■

**Proof of Proposition 5:** By repeated substitution, we can relate  $\Delta_t$  to any  $\Delta_{t+h}$  for any horizon  $h$ :

$$\Delta_t = \left( \prod_{j=0}^{h-1} \frac{1}{1 + r_{t+j}} \right) \Delta_{t+h}$$

From Lemma A1, we know that  $\lim_{t \rightarrow \infty} r_t = g$ . Hence, for any value of  $b$ , we have

$$f_t \equiv \lim_{t \rightarrow \infty} \sum_{j=0}^{\infty} \left( \prod_{i=1}^j \frac{1}{1 + r_{t+i}} \right) d \rightarrow \frac{d}{g}$$

By contrast,  $p_t = e_t - r_{t-1}b$ , which tends to  $e_t - gb$ . This implies that the bubble term  $\Delta_t = p_t - f_t$  tends to  $e_t - gb - \frac{d}{g}$ , which is decreasing in  $b$ . In other words, there exists a  $T$  such that for  $t \geq T$ ,  $\Delta_t$  is decreasing in  $b$ . Since  $r_{t+j}$  is increasing in  $b$  by Proposition 4, it follows that  $\Delta_t = \left(\prod_{j=0}^{h-1} \frac{1}{1+r_{t+j}}\right) \Delta_{t+h}$  will be decreasing in  $b$ . ■

**Proof of Proposition 6:** We show the statement holds for the fundamentals at date 0. The argument for other dates is similar. The fundamental value of the asset at date 0 if growth stops until a random date  $T_n$  is given by

$$f_0^n = \sum_{T=1}^n \frac{(1-\pi)^{T-1} \pi}{1-(1-\pi)^n} \left( \sum_{s=1}^{T-1} \left( \prod_{i=0}^s \frac{1}{1+r_i} \right) d + \left( \prod_{i=0}^{T-1} \frac{1}{1+r_i} \right) e_T \right)$$

Here we use the fact that  $f_t^n = e_t$  for  $t \geq n$ . The above summation can be written as two sums:

$$f_0^n = \sum_{T=1}^n \frac{(1-\pi)^{T-1} \pi}{1-(1-\pi)^n} \sum_{t=1}^{T-1} \left( \prod_{i=0}^s \frac{1}{1+r_i} \right) d + \sum_{T=1}^n \frac{(1-\pi)^{T-1} \pi}{1-(1-\pi)^n} \left( \prod_{i=0}^{T-1} \frac{1}{1+r_i} \right) e_T$$

Since for  $t < n$ , we have

$$r_t = (1-\pi)g + \frac{d}{e_t} > (1-\pi)g > 0$$

we can easily establish that the first term converges. As for the second term, once again using the expression for  $r_t$  when  $t < n$ , we have

$$\begin{aligned} \sum_{T=1}^n \frac{(1-\pi)^{T-1} \pi}{1-(1-\pi)^n} \left( \prod_{i=0}^{T-1} \frac{1}{1+r_i} \right) e_T &= \sum_{T=1}^n \frac{(1-\pi)^{T-1} \pi}{1-(1-\pi)^n} \left( \prod_{i=0}^{T-1} \frac{1+g}{1+(1-\pi)g+d/e_i} \right) e_0 \\ &= \sum_{T=1}^n \frac{\pi}{1-(1-\pi)^n} \left( \prod_{i=0}^{T-1} \frac{(1-\pi)(1+g)}{1+(1-\pi)g+d/e_i} \right) \frac{e_0}{1-\pi} \\ &< \sum_{T=1}^n \frac{\pi}{1-(1-\pi)^n} \left( \prod_{i=0}^{T-1} \frac{(1-\pi)(1+g)}{1+(1-\pi)g} \right) \frac{e_0}{1-\pi} \end{aligned}$$

Since the last term converges whenever  $\pi > 0$ , we know that  $\lim_{n \rightarrow \infty} f_0^n$  exists. Moreover, we know that  $f_0^n = e_0$ . This can be computed directly, but it also follows from a simple unravelling argument for bubbles with finite horizons. Hence,  $f_0 = \lim_{n \rightarrow \infty} f_0^n = e_0$ . From this, we can conclude that  $f_0 = e_0$ . A similar argument can be applied at any date. ■

**Proof of Proposition 8:** The implications of higher  $b$  for  $r_t$  and  $p_t$  can be established in the same way as in Proposition 4. From (30), we have that

$$\begin{aligned} R_t &= r_t + \frac{\pi g e_t}{e_t - b_t - \tau_t} \\ &= r_t + \frac{\pi g e_t}{e_t - (1+r_{t-1})b} \end{aligned}$$

Since both  $r_t$  and  $(1+r_{t-1})b$  are increasing in  $b$ , we can conclude that  $R_t$  is increasing in  $b$  as well. ■

**Proof of Proposition 9:** Observe that the price of the asset at date 0 is given by

$$\begin{aligned} p_0 &= e_0 - b_0 - \tau_0 - \kappa \\ &= e_0 - (1 + r_{-1})b - \kappa \end{aligned}$$

Next, from (31), we have

$$(1 + r_0)(e_0 - b_0 - \tau_0 - \kappa) = (1 - \pi)ge_0 + e_0 - b_1 - \tau_1 - \kappa + d + s_0\kappa$$

where

$$s_0 = \frac{\pi ge_0}{e_0 - b_0 - \tau_0}$$

Using the parameterization (13), we have

$$\tau_t = r_{t-1}b$$

and so

$$(1 + r_0)(e_0 - b - r_{-1}b - \kappa) = (1 - \pi)ge_0 + e_0 - (1 + r_0)b - \kappa + d + s_0\kappa$$

Rearranging, we have an expression for  $r_0$  in terms of  $b$  and other primitives.

$$1 + r_0 = \frac{(1 - \pi)ge_0 + r_{-1}b + d + s_0\kappa}{e_0 - r_{-1}b - \kappa}$$

The fundamental value at date 0 is given by

$$f_0 = \frac{d + (1 + (1 - \pi)g)e_0 - (1 + r_0)b - \kappa}{1 + r_0}$$

Substituting in for  $r_0$  and evaluating  $\Delta_0 = p_0 - f_0$  yields

$$\Delta_0 = \frac{\pi ge_0\kappa(r_{-1}b - e_0 + \kappa)}{(1 + r_{-1})b(d + (1 + (1 - \pi)g)e_0 - \kappa) - e_0(d + e_0(1 + (1 - \pi)g) - \kappa(1 - \pi g))}$$

Differentiating  $\Delta_0$  with respect to  $b$  and simplifying (with Mathematica) yields

$$\frac{d\Delta_0}{db} = \frac{(1 + (1 - \pi)g)e_0^2 - (2 + (1 + g)r_{-1} + g(1 - \pi))e_0\kappa + (1 + r_{-1})\kappa^2 + (e_0 - (1 + r_{-1})\kappa)d}{[(1 + r_{-1})b(d + (1 + (1 - \pi)g)e_0 - \kappa) - e_0(d + e_0(1 + (1 - \pi)g) - \kappa(1 - \pi g))]^2} \pi ge_0\kappa$$

The sign of this derivative corresponds to the sign of the numerator in the fraction above, which is independent of  $b$ , and represents a quadratic in  $\kappa$ . This quadratic is convex, since the coefficient on  $\kappa^2$  is  $1 + r_{-1} > 0$ . Evaluating the quadratic at  $\kappa = 0$  yields

$$(1 + (1 - \pi)g)e_0^2 + e_0d$$

which is positive, while evaluating the quadratic at  $\kappa = e_0$  yields

$$-e_0(d + ge_0)r_{-1}$$

which is negative. This tells us that the roots of the quadratic are both real, with one between 0 and  $e_0$  and the other above  $e_0$ . It follows that we can find a  $\kappa^* \in (0, e_0)$  such that  $\frac{d\Delta_0}{db} > 0$  if  $\kappa \in (0, \kappa^*)$  and  $\frac{d\Delta_0}{db} < 0$  if  $\kappa \in (\kappa^*, e_0)$ . ■

## Appendix B: Monetary Policy

In this Appendix, we describe a monetary OLG economy, building on the framework we sketch out in Section 3. The key difference between the model here and the one sketched out in Section 3 is that rather than assuming workers are yeoman farmers who operate their own technology, we now introduce producers who price their goods and hire labor, allowing us to incorporate the possibility of price rigidity. Our framework borrows elements from both Adam (2003) and Appendix 3 in Gali (2014). As in Gali, we assume agents hold money because they derive utility directly from money holdings without modelling why. However, we follow Adam in allowing for variable labor supply and in assuming that the entrepreneurs who hire workers are young rather than old. The implication of this is that all output generated within the period will be used to buy assets and not just a fraction of the output that accrues to the young.

Following Adam (2003), suppose each cohort consists of a unit mass of workers and a unit mass of entrepreneurs. This is in contrast to the model described in Section 3, where we assume there is only a single unit mass of individuals in each cohort. When agents are young, those who are workers supply their labor services to entrepreneurs who know how to deploy labor to produce goods. When they turn old, neither type is productive any more and each must rely on previous earnings to consume. We assume that the two cohorts equally bear the burden of lump sum taxes, so that when the government collects  $\tau_t$  from the young, it collects  $\frac{1}{2}\tau_t$  from workers and entrepreneurs.

As in the text, we use  $M_t$  to denote the amount of money circulating at date  $t$ ,  $\Pi_t$  to denote the gross inflation rate  $P_{t+1}/P_t$ , and

$$x_{t+1} = \frac{M_{t+1} - M_t}{P_{t+1}}$$

as the injection of money between dates  $t$  and  $t+1$  measured in terms of how much this amount could buy at date  $t+1$ . We assume the injection is split equally between old workers and old entrepreneurs, so each expects to receive  $\frac{1}{2}x_{t+1}$  when old.

We start with workers. Each worker is endowed with one unit of time and must decide how to allocate it. The cost of providing effort  $n_t$  for the cohort born at date  $t$  is given by a convex function  $v_t(n_t)$ . As in the text, we assume  $v_t(n_t) = A_t v(n_t)$  where  $\lim_{n \rightarrow 0} v'(n) = 0$  and  $\lim_{n \rightarrow 1} v'(n) = \infty$ . In addition to caring about consumption when old and leisure when young, workers derive utility from their money holdings as in Gali (2014). Specifically, we replace (1) with

$$u(c_t^w, c_{t+1}^w, m_t^w, n_t) = c_{t+1}^w + \frac{\theta}{2} \ln(m_t^w) - v_t(n_t) \quad (34)$$

The budget constraint of a worker is given by

$$c_{t+1}^w = (1 + r_t) \left( \frac{W_t}{P_t} n_t - m_t^w - \frac{1}{2} \tau_t \right) + \Pi_t^{-1} m_t^w + \frac{1}{2} x_{t+1} \quad (35)$$

where  $W_t$  denotes the nominal wage per unit labor. Since government bonds and the asset are perfect substitutes, agents will be indifferent between them. Hence, workers face only two non-trivial choices: How

hard to work and how much of their wealth to hold as money. The optimal effort level  $n_t$  will satisfy

$$A_t v'(n_t) = (1 + r_t) \frac{W_t}{P_t}$$

and the optimal level of money demand will satisfy

$$m_t^w = \frac{\theta}{2((1 + r_t) - \Pi_t^{-1})} = \frac{\theta \Pi_t}{2i_t}$$

Next, we turn to entrepreneurs. They are endowed not with labor but with the knowledge of how to deploy labor to produce goods. Each entrepreneur  $i \in [0, 1]$  can produce a different intermediate good  $i$ . These goods are sold to competitive final goods producers who produce the goods old agents consume. Entrepreneurs derive utility from consuming when old and from holding money, i.e.

$$u(c_t^e, c_{t+1}^e, m_t^e) = c_{t+1}^e + \frac{\theta}{2} \ln(m_t^e) \quad (36)$$

The budget constraint of an entrepreneur is given by

$$c_{t+1}^e = (1 + r_t) \left( \rho_{it} - m_t^e - \frac{1}{2} \tau_t \right) + \Pi_t^{-1} m_t^e + \frac{1}{2} x_{t+1} \quad (37)$$

where  $\rho_{it}$  denote the profits of entrepreneur  $i$  after paying the workers he hires. As will become clear below, entrepreneurs also face two non-trivial choices: How much of their earnings to hold as money, just like workers, and what price to charge for the particular good they produce. Their demand for money is the same as workers, i.e.

$$m_t^e = \frac{\theta \Pi_t}{2i_t}$$

As for what price to set, we first need to specify the production technology for both intermediate and final goods. Suppose that if entrepreneur  $i \in [0, 1]$  hires  $n_{it}$  units of labor to work at date  $t$ , he will produce

$$y_{it} = A_t n_{it}$$

units of good  $i$ , where  $A_t = A_0 (1 + g)^t$ . These goods can be combined to produce final goods according to a Dixit-Stiglitz production function, i.e.  $y_{it}$  of each good  $i \in [0, 1]$  combine to yield  $Y_t$  of final goods, where

$$Y_t = \left( \int_0^1 y_{it}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}}$$

The production of final goods is competitive. Final goods producers will therefore choose intermediate goods  $y_{it}$  to produce the  $Y_t$  final goods at the lowest possible cost. That is, they solve

$$\max P_t \left( \int_0^1 y_{it}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} - \int_0^1 P_{it} y_{it} di$$

The first-order condition with respect to  $y_{it}$  for final goods producers yields the demand for each intermediate good as follows:

$$y_{it} = Y_t \left( \frac{P_t}{P_{it}} \right)^{\frac{1}{\sigma}} \quad (38)$$

where recall  $Y_t$  denotes the output of final goods in equilibrium. Substituting in for  $y_{it}$ , we can compute the cost of producing a unit of final goods. Since the market for final goods is competitive, the price of final goods  $P_t$  must equal this cost. Equating the two yields the familiar Dixit-Stiglitz price index:

$$P_t = \left( \int_0^1 P_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

Each entrepreneur chooses the price  $P_{it}$  of his good to maximize profits  $\rho_{it}$  given demand (38) for his good,

$$\left( P_{it} - \frac{W_t}{A_t} \right) Y_t \left( \frac{P_t}{P_{it}} \right)^{\frac{1}{\sigma}}$$

From the first-order condition for this problem, we have

$$P_{it} = \frac{W_t}{(1-\sigma)A_t} \quad (39)$$

Since the price  $P_{it}$  will determine demand, the choice of price will fully determine how much labor effort entrepreneur  $i$  will hire. The equilibrium nominal wage  $W_t$  will equate this demand with the amount workers are willing to supply.

Finally, we turn to the government sector. The government budget constraint still corresponds to (19). By repeated substitution, we get

$$\begin{aligned} (1+r_{t-1})b_{t-1} &= \sum_{s=0}^{\infty} \left( \prod_{i=0}^{s-1} \frac{1}{1+r_{t+i}} \right) \left[ \tau_{t+s} + \left( \frac{M_{t+s} - M_{t+s-1}}{P_{t+s}} \right) \right] \\ &= \sum_{s=0}^{\infty} \left( \prod_{i=0}^{s-1} \frac{1}{1+r_{t+i}} \right) \left[ \tau_{t+s} + \frac{1}{P_t} \frac{M_{t+s} - M_{t+s-1}}{\prod_{i=0}^{s-1} \Pi_{t+i}} \right] \end{aligned} \quad (40)$$

Equation (40) states that the outstanding government liability  $(1+r_{t-1})b_{t-1}$  at date  $t$  must equal the present discounted value of taxes and seniorage revenue the government is set to collect. We define a monetary intervention as a change in the path  $\{M_t\}_{t=0}^{\infty}$  holding the path of lump-sum taxes  $\{\tau_t\}_{t=0}^{\infty}$  fixed.

We can now examine the effects of monetary policy. Let  $p_t$  denote the real price of the asset relative to the final good. An equilibrium is a path for prices  $\{p_t, P_t, W_t, r_t\}_{t=0}^{\infty}$  and quantities  $\{n_t, c_t^w, c_t^e\}_{t=0}^{\infty}$  such that agents optimize and markets clear. We first consider the case where prices are flexible, i.e. where entrepreneurs set prices  $P_{it}$  knowing the full path of  $\{M_t\}_{t=0}^{\infty}$  and the implied equilibrium nominal wage  $W_t$  associated with this path. We then contrast this with the case where entrepreneurs set their prices before the monetary authority and thus before seeing the realized nominal wage  $W_t$ .

## Case I: Flexible Prices

We begin with the case where prices are flexible. Since intermediate goods producers all choose the same price according to (39), the price of final goods will be given by

$$P_t = \left( \int_0^1 P_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} = \frac{W_t}{(1-\sigma)A_t}$$

The real wage will then equal  $W_t/P_t = (1 - \sigma) A_t$ , and so the equilibrium amount of labor will solve

$$A_t v'(n_t) = (1 + r_t)(1 - \sigma) A_t$$

Since entrepreneurs each set the same price and face the same technology, they will hire the same amount of labor in equilibrium. Since there is a unit mass of entrepreneurs, this implies each entrepreneur will hire  $n_t$  units of labor where  $n_t$  denotes aggregate employment. The total output of final goods will be given by

$$Y_t = A_t \left( \int_0^1 n_t^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} = A_t n_t$$

We now argue that when prices are flexible, the equilibrium real interest rate  $r_t$ , employment  $n_t$ , output of final goods  $Y_t = A_t n_t$ , and total consumption  $C_t = c_t^e + c_t^w$  are all independent of monetary policy, i.e. they will not depend on  $M_{-1}$  or  $\{M_t\}_{t=0}^\infty$ . This feature follows from our assumption that agents have log preferences over real money balances. These preferences ensure that even though changes in  $\{M_t\}_{t=0}^\infty$  affect equilibrium real money balances, changes in real balances do not influence consumption and employment decisions.

Formally, the equilibrium real return on the asset must equal the return on government debt  $1 + r_t$ , i.e.

$$(1 + r_t) p_t = d + p_{t+1}$$

Since agents will not use storage in equilibrium, we can substitute in for the price of the asset  $p_t$ , i.e.

$$(1 + r_t)(A_t n_t - \tau_t - b_t - m_t) = d + (A_{t+1} n_{t+1} - \tau_{t+1} - b_{t+1} - m_{t+1})$$

where aggregate real money balances under log utility are given by

$$m_t = m_t^e + m_t^w = \frac{\theta \Pi_t}{i_t}$$

Substituting the government budget constraint (19) into the equilibrium condition above implies

$$\begin{aligned} (1 + r_t)(A_t n_t - \tau_t) &= d + (A_{t+1} n_{t+1} + (1 + r_t - \Pi_t^{-1}) m_t) \\ &= d + \left( A_{t+1} n_{t+1} + i_t \frac{m_t}{\Pi_t} \right) \\ &= d + (A_{t+1} n_{t+1} + \theta) \end{aligned}$$

and so

$$1 + r_t = \frac{d + A_{t+1} n_{t+1} + \theta}{A_t n_t - \tau_t} \quad (41)$$

From the worker's first order condition, we know that they will choose  $n_t$  so that

$$v'(n_t) = (1 + r_t)(1 - \sigma) \quad (42)$$

Conditions (41) and (42) together yields a difference equation in  $n_t$ :

$$\frac{d + A_{t+1} n_{t+1} + \theta}{A_t n_t - \tau_t} = \frac{v'(n_t)}{1 - \sigma}$$

In the limit as  $t \rightarrow \infty$ , this converges to the condition

$$n_{t+1} = \frac{v'(n_t) n_t}{(1+g)(1-\sigma)}$$

This condition has a fixed point at  $n_* = 0$  and one other fixed point at  $n^* = v'^{-1}[(1+g)(1-\sigma)]$ , and the latter is an unstable fixed point. Hence, the limiting condition  $\lim_{t \rightarrow \infty} n_t = n^*$  provides a boundary condition for the difference equation in  $n_t$ . It follows that  $n_t$  is independent of  $\{M_t\}_{t=0}^\infty$ , and from (42) is as well.

Next, we turn to the variables that do respond to monetary policy. Consider first the price of goods relative to money,  $P_t$ . From the demand for money balances, we have

$$\frac{M_t}{P_t} = \frac{\theta \Pi_t}{i_t} = \frac{\theta \Pi_t}{(1+r_t)\Pi_t - 1}$$

Since  $\Pi_t = P_{t+1}/P_t$  for all  $t$ , we can rearrange this condition into a difference equation where  $P_{t+1}$  can be expressed as a function of  $P_t$ :

$$P_{t+1} = \frac{1}{(1+r_t)/P_t - \theta/M_t}$$

The boundary condition for this difference equation comes from the government budget constraint (19) evaluated at date 0. That is, given a path for inflation  $\{\Pi_t\}_{t=0}^\infty$ , the initial price  $P_0$  must be such that the amount of seigniorage revenue just offsets the difference between the initial obligation  $(1+r_{-1})b_{-1}$  and the present discounted of lump-sum taxes the government collects. This allows us to solve for the path of prices  $\{P_t\}_{t=0}^\infty$ . We then can easily back out nominal wages and the real price of the asset:

$$\begin{aligned} W_t &= (1-\sigma)A_t P_t \\ p_t &= A_t n_t - \tau_t - b_t - \frac{M_t}{P_t} \end{aligned}$$

In short, when prices are flexible, monetary policy has no effect on the real interest rate, employment, or output, but it can affect the real price of the asset  $p_t$ . Consider the price  $p_t$  at date  $t = 0$ . From the intertemporal government budget constraint in (19), we have

$$b_0 + \tau_0 + \frac{M_0}{P_0} = (1+r_{-1})b_{-1} + \frac{M_{-1}}{P_0}$$

Substituting this into the expression for  $p_t$ , the real price of the asset at date 0 is given by

$$p_0 = A_0 n_0 - (1+r_{-1})b_{-1} - \frac{M_{-1}}{P_0}$$

Since  $n_t$  is independent of monetary policy, while  $(1+r_{-1})b_{-1}$  and  $M_{-1}$  are fixed, the effect of monetary policy on the price of the asset at date 0 works entirely through the initial price level  $P_0$ . In particular, a policy that lowers the nominal price of goods, or alternatively that increases the real value of money, will lower the real price of the asset. This is also what we argue in the text.

Note that a monetary policy intervention that drives down the price level  $P_0$  will also depress any bubble in the asset. This is because the real interest rate  $r_t$  is independent of monetary policy. Hence, the

fundamental value of the asset, i.e. the present discount value of the dividends it yields evaluated at the market real interest rate, will be unchanged, and so the gap between the price and the fundamentals will be lower.

Reducing the price level arguably represents a contractionary monetary policy, since it makes money more valuable. However, this policy will not result in a higher real interest rate, since recall the real interest rate  $1 + r_t$  is independent of  $\{M_t\}_{t=0}^{\infty}$  when prices are flexible. It has ambiguous implications for the nominal rate: (40) suggests that the price level generally depends on the entire future path of money, so a lower initial price level  $P_0$  could in principle be associated with either a higher or lower nominal interest rate  $1 + i_0$ . To generate an example of contractionary monetary policy that both raises the real interest rate and depresses the bubble, we now turn to the case where goods prices are set in advance of monetary policy.

## Case II: Sticky Prices

Suppose entrepreneurs must set their prices  $P_{it}$  before they get to observe monetary policy, based only on their expectation of the nominal wage  $W_t$ . We want to study what would happen if at date  $t$  the monetary authority unexpectedly announced a different path for money from what entrepreneurs expected. For ease of exposition, we will refer to any variables after the intervention with a hat whenever these variables might differ from what would have happened without the change in monetary policy. Hence, the new path for monetary policy will be denoted by  $\{\widehat{M}_t\}_{t=0}^{\infty}$ . The change in path is unanticipated as of date  $t$ , but is perfectly anticipated from date  $t$  on. Hence, any changes in the path of money beyond date  $t$  are anticipated and incorporated into entrepreneurs' pricing decisions.

We want to show that there exists an unanticipated monetary policy intervention starting from date  $t$  that both raises the real interest rate and reduces the bubble. In particular, suppose the government shrinks the amount of money by a small amount  $\Delta$  at date  $t$ , i.e.

$$\widehat{M}_t = M_t - \Delta$$

From the government budget constraint (19), we know that

$$\widehat{b}_t = (1 + r_{t-1}) b_{t-1} - \tau_t - \frac{M_t - \Delta - M_{t-1}}{P_t}$$

Since  $(1 + r_{t-1}) b_{t-1}$ ,  $\tau_t$ ,  $M_{t-1}$ , and  $P_t$  are all preset variables, it follows that as a result of this intervention, the government at date  $t$  must issue an additional  $\Delta/P_t$  units of real debt, i.e.

$$\widehat{b}_t = b_t + \frac{\Delta}{P_t} \tag{43}$$

We then assume that the remaining path  $\{\widehat{M}_{t+1}, \widehat{M}_{t+2}, \dots\}$  will be set to ensure that the real price of the asset at date  $t + 1$  is unchanged, i.e. they will be set to ensure

$$\widehat{p}_{t+1} = p_{t+1}$$

To appreciate what this condition implies for the path of money balances, recall that the equilibrium price of the asset is given by

$$p_t = An_t - \tau_t - b_t - m_t$$

Since  $\{\widehat{M}_{t+1}, \widehat{M}_{t+2}, \dots\}$  is anticipated, our analysis of the case where prices are flexible applies. This implies  $\widehat{n}_{t+1} = n_{t+1}$ , i.e. employment at date  $t+1$  would be the same as if there was no surprise change in money. The lump-sum tax  $\tau_{t+1}$  is unaffected by monetary policy by assumption. Hence,  $\widehat{p}_{t+1} = p_{t+1}$  implies

$$b_{t+1} + m_{t+1} = \widehat{b}_{t+1} + \widehat{m}_{t+1}$$

Substituting in from (43), we can rewrite this as

$$\widehat{m}_{t+1} - m_{t+1} = (r_t - \widehat{r}_t) b_t - (1 + \widehat{r}_t) \frac{\Delta}{P_t}$$

We will argue below that  $\widehat{r}_t > r_t$ , so this implies that real money balances at date  $t+1$  must be lower than they would have been absent the shock.

Given a path  $\{\widehat{M}_t, \widehat{M}_{t+1}, \dots\}$  that satisfies these conditions, we can now characterize the effect on certain equilibrium prices at date  $t$ . Since the old exchange all of their resources for goods, they will want to consume

$$\frac{M_t - \Delta}{P_t} + (1 + r_{t-1}) b_{t-1} + (\widehat{p}_t + d)$$

Since the old are the only ones who consume, the amount of final goods  $Y_t$  produced at date  $t$  must equal the above amount net of the  $d$  units of consumption available as a dividend. That is,

$$A_t \widehat{n}_t = \frac{M_{t-1} - \Delta}{P_t} + (1 + r_{t-1}) b_{t-1} + \widehat{p}_t \quad (44)$$

Note that most of the variables on the right-hand side of (44) are pre-determined, including the price level  $P_t$ . The only endogenous variable on the right-hand side is the price of the asset  $\widehat{p}_t$ . But we can rewrite this as

$$\begin{aligned} A_t \widehat{n}_t &= \frac{M_{t-1} - \Delta}{P_t} + (1 + r_{t-1}) b_{t-1} + \frac{d + \widehat{p}_{t+1}}{1 + \widehat{r}_t} \\ &= \frac{M_{t-1} - \Delta}{P_t} + (1 + r_{t-1}) b_{t-1} + (d + \widehat{p}_{t+1}) \left( \frac{A_t n_t - \tau_t}{A_{t+1} n_{t+1} + d + \theta} \right) \end{aligned}$$

where the second equation uses the fact that  $1 + \widehat{r}_t$  must satisfy (41) whether prices are flexible or not. Since young agents at date  $t+1$  must be able to afford the asset, we know  $\widehat{p}_{t+1} \leq A_{t+1} n_{t+1}$ . Hence,

$$\frac{d + \widehat{p}_{t+1}}{A_{t+1} n_{t+1} + d + \theta} < 1$$

Since we considered a path that ensures  $\widehat{p}_{t+1} = p_{t+1}$ , then  $\widehat{p}_{t+1}$  is by construction independent of  $\Delta$ . This implies we have

$$\left[ 1 - \frac{d + \widehat{p}_{t+1}}{A_{t+1} n_{t+1} + d + \theta} \right] A_t \widehat{n}_t = \frac{M_t - \Delta}{P_t} + (1 + r_{t-1}) b_{t-1} - \frac{\tau_t (d + \widehat{p}_{t+1})}{A_{t+1} n_{t+1} + d + \theta} \quad (45)$$

where all terms above are independent of  $\Delta$ . From this, it follows that  $\widehat{n}_t$  is decreasing in  $\Delta$ , since

$$\frac{d\widehat{n}_t}{d\Delta} = - \left[ 1 - \frac{d + \widehat{p}_{t+1}}{A_{t+1}n_{t+1} + d + \theta} \right]^{-1} \frac{P_t}{A_t} < 0$$

This shows that an unexpected reduction in money, followed by an adjustment in money balances that sterilizes the affect on asset prices at date  $t + 1$ , would reduce employment and thus output. Intuitively, the shock leaves old agents with fewer resources to consume, and so less will be produced to meet their demand.

From (41), we can further deduce that the real interest rate  $1 + \widehat{r}_t$  at date  $t$  will be higher than  $1 + r_t$ . This is because all date  $t + 1$  variables will be unchanged, while  $\widehat{n}_t < n_t$ . As for the price of the asset, (45) reveals that output  $A_t \widehat{n}_t$  falls more than one for one with  $\Delta/P_t$ , implying the real price of the asset  $\widehat{p}_t < p_t$ . Appealing to a similar argument as in the case of fiscal policy, we can use the fact that any bubble would not change asymptotically as a result of this policy, and yet the interest rate  $1 + \widehat{r}_t$  is higher than  $1 + r_t$ , to argue that the bubble at date  $t$  must fall. Hence, an unexpected monetary contraction would not only temporarily raise the real interest rate, but it would temporarily depress the bubble. This is precisely what we wanted to demonstrate.

Finally, we note that from the equation for labor supply, we have

$$v'(\widehat{n}_t) = (1 + \widehat{r}_t) \frac{\widehat{W}_t}{\widehat{P}_t}$$

Since  $\widehat{n}_t < n_t$  while  $1 + \widehat{r}_t > 1 + r_t$ , we can conclude that the real wage  $\widehat{W}_t/\widehat{P}_t < W_t/P_t$ . That is, since employment must fall when goods prices are sticky, the real wage must fall to induce workers to put in less effort. Since  $\widehat{P}_t = P_t$  when prices are set in advance, it follows that the nominal wage  $\widehat{W}_t < W_t$ , i.e. the monetary policy we consider will cause the nominal wage to fall. But this means that if some producers could respond to monetary policy, they would want to lower their prices. This suggests that if some producers had flexible prices while others were not, then the policy we consider would lead the price level to fall. Thus, when some prices are flexible, we would expect both a fall in output, meaning young agents have fewer resources with which to buy the asset, and an increase in the amount of debt issued to rise as the value of previous nominal government obligations increased. Thus, in a version where some but not all prices were flexible, contractionary monetary policy would depress the bubble both because there are fewer resources agents can use to buy the asset and because there is more public debt that agents must hold instead of the asset.

## Appendix C: Credit-Driven Bubbles and Variable Supply of Assets

In this Appendix, we provide an example of credit-driven bubbles in which an intervention that raises the interest rate and mitigates a bubble can affect the amount of aggregate risk in the economy. This example requires two departures from the model we analyzed in Section 4. First, we need to modify the model so that additional units of the asset can be created. This allows an intervention that mitigates the bubble to affect the quantity of bubble assets created. Second, for a change the number of assets created to influence aggregate risk, we need the source of risk in the economy to come from the asset rather than the aggregate endowment. If the source of risk in the economy concerns the aggregate endowment as in the model we analyze in Section 4, the amount of assets created would have no effect on the risk the economy is exposed to.

Our formulation follows the setup in Section 4 in which young agents who want to save can buy government bonds, buy the asset, or lend to a combination of entrepreneurs or speculators. We consider two modifications. First, we assume  $g = 0$ , i.e.  $e_t = e_0$  for all  $t$  and there is no uncertainty about next period's endowment. Second, rather than assume that the initial old are endowed with a fixed stock of assets that yields a fixed dividend  $d$ , we assume that the initial old can create assets by converting output into assets, and that the dividend on these assets is stochastic. The production technology for assets features increasing marginal cost. In particular, we assume that creating the  $q$ -th unit of the asset requires  $c(q)$  units of output. For simplicity, suppose  $c(q) = q^a$  for some constant  $a > 0$ . As before, we use  $p_0$  to denote the real price of the asset at date 0. Since producers earn  $p_0$  on each unit they produce, they will create assets up to the point where the cost of the last asset is equal to the price at which they can sell the asset, i.e.

$$p_0 = c(q) = q^a$$

We continue to assume the asset yields a constant dividend. However, the value of this dividend is only revealed at date  $t = 1$ , when those who purchased the asset at date 0 are old. Thus, only the initial cohort who buys the asset is uncertain about its payoff. To keep things simple, suppose the dividend  $d$  has a binomial distribution and will equal  $\underline{d} > 0$  with probability  $1 - \pi$  and  $\bar{d} > \underline{d}$  with probability  $\pi$ .

Since  $\underline{d} > 0$ , agents who are young at date 0 will not rely on storage. Since we assume the return  $y$  on production is large, all entrepreneurs will wish to borrow. They will therefore receive an amount  $\kappa$  of the resources available at date 0. This implies

$$e_0 - b_0 - \tau_0 - \kappa = p_0 q$$

That is, total spending on the asset at date 0 is equal to the endowment of agents at date date net of the amount they use to pay taxes, buy bonds, and finance entrepreneurs. Substituting in  $q^a = p_0$  yields an expression for the initial price of the asset as well as the quantity produced:

$$\begin{aligned} p_0 &= (e_0 - b_0 - \tau_0 - \kappa)^{\frac{a}{a+1}} \\ q &= (e_0 - b_0 - \tau_0 - \kappa)^{\frac{1}{a+1}} \end{aligned}$$

The higher is  $a$ , the more an increase in  $p_0q$  will be reflected in a higher price as opposed to a higher quantity. When  $a = 1$ , the two effects are equal.

To show that the model still gives rise to a bubble, let us begin at date 1 when the value of the dividend  $d$  is revealed. The total amount spent on the asset at this date will equal

$$p_1q = e_0 - b_1 - \tau_1 - \kappa$$

This amount will be the same whether  $d = \bar{d}$  or  $d = \underline{d}$ . However, the realization of dividends will instead affect the equilibrium interest rate. In particular, for  $t \geq 1$ , we have

$$1 + r_t = \frac{d + p_{t+1}}{p_t} \quad (46)$$

where  $d \in \{\underline{d}, \bar{d}\}$ . Thus, if dividends are low, the interest rate will be low as well. Indeed, the interest rate will fall by exactly the amount needed to push up the present discounted value of dividends to be the same as when dividends are high. In other words, if we denote  $1 + r_t$  as the interest rate at date  $t$  if  $d = \underline{d}$  and  $1 + \bar{r}_t$  if  $d = \bar{d}$ , then

$$\sum_{t=1}^{\infty} \left( \prod_{s=1}^t \frac{1}{1 + r_s} \right) \underline{d} = \sum_{t=1}^{\infty} \left( \prod_{s=1}^t \frac{1}{1 + \bar{r}_s} \right) \bar{d} = p_1 = \frac{e_0 - b_1 - \tau_1 - \kappa}{q}$$

To prove this result formally, we use (46) to obtain

$$\begin{aligned} f_1 &= \sum_{t=1}^{\infty} \left( \prod_{s=1}^t \frac{1}{1 + r_s} \right) d \\ &= d \left[ \frac{p_1}{p_2 + d} + \frac{p_2}{p_3 + d} \frac{p_1}{p_2 + d} + \frac{p_3}{p_4 + d} \frac{p_2}{p_3 + d} \frac{p_1}{p_2 + d} + \dots \right] \\ &= d \times \frac{p_1}{d} \times \left[ \frac{d}{p_2 + d} + \frac{p_2}{p_3 + d} \frac{d}{p_2 + d} + \frac{p_3}{p_4 + d} \frac{p_2}{p_3 + d} \frac{d}{p_2 + d} + \dots \right] \\ &= d \times \frac{p_1}{d} \times \left[ \frac{d}{p_2 + d} + \frac{d}{p_3 + d} \frac{p_2}{p_2 + d} + \frac{d}{p_4 + d} \frac{p_3}{p_3 + d} \frac{p_2}{p_2 + d} + \dots \right] \end{aligned}$$

We now argue that the last expression in brackets is equal to 1. Consider a random variable that describes the date of an arrival, where the arrival can occur at any date  $t \geq 2$ , and where the probability of arrival at date  $t$  conditional on no arrival up to date  $t$  is equal to  $\frac{d}{p_t + d} \in (0, 1)$ . Then the expression in brackets is the probability of an arrival at any date, which is 1. Hence,  $f_1 = p_1$ , regardless of the realization of  $d$ .

At date 0, the fundamental value of the asset is equal to

$$f_0 = \frac{E[d + f_1]}{1 + r_0} = \frac{E[d] + p_1}{1 + r_0}$$

We now want to show that the price of the asset at date 0 exceeds  $f_0$ . To see this, note that because there is free entry into speculation, the interest rate on loans  $R_0$  must ensure speculators earn zero profits from borrowing if the high state is realized, i.e.

$$1 + R_0 = \frac{\bar{d} + p_1}{p_0}$$

Since creditors could have purchased riskless government bonds and earned  $1 + r_0$  on the resources they lend out, and since lenders are risk-neutral, the expected profits from lending out resources must yield the same return as the risk-free rate, i.e.

$$(1 + r_0)(\kappa + p_0q) = (1 + R_0)(\kappa + (1 - \pi)p_0q) + \pi(\underline{d} + p_1)q \quad (47)$$

By adding and subtracting  $\pi(\bar{d} + p_1)q$  to the RHS implies

$$\begin{aligned} (1 + r_0)(\kappa + p_0q) &= (1 + R_0)(\kappa + (1 - \pi)p_0q) + \pi(\bar{d} + p_1)q - \pi(\bar{d} + p_1)q + \pi(\underline{d} + p_1)q \\ &= (1 + R_0)(\kappa + p_0q) - \pi(\bar{d} - \underline{d})q \end{aligned}$$

where the second step uses the fact that  $1 + R_0 = (\bar{d} + p_0)/p_0$ . Rearranging, we can derive an expression for the risk free interest rate on government debt  $r_0$  in terms of the loan rate  $R_0$ :

$$r_0 = R_0 - \frac{\pi(\bar{d} - \underline{d})q}{\kappa + p_0q}$$

This implies  $R_0 > r_0$ . Next, observe that from (47), we have

$$\begin{aligned} (1 + r_0)(\kappa + p_0q) &= \frac{\bar{d} + p_1}{p_0}(\kappa + (1 - \pi)p_0q) + \pi(\underline{d} + p_1)q \\ &= \frac{\bar{d} + p_1}{p_0}\kappa + (1 - \pi)(\bar{d} + p_1)q + \pi(\underline{d} + p_1)q \\ &= (1 + R_0)\kappa + [E[d] + p_1]q \end{aligned}$$

Dividing both sides by  $1 + r_0$  and rearranging the above equation yields

$$(p_0 - f_0)q = \frac{R_0 - r_0}{1 + r_0}\kappa$$

Since we argued above that  $R_0 > r_0$ , this implies that if  $\kappa > 0$ , there will be a bubble in the asset market.

Finally, we want to study the effects of intervention. As before, we parameterize policy so that  $b_t = b$  for  $t = -1, 0, 1, \dots$ . From the government budget constraint, we have

$$b_0 + \tau_0 = (1 + r_{-1})b_{-1} = (1 + r_{-1})b$$

Substituting this into the expression for the price of the asset at date 0 yields an expression in terms of  $b$  and other primitives:

$$p_0 = (e_0 - (1 + r_{-1})b - \kappa)^{\frac{\alpha}{\alpha+1}} \quad (48)$$

Next, we turn to the real interest rate. Using the expression for the spread between  $R_0$  and  $r_0$  above, as well as the expression for  $R_0$  above, we can solve for  $r_0$  as the value which solves

$$\begin{aligned} 1 + r_0 &= \frac{\bar{d}q + p_1q}{p_0q} - \frac{\pi(\bar{d} - \underline{d})q}{\kappa + p_0q} \\ &= \frac{d_1q + (e_0 - (1 + r_0)b - \kappa)}{e_0 - (1 + r_{-1})b - \kappa} - \frac{\pi(d_1 - d_0)q}{e_0 - b - r_{-1}b} \\ &= \frac{d_1(e_0 - (1 + r_{-1})b - \kappa)^{\frac{1}{\alpha+1}} + (e_0 - (1 + r_0)b - \kappa)}{e_0 - (1 + r_{-1})b - \kappa} \\ &\quad - \frac{\pi(d_1 - d_0)(e_0 - (1 + r_{-1})b - \kappa)^{\frac{1}{\alpha+1}}}{e_0 - b - r_{-1}b} \end{aligned}$$

The analytical expression for  $r_0$  in terms of primitives is messy, and differentiating it with respect to  $b$  does not offer much insight. However, we can easily produce numerical examples in which an increase in  $b$  will raise the risk-free interest rate  $r_0$ , reduce the initial price of the asset  $p_0$ , lower the bubble  $\Delta_0 = p_0 - f_0$ , and reduce the quantity of the asset supplied  $q$ . For example, suppose we set the initial endowment  $e_0 = 1$ . For the asset, we assume  $\bar{d} = 0.1$  and  $\underline{d} = 0.05$ , each equally likely so  $\pi = 0.5$ . We set  $\kappa = 0.99$ , so most lending finances entrepreneurial activity rather than speculation. This is consistent with our finding in Proposition 8 that a high  $\kappa$  is more compatible with bond issuance reducing the size of the bubble. We set the initial interest rate at  $r_{-1} = 0.15$ . Finally, we set  $a = 9$  so the effect of bonds on total spending on the asset primarily affects the price of the asset  $p_0$  rather than the quantity of the asset created  $q$ . In this case, we can compute numerically that for all values of  $b$  that ensure the asset price  $p_0$  and  $p_1$  will be positive, a higher  $b$  will in fact raise the real interest rate, reduce the price and quantity of the asset produced at date 0, and reduce the bubble component at date 0.