Why Does U.S Public Debt Flow to China?

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We show that the massive flows of U.S public debt to China can arise as an equilibrium outcome of a model where governments issue debt to help domestic entrepreneurs insure against idiosyncratic investment risks. Precautionary motive of entrepreneurs pushes down equilibrium interest rate. Hence in autarky, the country with lower investment risks (the U.S) has higher interest rate and lower stock of debt. When it integrates with a country with higher investment risks (China), the extra precautionary demand drives the interest rate down further, lowering the borrowing cost. As a result, the U.S issues more debt, and much of these debt flows to China. (JEL E62, F65, H63)

Starting from the mid-1990s, foreign holdings of U.S securities have grown markedly from 1.2 trillion in 1994 to 17 trillion in 2015. Meanwhile, the net foreign asset (NFA, henceforth) of China also increases significantly from $33 billion in 2001 to $1.5 trillion in 2011. Moreover, the growth of both series accelerates in the 2000s. These patterns are shown in Figure 1, where the left panel shows the foreign holdings of U.S securities over time and the right panel shows that of the NFA of China. Further decomposition of data reveals that China’s purchase of the U.S Treasury debt plays an important role in driving these facts. In this paper, we propose a theory in which government internalizes the impact on interest rate when borrowing to explain the U.S-China sovereign debt transaction, and subsequently evaluates its quantitative relevance.

Our focus on the U.S-China sovereign debt is motivated by two pairs of observations.

1. (a) Treasury debt accounts on average 32% for the total U.S securities held by foreign countries. The left panel of Figure 2, which is calculated using data from the U.S Treasury International Capital Reporting System (TIC, henceforth), shows a marked increase of the fraction of Treasury debt in total U.S foreign liabilities in last decade.

(b) The fraction of U.S Treasury debt held by China increases sharply from 10% in 2002 to about 1/4 in 2011. The right panel of Figure 2 plots this using also the TIC data.

2. (a) After the 2000s, official foreign reserves accounts for more than 2/3 of China’s foreign debt assets. This is plotted in the left panel of Figure 3 using data from Lane and Milesi-Ferretti (2007). We see from the Figure that the ratio of official foreign reserves in China’s total foreign debt asset increases from 40% in 2000 to nearly 70% in 2011.

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(b) Around 35% of China’s official foreign reserves is the U.S Treasury debt. As is shown in the right panel of Figure 3, except for a temporary decline in the 2008 financial crisis, the fraction of China’s foreign reserves from the U.S stays mostly above 35%.

Together, these facts suggest that in order to understand the general capital flows between the U.S and China, public debt flows are a very important factor.

We study a two-country model where public debt is used to smooth the consumption of domestic residents with uninsurable idiosyncratic investment risks. The elements of the model are identical to those of Azzimonti, de Francisco and Quadrini (2014) except that the investment risk is higher in China, allowing us to investigate capital flows between them. To keep tractability, the model adopts the Angeletos (2007) framework. In particular, there are two types of agents: those who face idiosyncratic investment risks (entrepreneurs), and those who face no risks (workers). Government issues risk free public debt, which we assume can only be bought by entrepreneurs. As in Aiyagari (1994), the consumption smoothing motive of the entrepreneurs pushes the equilibrium interest rate down to a level lower than the intertemporal discount rate. Because of the lower interest rate, workers would prefer the government borrowing more on their behalf to transfer future resources for current consumption. As government borrows more, the interest rate increases gradually. Under such scenario, entrepreneurs prefer more borrowing since the cost of borrowing declines. However, the benefits for workers gradually fade away, and workers eventually stop supporting further increase of public debt when the interest rate is higher than the intertemporal discount rate. Therefore, a utilitarian government who maximizes a weighted average of the utility of all domestic residents chooses a finite level of public debt in equilibrium.

How does the integration of China into the global economy affect the flow of U.S public debt?
The core mechanism is the different consumption smoothing incentive of entrepreneurs (and hence governments) in the two countries. Following Bai, Hsieh and Qian (2006) and Wang, Wen and Xu (2016 forthcoming), we calculate the returns to private investment for both China and the U.S. We find that the rate of return to private investment in China has higher standard deviation. As shown in the left panel of Figure 4, the standard deviation of the rate of return in China is 2.92% which is 1.7 times the U.S level of 1.72%. Therefore, entrepreneurs in China have higher demand for risk free assets. When China is financially integrated with the U.S, entrepreneurs in China provide additional demand for U.S government debt. This drives down the equilibrium interest rate, and as a result, the U.S government issues more debt which flows to China. The right panel of Figure 4 shows that the Treasury bill rate in China gradually increases during last decade, while that of the U.S decreases.

**Related Literature.**—Our paper contributes to the large literature studying the Lucas Paradox of capital flows between rich and poor countries. In particular, the seminal paper by Lucas (1990) points out that the actual capital flows from rich to poor countries are much smaller than those predicted by a neoclassical growth model. Recent empirical literature has found that there also exists a dynamic version of the Lucas Paradox that is named the “allocation puzzle,” which shows that capital is flowing out from fast growing countries [Prasad, Rajan and Subramanian (2007), Alfaro, Kalemli-Ozcan and Volosovych (2014), and Gourinchas and Jeanne (2013)]. A burgeoning of literature is devoted to understanding the Lucas Paradox and the allocation puzzle. Empirical investigation includes Hsieh and Klenow (2007), Caselli and Feyrer (2007), and Benhima (2013) which emphasize the higher relative cost of investment, and Alfaro, Kalemli-Ozcan and Volosovych (2008) which studies the role of political risk. A recent paper by Azzimonti (2016) elaborates the latter point with a neoclassical growth model featuring political turnover.
A number of papers study the role of heterogeneous degrees of financial development across countries in driving the global imbalances. Caballero, Farhi and Gourinchas (2008) interpret financial development as the ability of an economy to provide saving instruments that can be used to carry wealth. They argue that capital flows out from these countries because financial markets in poor countries cannot generate enough financial assets. Song, Storesletten and Zilibotti (2011), Buera and Shin (2016 forthcoming), and Wang, Wen and Xu (2016 forthcoming) stress that financial frictions impede the ability of the domestic economy to absorb all domestic savings, thereby forcing savings to be invested abroad. Our paper differs from these studies in that we do not impose any restrictions on the quantity of capital that can be invested domestically, and interpret financial development of an economy as its capability to insure against investment risks.

In this regard, our paper relates closer to several studies that also model financial development as the extent to which residents can insure against idiosyncratic risks [Mendoza, Quadrini and Ríos-Rull (2009), Angeletos and Panousi (2011), and Sandri (2014)]. A major difference between our paper and these studies is that we model the optimal policy of the government explicitly. In our model, the government internalizes the impact of debt issuance on equilibrium interest rate, while in all these studies a competitive equilibrium where agents behave as price takers are analyzed. This also allows us to study the gross stock of debt instead of just the net volumes. We also differ from Sandri (2014) in that we study a two-country general equilibrium model, while his paper uses a small open economy model where the interest rate of foreign asset is given exogenously.

Another difference of our paper with all the papers above is that we focus on the flows of public debt, while they all study private asset. We choose to focus on public debt because of the stylized facts introduced early in this section. The empirical literature has also found that the allocation puz-
The returns to private investment are mainly driven by the flows of public debt. Those of the private debt however, are consistent with the prediction of standard neoclassical growth model [Alfaro, Kalemli-Ozcan and Volosovych (2014), and Gourinchas and Jeanne (2013)]. Two papers that also study the flows of public debt are Aguiar and Amador (2011) and Azzimonti, de Francisco and Quadrini (2014). Aguiar and Amador (2011) studies the debt repayment scheme of a small open economy with political frictions, while we focus on the role of insurance against investment risks. By assuming a small open economy, foreign bonds are supplied inelastically and thus all the results are driven by bond demand. On the contrary, we model the supply side explicitly and answer the related question that why developed countries choose to issue a large amount of debt. Our paper relates closest to Azzimonti, de Francisco and Quadrini (2014), but relaxing the symmetric countries assumption. This allows us to investigate the implication of the model on cross-country capital flows which are ruled out in their paper.

The rest of this notes is organized as follows. Section I introduces a two-period model to explain the model mechanisms. Section II lays out the infinite horizon model. Section III explains the computational algorithm.

I. Capital Flows in the Two-Period Model

We start by analyzing capital flows in a version of the model with only two periods. We proceed as follows. First, we describe the model environment. We then characterize the optimal government policy in financial autarky. In the end, we use a numerical example to show qualitatively the effect of financial integration.
A. The Environment

Consider an economy with two countries indexed by $j = 1, 2$. For the ease of reference, we refer later in this paper country 1 as the U.S and country 2 as China. In each country, there are two types of agents: a measure $\Phi$ of workers and a measure one of entrepreneurs. There are two types of assets in the economy: a risk free debt $B_j$ issued by the government, and a unit supply of a Lucas-tree type risky asset $k_j$ (land). We assume that workers are excluded from the financial markets, but entrepreneurs are allowed to save. Hence there are no risks involving the workers, while the entrepreneurs face idiosyncratic investment risks. Public debt is thus used by entrepreneurs to insure against the risks in the yield of $k_j$. Because workers and entrepreneurs are exposed to risks differently, they have distinct preference for public debt. A utilitarian government which maximizes a weighted average of welfare of workers and entrepreneurs then chooses the optimal debt based on the relative size of these two groups.

The utility functions for both agents are the same:

\[
\mathbb{E}_0 [\ln c_1 + \beta \ln c_2]
\]

where $c_t$ is consumption and $\beta \in (0, 1)$ in the discount factor.

Let internationally immobile land be traded at $p_{j,t}$. Labor is also internationally immobile. Each worker has $1/\Phi$ units of labor which is supplied inelastically for wage $w_{j,t}$.

Firms are owned by entrepreneurs. Each firm operates according to a production function

\[
F(z, k, l) = z^\theta k^\theta l^{1-\theta},
\]

which combines idiosyncratic TFP $z$, land $k$, and labor $l$ to produce the numeraire good. Among all the factors, $k$ is chosen in the previous period, $z$ is observed at the beginning of the current period, and $l$ is determined after $z$ and $k$ are observed. $z$ is the only source of uncertainty in the model. We assume that it is i.i.d among agents and over time with discrete support $\{z_1, \cdots, z_m\} \triangleq Z$ and probability measure $\{\mu_1, \cdots, \mu_m\}$. We further assume that the idiosyncratic shocks in China are more volatile, meaning that $Z_2$ is a mean preserving spread of $Z_1$. This can be interpreted as lower degree of financial development in China, which limits the ability that entrepreneurs insure against investment risks. The timing of the model suggests that entrepreneurial profits are linear in land $k$:

\[
\pi(z, k, w) = F(z, k, l) - wl \triangleq A(z, w)k.
\]

Since the shocks are i.i.d and the distribution is stationary over time, the equilibrium wage $\bar{w}$ is constant, and can be calculated from firm’s first order condition and labor market clear condition.

In each period $t$, entrepreneur $i$ in country $j$ begins with risk free bonds $b_{j,t}$, land $k_{j,t}$, TFP shock $z_{j,t}$, and receives lump-sum transfers $\tau_{j,t}$. The entrepreneur then makes consumption decision $c_{j,t}$ and portfolio choice decisions $b_{j,t+1}$ and $k_{j,t+1}$. Risk free bonds $B_{j,t}$ are issued by governments.

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1This assumption is a normalization that keeps the total labor supply equal to 1. Because labor is internationally immobile, in equilibrium it only affects the equilibrium wage. The total labor expenditure by the firm $w \cdot l$, however, is not affected.
The governments use bonds proceeds to pay lump-sum transfers and to repay outstanding debt. If we let \( R_{j,t} \) denote the interest rate, then the government budget constraint is

\[
(1 + \Phi) \tau_{j,t} + B_{j,t} = \frac{B_{j,t+1}}{R_{j,t}}.
\]

Suppose that in period 1 governments start with zero debt \( B_{j,1} = 0 \), the transfers in period 1 and 2 are then respectively

\[
\tau_{j,1} = \frac{B_{j,2}/R_{j,1}}{1 + \Phi},
\tau_{j,2} = -\frac{B_{j,2}}{1 + \Phi}.
\]

To further simplify the analysis, we assume that in period 1, all entrepreneurs start with the same amount of land, zero bonds \( b_{j,1} = 0 \), and receive the same productivity \( z_{i,j,1} = \bar{z} = \int z_m d\mu \). Since when \( t = 1 \), all entrepreneurs are homogeneous, land market clearing condition yields \( k_{j,1} = k_{j,2} = 1 \). As a result, the equilibrium wage is given by

\[
(1 - \theta)z^\theta = \bar{w}.
\]

Substitute back into firm’s first order condition, we get

\[
A(z) = \frac{\theta z}{z^{1-\theta}}.
\]

Thus, the initial wealth of entrepreneurs (including transfers) is

\[
a_{j,1} = \bar{A} + p_{j,1} + \frac{B_{j,2}/R_{j,1}}{1 + \Phi},
\]

where

\[
\bar{A} = \int \theta z/\bar{z}^{1-\theta} d\mu = \theta \bar{z}^\theta.
\]

Because land has no value after production, \( p_{j,2} = 0 \). Thus the second period wealth is

\[
a_{j,2} = A(z_{j,2}^i) + b_{j,2} - \frac{B_{j,2}}{1 + \Phi} \triangleq A(z_{j,2}^i) + \tilde{b}_{j,2}^i,
\]

where \( \tilde{b}_{j,2}^i \) is the excess demand for bonds, which is the net saving in risk free bonds by entrepreneurs.

**B. Financial Autarky**

We first simplify the notations. Since all entrepreneurs are homogeneous in period 1, we drop the individual superscript \( i \). We also ignore country and time subscripts, and let \( k \) and \( b \) denote the individual land and bond purchased at period 1. We further use \( p, R \) and \( B \) to denote the price
of land, the gross interest rate, and the bond issued in period 1. The realized shock in period 2 is denoted by \( z \), which uniquely identifies an entrepreneur. In period 2, Entrepreneurs are ex-ante homogenous but ex post heterogeneous, with \( A(z) \) being the only source of heterogeneity. Bond market clearing condition then implies \( b = B \) and \( \tilde{b} = \nu B \), with \( \nu = \Phi / (1 + \Phi) \). The consumption of entrepreneur in period 1 is therefore

\[
(3) \quad c^e_1 = A + p + \frac{B/R}{1 + \Phi} - p - \frac{B}{R} = A - \frac{\nu B}{R}.
\]

Because entrepreneurs do not save in period 2, the consumption is

\[
(4) \quad c^e_2 = A(z) + \nu B.
\]

Notice that though we have pinned down the value of \( \tilde{b} = \nu B \) using the market clearing condition, it still has to be consistent with the optimization problem of the entrepreneurs. This helps us find out the interest rate as a function of public debt. More specifically, the entrepreneurs’ problem is

\[
(5) \quad V = \max_b \left\{ \ln \left( A - \frac{1}{R} \tilde{b} \right) + \beta \mathbb{E} \ln \left( A(z) + \tilde{b} \right) \right\},
\]

where we have substituted in the budget constraints in the two periods, since they all hold with equality. The first order condition of problem (5) yields

\[
(6) \quad \frac{1}{A - \tilde{b}/R} = \beta \mathbb{E} \ln \left( \frac{R}{A(z) + \tilde{b}} \right).
\]

Rearranging terms, and substitute in the equilibrium condition \( \tilde{b} = \nu B \), we have the following expression for the interest rate as a function of public debt:

\[
(7) \quad R(B) = \frac{\nu B [1 + \beta (1 - \phi(B))]}{\beta \bar{A} (1 - \phi(B))},
\]

where

\[
\phi(B) = \mathbb{E} \left[ \frac{A(z)}{A(z) + \nu B} \right]
\]

is the share of risky assets in entrepreneurs’ portfolio. Therefore, entrepreneurs’ indirect utility function can be written as

\[
(8) \quad V(B) = \ln \left( \bar{A} - \frac{\nu B}{R(B)} \right) + \beta \mathbb{E} [\ln (A(z) + \nu B)].
\]

On the other hand, workers earn wage \( \bar{w} \) every period on labor \( 1/\Phi \), receive transfer \( \tau_1 \) and pay taxes \( \tau_2 \). Hence consumptions in period 1 and 2 are respectively

\[
\begin{align*}
    c^w_1 &= \frac{1}{\Phi} \left( \bar{w} + \frac{\nu B}{R} \right), \\
    c^w_2 &= \frac{1}{\Phi} (\bar{w} - \nu B),
\end{align*}
\]
which gives the indirect utility function of workers

\[ W(B) = \chi + \ln \left( \frac{\nu B}{R(B)} \right) + \beta \ln(\bar{w} - \nu B), \]

where \( \chi = -(1 + \beta) \ln \Phi \) is a constant.

With some algebra, we can prove the following properties about the two indirect utility functions. First, the function \( V(B) \) is strictly increasing in \( B, \forall B \geq 0 \). Second, the function \( W(B) \) is strictly concave in \( B \) with a unique maximum in the interval \( (0, \bar{w}/\nu) \). We assume that the government’s objective function is a weighted sum of the utilities of workers and entrepreneurs:

\[ G(B) = \max_B \{ \Phi W(B) + V(B) \}. \]

Because \( V(B) \) is not concave, \( G(B) \) is concave only when \( \Phi \) is large. However, since

\[ \lim_{B \to \bar{w}/\nu} W(B) = -\infty, \]

continuity of \( G(B) \) indicates that the problem still warrants an internal optimum. Furthermore, if we compare \( V(B) \) and \( W(B) \), we find that public debt redistributes consumption intertemporally between workers and entrepreneurs. More specifically, it redistributes consumption from entrepreneurs to workers in period 1 and does the reverse in period 2. From a time perspective, public debt helps entrepreneurs and workers move resources to and from the future respectively. From the left to the right panel, Figure 5 illustrates numerically the property of \( W(B), V(B) \) and \( G(B) \).

C. Financial Integration

When the two countries integrate with each other on the financial market, the market clearing condition of public debt changes. Domestic entrepreneurs now can also buy bonds issued by foreign
government. Notice that since here we assume \( Z^2 \) is a mean preserving spread of \( Z^1 \), the excess debt held by entrepreneurs in the two countries is no longer the same. Hence we do not have
\[
\tilde{b}_1 = \tilde{b}_2 = \frac{\nu}{2} (B_1 + B_2),
\]
as in Azzimonti, de Francisco and Quadrini (2014). We need to combine the first order conditions of the entrepreneurs (6) and the market clearing condition together to solve for the equilibrium interest rate and excess debt holdings. Notice that in the open economy case, the domestic allocation of immobile resources is the same as in autarky. Because now the equilibrium interest rate depends on public debt issued by both countries, we start by characterizing the \( R(B) \) where \( B = [B_1, B_2] \).

**Equilibrium Interest Rate.**—For each country \( j \), first order condition (6) holds:
\[
\frac{1}{A_j - b_j / R} = \beta \mathbb{E} \left[ \frac{R}{A(z_j) + b_j} \right].
\]
Rearranging terms, we have
\[
R = \frac{1}{A_j} \left( \beta \mathbb{E} \left[ \frac{1}{A(z_j) + b_j} \right] + \tilde{b}_j \right)^{-1}, \quad j = 1, 2.
\]
To solve for \( R \), we need first to find out the value of \( \tilde{b}_j \). The previous equation and the international bond market condition provide two equations to solve for the two unknowns:
\[
\frac{1}{A_1} \left( \beta \mathbb{E} \left[ \frac{1}{A(z_1) + \tilde{b}_1} \right] + \tilde{b}_1 \right)^{-1} = \frac{1}{A_2} \left( \beta \mathbb{E} \left[ \frac{1}{A(z_2) + \tilde{b}_2} \right] + \tilde{b}_2 \right)^{-1},
\]
\[
\tilde{b}_1 + \tilde{b}_2 = \nu(B_1 + B_2).
\]
System (13) implies that \( \tilde{b}_j \) depends only on the total supply of debt \( B_1 + B_2 \). The distribution of public debt among the two countries is determined by the mean and standard deviation of \( Z^j \). Once we have \( \tilde{b}_j \), we can then use Equation (12) to solve for the equilibrium interest rate. Because the dependence of \( R \) on \( B \) comes only from \( \tilde{b}_j \), \( R(B) \) is also a function of \( B_1 + B_2 \). However, since we do not have closed form solution for \( \tilde{b}_j \), we have to solve for \( R(B) \) numerically, even in the two-period model.

**Bond Demand.**—The excess demand \( \tilde{b}_j \) reflects the true holdings of risk free asset by entrepreneurs net of transfers and taxes. Put differently, it indicates the precautionary motive of entrepreneurs. We start by considering the case where \( A_1 = A_2 \), that is, there is no extra risk premium in country 2. To simplify notation, denote
\[
\psi_j = \mathbb{E} \left[ \frac{1}{A(z_j) + \tilde{b}_j} \right].
\]
Then the first Equation in System (13) becomes
\[
\tilde{b}_1 - \tilde{b}_2 = \frac{1}{\beta} \left[ \frac{1}{\psi_2} - \frac{1}{\psi_1} \right].
\]
Because $1/(A(z) + \tilde{b})$ is convex in $A(z)$, by Jensen’s inequality, when $\tilde{b}_1 = \tilde{b}_2$, $\psi_1 < \psi_2$. Therefore, to restore the equality in (14), we need $\tilde{b}_2 > \tilde{b}_1$. Since both sides of (14) are continuous in $\tilde{b}_j$, there exist $\tilde{b}_2 > \tilde{b}_1$ such that System (13) holds. To summary, when $\bar{A}_1 = \bar{A}_2$, if $\text{Var}(z_1) < \text{Var}(z_2)$, entrepreneurs in country 2 have stronger precautionary motive. That is, $\tilde{b}_2 > \tilde{b}_1, \forall R$. However, this does not necessarily mean that capital flows from country 2 to country 1, because what defines capital flows is $\tilde{b}_j - \nu B_j$. Thus it could well be the case that though $\tilde{b}_2 > \tilde{b}_1$, because $B_2$ surpasses $B_1$ by a larger extent, in net capital flows into country 2. As a result, though entrepreneurs in country 2 have unambiguously stronger precautionary motive, the actual capital flow has to be determined quantitatively.

When $\bar{A}_1 < \bar{A}_2$. Again, to simplify notation, let $\kappa = \bar{A}_1/\bar{A}_2 < 1$, then Equation (14) becomes

$$\tilde{b}_1 - \kappa \tilde{b}_2 = \frac{1}{\beta} \left[ \frac{\kappa}{\psi_2} - \frac{1}{\psi_1} \right].$$

Because we cannot establish in general the relationship between $\psi_1$ and $\psi_2$, qualitatively that between $\tilde{b}_1$ and $\tilde{b}_2$ is also undetermined.

**Optimal Policy.**—Since now $R$ is a function of $B$ (more specifically $B_1 + B_2$), the indirect utility functions change mildly to:

$$V(B) = \ln \left( \bar{A} - \frac{\tilde{b}_j(B)}{R(B)} \right) + \beta \mathbb{E} \left[ \ln \left( A(z_j) + \tilde{b}_j(B) \right) \right],$$

$$W(B) = \chi + \ln \left( \bar{w} + \frac{\nu B_j}{R(B)} \right) + \beta \ln(\bar{w} - \nu B_j),$$

where $\tilde{b}_j(B)$ is the solution of System (13), and we still have $\chi = -(1 + \beta) \ln \Phi$. It is clear that $B$ affects the redistribution only through the channel of $B_1 + B_2$.

In the open economy, governments’ objective functions become

$$G_j(B) = \max_{\tilde{B}_j} \{ \Phi W(B) + V(B) \}, \quad j = 1, 2.$$  

We choose Nash equilibrium as the solution concept. More specifically, if we let $B_{-j}$ be the debt issued by the other country, then Problem (16) leads to a best response function of $\tilde{B}_j$, $\tilde{B}_j = \phi_j(B_{-j})$. The Nash equilibrium is then defined as $B^* = [B^*_1, B^*_2]$ such that

$$B^*_j = \phi_j(B^*_{-j}), \quad j = 1, 2.$$  

**Capital Flows.**—Figure 6 illustrates the basic mechanism of the paper. The left panel plots the autarky equilibria and the right panel plots the open economy equilibrium. We want to show that our model mechanism can generate the stylized facts introduced in Section I. More specifically, we want to show that given that country 1 provides better insurance against investment risks, in autarky, $R_1 > R_2, B_1 < B_2$; and after the two countries integrate, $R_2 < R_f < R_1$, capital flows from country 2 to country 1, and the average debt level of the world increases.
For the ease of exposition, in the left panel we assume that \( R(B) \) is the same for both countries. This obviously contradicts Equation (7). However, it keeps us focusing on the effect of risk exposure. Since entrepreneurs in country 2 are subject to higher investment risks, they have higher precautionary motives as can be seen from Equation (6). So for any level of \( R \), they have higher demand for bonds. Graphically, this shows up in the figure as curve \( D_2 \) (red) positions right to \( D_1 \) (blue). Because \( R(B) \) is downward sloping, in autarky, country 2 issues more debt and supports a lower interest rate. When the two countries integrate, interest rates converge to one international interest rate \( R_f \). Entrepreneurs in country 2 demand more debt than government 2 issues. The \( R_f \) is such that \( b_2 - B_2 = B_1 - b_1 \), meaning that the extra demand is met by the bonds supplied by country 1. Thus capital flows from country 2 to country 1. In the right panel, we have assumed symmetric Nash equilibrium with \( B_1 = B_2 = B \). If we assume function \( R(\cdot) \) remains the same, then we have \( R_2 < R_f < R_1 \). The average debt level however does not necessarily increase, which depends on the curvature of \( D_1 \) and \( D_2 \). Furthermore, as in Azzimonti, de Francisco and Quadrini (2014), the elasticity of interest rate to bond decreases in open economy. This shows up in the figure as a flattening of \( R(B) \) in open economy. It tends to increase both the equilibrium bond level and interest rate. The increase in the debt level is consistent with the data. However, if \( R(B) \) flattens by too much, it is possible that \( R_f > R_1 \). Thus, whether the prediction of the model on interest rate dynamics is consistent with the data has to be examined quantitatively.

We therefore compute a numerical example to investigate the property of the model. The results are shown in Table 1. The parameterization is the same as in Azzimonti, de Francisco and Quadrini

![Figure 6. Capital Flows](image-url)

### Table 1—Capital Flows in the Two-Period Model

<table>
<thead>
<tr>
<th>Country</th>
<th>( B_j )</th>
<th>( B_f )</th>
<th>( R_j )</th>
<th>( R_f )</th>
<th>CA</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S (L. Volatility)</td>
<td>0.0048</td>
<td>0.0064</td>
<td>1.0297</td>
<td>1.0218</td>
<td>-0.002</td>
</tr>
<tr>
<td>CHN (H. Volatility)</td>
<td>0.0120</td>
<td>0.0121</td>
<td>0.9975</td>
<td>1.0218</td>
<td>0.002</td>
</tr>
</tbody>
</table>

*[Parameterization: \( \beta = 0.95, \theta = 0.2, z^1 \in \{0.75, 1.25\}, z^2 \in \{0.6, 1.4\} \) with equal probabilities, and \( \nu = 0.85 \).]*


(2014) except that we set the standard deviation of \( z \) in country 2 to be 0.8, 1.6 times higher than that in the U.S. We see from the table that all results are qualitatively consistent with the data. More specifically, in autarky, China has lower interest rate \((0.9975 < 1.0297)\) and higher public debt \((0.012 > 0.0048)\) than the U.S. When the two countries integrate together, both countries issue more debt. Further, the average debt level increases as well. The interest rate in the open economy lies between the two autarkic interest rates \((0.9975 < 1.0218 < 1.0297)\). Finally, the last column shows the current account surplus, which indeed suggests that capital flows from China to the U.S.

II. General Model

In this section, we extend to two period model to infinite horizon to examine the full quantitative potential of the model. In the Angeletos (2007) framework, entrepreneurs decisions are linear in their wealth despite that they have precautionary motives. The linear decision rules thus allow for aggregation, meaning that at the aggregate level, the economy with heterogeneous agents behaves the same as an economy with a representative worker and a representative entrepreneur. We do not need to track the wealth distribution of the entrepreneurs.

**Competitive Equilibrium.**—Now both countries last \( T \) periods. We use the limit at \( T \to \infty \) to approximate infinite horizon. Entrepreneurs and workers have the same preference

\[
E_0 \sum_{t=1}^{T} \beta^t \ln c_t.
\]

The budget constraint for entrepreneur \( i \) is

\[
 c_{j,t}^i + p_{j,t} k_{j,t+1} + \frac{b_{j,t+1}^i}{R_{j,t}} = A(z_{j,t}^i, w_{j,t}) k_{j,t}^i + b_{j,t}^i + \tau_{j,t},
\]

with terminal condition \( b_{j,T+1}^i = 0 \). To be consistent with the empirical evidence in the left panel of Figure 4, we allow \( z_1 \) and \( z_2 \) to have different mean and variance. We only require that their realizations are i.i.d across agents and over time.

On the other hand, workers’ consumption is equal to

\[
c_{j,t}^w = w_{j,t} \left( \frac{1}{\Phi} \right) + \tau_{j,t}.
\]

Further, governments’ budget constraint is

\[
(1 + \Phi) \tau_{j,t} + B_{j,t} = \frac{B_{j,t+1}}{R_{j,t}},
\]

with terminal condition \( B_{j,T+1} = 0 \).

If we let the vector of individual shock histories to be \( z_{j,t}^i = \{z_{j,1}^i, \cdots, z_{j,t}^i\} \), then the competitive equilibrium with financial integration is defined as follows.
Definition 1. (Open Economy Equilibrium) Given a sequence of government debt \( \{B_{j,t}\}_{t=1}^T \) satisfying terminal conditions \( B_{j,T+1} = 0 \), a competitive equilibrium under financial integration is composed of price sequences \( \{w_{j,t}, p_{j,t}, R_t\}_{t=1}^T \), entrepreneurs’ decisions \( \{c_{j,t}^i(z_{j,t}), l_{j,t}^i(z_{j,t}), k_{j,t+1}^i(z_{j,t}), b_{j,t+1}^i(z_{j,t})\}_{t=1}^T \), consumption of workers \( \{c_{w,j,t}\}_{t=1}^T \), and transfers \( \{\tau_{j,t}\}_{t=1}^T \) for \( j = 1,2 \), such that

(i) Entrepreneurs maximize utility given \( k_{j,0}^i \) and \( b_{j,0}^i \), and workers’ budget constraint is satisfied.

(ii) Domestic labor market clears:
\[
\int_i l_{j,t}^i(z_{j,t}) di = 1, \quad \text{for } j = 1,2.
\]

Domestic land market clears:
\[
\int_i k_{j,t+1}^i(z_{j,t}) di = 1, \quad \text{for } j = 1,2.
\]

International bond market clears:
\[
\sum_{j=1}^2 \int_i b_{j,t+1}^i(z_{j,t}) di = \sum_{j=1}^2 B_{j,t+1}, \quad \text{for } j = 1,2.
\]

(iii) Domestic bonds and transfers satisfy governments’ budget.

We continue to use the notation
\[
\tilde{b}_{j,t}^i = b_{j,t}^i - \frac{B_{j,t}}{1 + \Phi},
\]
to represent the excess demand of bonds. The aggregate excess demand is then defined as \( \tilde{b}_{j,t} = \int_i \tilde{b}_{j,t}^i di \). The entrepreneurs’ budget constraint can thus be written as
\[
c_{j,t}^i + p_{j,t} k_{j,t+1}^i + \frac{\tilde{b}_{j,t+1}^i}{R_{j,t}} = A(z_{j,t}, w_{j,t}) k_{j,t}^i + p_{j,t} k_{j,t}^i + \tilde{b}_{j,t}^i.
\]

Notice that the aggregation results in Angeletos (2007) hold even when the two countries are asymmetric. Therefore, entrepreneurs’ decision rules are still linear in wealth
\[
a_{j,t}^i = A(z_{j,t}, w_{j,t}) k_{j,t}^i + p_{j,t} k_{j,t}^i + \tilde{b}_{j,t}^i.
\]

This is formally stated in Lemma 1.

Lemma 1. (Linear Policy Rules) Given a sequence of prices \( \{w_{j,t}, p_{j,t}, R_t\}_{t=1}^T \), entrepreneurs’ policies are
\[
c_{j,t}^i = (1 - \eta_t) a_{j,t}^i,
\]
\[
p_{j,t} k_{j,t+1}^i = \phi_{j,t} \eta_t a_{j,t}^i,
\]
\[
\frac{\tilde{b}_{j,t+1}^i}{R_{j,t}} = (1 - \phi_{j,t}) \eta_t a_{j,t}^i,
\]
where $\eta_t$ is the savings rate

$$
\eta_t = \frac{\beta}{1 + \beta^{T-t}/\sum_{s=1}^{T-t} \beta^{s-t}},
$$

and $\phi_{j,t}$ is the risky share

$$
E_t \left[ \frac{R_t}{\left( \frac{A_j(z_{j,t+1}^i, w_j,t+1) + p_{j,t+1}}{p_{j,t}} \right) \phi_{j,t} + R_t(1 - \phi_{j,t})} \right] = 1.
$$

An important feature of Lemma 1 is that the savings rate $\eta_t$ and risky share $\phi_{j,t}$ are independent of wealth $a_{i,j,t}$. This is the key property of the model that allows for aggregation. Formally, the aggregation results are shown in Proposition 1.

**Proposition 1.** (Aggregation) Given a sequence of public debt $\{B_{j,t}\}_{t=1}^T$ and $B_{j,T+1} = 0$, the equilibrium wage is $w_{j,t} = \overline{w}_j$, and the remaining prices and aggregate allocations are

$$
\phi_{j,t} = \mathbb{E}_t \left[ \frac{A_j(z_{j,t+1}^i) + p_{j,t+1}}{A_j(z_{j,t+1}^i) + p_{j,t+1} + \bar{b}_{j,t+1}} \right],
$$

$$
p_{j,t} = \frac{\eta_t \phi_{j,t}(A_j + \bar{b}_{j,t})}{1 - \eta_t \phi_{j,t}},
$$

$$
R_{j,t} = \frac{(1 - \eta_t \phi_{j,t}) \bar{b}_{j,t+1}}{\eta_t (1 - \phi_{j,t})(A_j + \bar{b}_{j,t})},
$$

$$
c_{j,t}^e = \int_i c_{j,t}^i di = \overline{A}_j + \bar{b}_{j,t} - \frac{\bar{b}_{j,t+1}}{R_t},
$$

$$
c_{j,t}^w = \overline{w}_j + \nu \left( \frac{B_{j,t+1}}{R_t} - B_{j,t} \right),
$$

where we suppress the dependence of $A(\cdot)$ on $\overline{w}_j$ to $A_j(z_j^i)$, and $\overline{A}_j = \sum_i A_j(z_i)\mu_i$. The excess demands sequence $\{\bar{b}_{j,t}\}_{t=1}^T$ satisfies $R_{1,t} = R_{2,t}$ and

$$
(20) \quad \bar{b}_{1,t} + \bar{b}_{2,t} = \nu(B_{1,t} + B_{2,t}).
$$

**Politico-Economic Equilibrium.**—Notice that since we allow the countries to be heterogeneous in their shock structure, initial debt $(B_{1,1}, B_{2,1})$ are no longer the sufficient states. Here we also need the initial aggregate excess holdings $(\bar{b}_{1,1}, \bar{b}_{2,1})$. Because of the bond market clearing condition (20), we need only one of the excess demands, be it $\bar{b}_{1,1}$. The sufficient states are then $B_t = (B_{1,t}, B_{2,t}, \bar{b}_{1,t})$ and the politico-economic equilibrium is characterized by a sequence of policy functions $B_{t+1} = B_t(B_t)$ for $t = 1, \cdots, T$. The objective function of the governments is defined in Proposition 2.

**Proposition 2.** (Governments’ Objective Function) Given current states $B_t$ and future policy function $B_{t+1}(B_{t+1})$, the objective function of the governments is

$$
(21) \quad \max_{B_{j,t+1}} \{ \Phi W_{j,t}(B_t, B_{t+1}) + V_{j,t}(B_t, B_{t+1}) \},
$$
where the functions $W_{j,t}$ and $V_{j,t}$ are defined recursively as

\[
W_{j,t}(B_t, B_{t+1}) = \ln \left( \frac{\nu B_{j,t+1}}{R_t} - \nu B_{j,t} \right) + \beta W_{j,t+1}(B_{t+1}, B_{t+1}(B_{t+1})),
\]

\[
V_{j,t}(B_t, B_{t+1}) = \ln(1 - \eta_t) + \left( 1 - \eta_t \right) \left\{ \mathbb{E}[\ln A_j(z_{j,t}^i) + \tilde{b}_{j,t} + p_{j,t}] + \eta_t \ln \left( \frac{\eta_{j,t} \phi_{j,t}}{p_{j,t}} \right) \right\} + \beta V_{j,t+1}(B_{t+1}, B_{t+1}(B_{t+1})).
\]

In solving Problem (21), government $j$ takes as given the debt chosen by the other government $B_{-j,t+1}$. This yields the best response function of government $j$:

\[
B_{j,t+1} = \phi_{j,t}(B_t, B_{-j,t+1}).
\]

The Nash policy game can thus be defined as follows.

**Definition 2.** (Nash Policy Game) For given states $B_t$, the solution to the Nash policy game at time $t$ is $(B_{j,t+1}^*, B_{-j,t+1}^*)$ that satisfies $B_{j,t+1}^* = \phi_{j,t}(B_t, B_{-j,t+1}^*)$ for $j = 1, 2$.

Once we have the optimal debt chosen by the governments $(B_{j,t+1}^*)$, Proposition 1 can be used to solve for $\tilde{b}_{j,t+1}$. The Nash policy game thus defines the law of motion for aggregate states $B_{t+1} = B_t(B_t)$. Starting from the terminal condition $B_T(B_T) = 0$, we can then solve the whole sequence of policy functions backward.

### III. Computational Algorithm

In this section, we explain in details the computational algorithm that solves the general $T$-period model. The general procedure of the algorithm is the same as in Azzimonti, de Francisco and Quadrini (2014), with the only difference being that now an additional state variable $\tilde{b}_1$ needs to be included.

The state space is $(B_1, B_2, \tilde{b}_1)$. We discretize the state space and denote the resulting three-dimensional grid net to be $S$. We then solve the model at any point $(B_1, B_2, \tilde{b}_1) \in S$ backward, starting from the terminal period $t = T$. Before laying out the pseudo-code, we first show in Proposition 3 that the equilibrium of the economy can be characterized by combining Lemma 1, Proposition 1 and 2.

**Proposition 3.** (Computing the Equilibrium) Given $B_{j,t}, B_{j,t+1}, \tilde{b}_{j,t}, p_{j,t+1}, V_{j,t+1}, W_{j,t+1}, j = 1, 2$, all variables at time $t$ can be solved sequentially as follows. If we let $\bar{A}_j = \sum_l A_j(z_{i_l}) \mu_l$ and $\bar{w}_j = (1 - \theta) \bar{z}_j^\theta$, then

1. Savings rate

   \[
   \eta_t = \frac{\beta}{1 + \beta^T-t/\sum_{s=1}^{T-t} \beta^{s-1}}.
   \]
2. Excess bond holdings $\tilde{b}_{j,t+1}$ are solved from the following system of equations.

$$\frac{(1 - \eta_t \phi_{1,t}) \tilde{b}_{1,t+1}}{(1 - \phi_{1,t}) (A_1 + \tilde{b}_{1,t})} = \frac{(1 - \eta_t \phi_{2,t}) \tilde{b}_{2,t+1}}{(1 - \phi_{2,t}) (A_1 + \tilde{b}_{2,t})},$$

$$\tilde{b}_{1,t+1} + \tilde{b}_{2,t+1} = \nu (B_{1,t+1} + B_{2,t+1}),$$

where $\tilde{b}_{2,t} = \nu (B_{1,t} + B_{2,t}) - \tilde{b}_{1,t}$ and

$$\phi_{j,t} = \mathbb{E}_t \left[ \frac{A_j (z_{j,t+1}^i) + p_{j,t+1}}{A_j (z_{j,t+1}^i) + p_{j,t+1} + \tilde{b}_{j,t+1}} \right].$$

3. With $\tilde{b}_{1,t+1}, \tilde{b}_{2,t+1}$, we can solve for the risky share, land price, interest rate and wealth.

$$\phi_{j,t} = \mathbb{E}_t \left[ \frac{A_j (z_{j,t+1}^i) + p_{j,t+1}}{A_j (z_{j,t+1}^i) + p_{j,t+1} + \tilde{b}_{j,t+1}} \right],$$

$$p_{j,t} = \frac{\eta_t \phi_{j,t} (A_j + \tilde{b}_{j,t})}{1 - \eta_t \phi_{j,t}},$$

$$R_t = \frac{(1 - \eta_t \phi_{j,t}) \tilde{b}_{j,t+1}}{\eta_t (1 - \phi_{j,t}) (A_j + \tilde{b}_{j,t})},$$

$$\hat{a}_{j,t} = A_j (z_{j,t}^i) + p_{j,t} + \tilde{b}_{j,t},$$

where $\hat{a}_{j,t}$ is defined as

$$a_{j,t} = A_j (z_{j,t}^i) k_{j,t} + p_{j,t} k_{j,t} + \tilde{b}_{j,t} \triangleq \hat{a}_{j,t}.$$  

4. Value functions

$$V_{j,t} = \ln (1 - \eta_t) + \left( \frac{1}{1 - \eta_t} \right) \left[ \eta_t \ln \left( \frac{\eta_t \phi_{j,t}}{p_{j,t}} + \mathbb{E}_t \ln \hat{a}_{j,t} \right) \right] + \beta V_{j,t+1},$$

$$W_{j,t} = \ln \left[ w_j + \nu \left( \frac{B_{j,t+1}}{R_{j,t}} - B_{j,t} \right) \right] + \beta W_{j,t+1}.$$  

For the convenience of exposition, we define the following two vectors

$$X_t = \{ B_{j,t}, B_{j,t+1}, \tilde{b}_{1,t}, p_{j,t+1}, V_{j,t+1}, W_{j,t+1} \},$$

$$Y_t = \{ p_{j,t}, V_{j,t}, W_{j,t} \}.$$  

We can see that Proposition 3 defines a function between $X_t$ and $Y_t$

$$Y_t = \Upsilon(X_t).$$  

With these notation, we start by describing the solution at the terminal period $T$. We then move to the general case $t < T$. 


• Solution at $t = T$:

1. When $t = T$, by the model assumptions, we have $B_{j,T+1} = p_{j,T+1} = V_{j,T+1} = W_{j,T+1} = 0$. Therefore, $X_t = \{B_{j,T}, \tilde{b}_{1,t}, 0, 0, 0\}$.

2. For each grid $(B_1, B_2, \tilde{b}_1) \in S$, we can then solve for $Y_T = \{p_{j,T}, V_{j,T}, W_{j,T}\}$. Notice that now, $Y_T$ is defined on $(B_{1,T}, B_{2,T}, \tilde{b}_{1,T})$.

• Solution at $t < T$: To apply Proposition 3, we need values for all vector elements of $X_t$. Of all the elements, $B_{j,t}, \tilde{b}_{1,t}$ belong to the state space and are thus given exogenously. $Y_{t+1}$ can be calculated once we have $B_{j,t+1}, \tilde{b}_{1,t+1}$. Further, by item 2 of Proposition 3, given $B_{j,t}, \tilde{b}_{j,t}, B_{j,t+1}$, we can compute a corresponding $\tilde{b}_{1,t+1}$. Hence, the key step is to solve for the optimal debt $(B_{j,t+1}^*)$ at each grid $(B_{j,t}, \tilde{b}_{1,t})$. This is done by the following steps.

1. We first characterize the best response functions. To find the optimal response function, we need to first characterize governments objective function

$$G_{j,t}(B_{1,t+1}, B_{2,t+1}) = \Phi W_{j,t} + V_{j,t},$$

where we have suppressed the dependence on $B_t$, using notation from the last section. The best response functions are then defined as

$$\varphi_{1,t}(B_{2,t+1}) = \max_{B_{1,t+1}} G_{1,t}(B_{1,t+1}, B_{2,t+1}),$$

$$\varphi_{2,t}(B_{1,t+1}) = \max_{B_{2,t+1}} G_{2,t}(B_{1,t+1}, B_{2,t+1}).$$

We solve for $\varphi_{j,t}$ using also discretization. This is implemented as follows.

   a) Discretize the state space $(B_{1,t+1}, B_{2,t+1})$ and denote it to be $P$. We assume that $P$ has a finer topology than $S_1 \times S_2$, where $S_1$ and $S_2$ are subspaces of $S$ along $(B_1, B_2)$. We use multi-dimensional piecewise linear interpolation whenever needed.

   b) For any $(B_{1,t+1}, B_{2,t+1}) \in P$, we can calculate a corresponding $\tilde{b}_{1,t+1}$. We can then solve for a $Y_{t+1}$ using results from the previous iteration.

   c) Now we have all the elements in $X_t$, and hence Proposition 3 can be invoked to solve for $G_{j,t}$, which are defined on $(B_{1,t+1}, B_{2,t+1})$. Notice here we again suppress the dependence on $(B_{j,t}, \tilde{b}_{1,t})$.

   d) Once we finished looping over all $(B_{1,t+1}, B_{2,t+1}) \in P$, we get the Lagrangian table that numerically characterizes $G_{j,t}$. With $G_{j,t}$, we can solve for the best response functions $\varphi_{1,t}(B_{2,t+1})$ and $\varphi_{2,t}(B_{1,t+1})$.

2. We then can use the best response functions to solve for the optimal public debt, which are the solution $(B_{1,t+1}^*, B_{2,t+1}^*)$ of the following nonlinear system:

$$B_{1,t+1}^* = \varphi_{1,t}(B_{2,t+1}^*),$$

$$B_{2,t+1}^* = \varphi_{2,t}(B_{1,t+1}^*).$$

$(B_{1,t+1}^*, B_{2,t+1}^*)$ are thus the optimal public debt corresponding to (defined on) state grid $(B_{1,t}, B_{2,t}, \tilde{b}_{1,t})$. 

3. For any \((B_{1,t}, B_{2,t}, \tilde{b}_{1,t}) \in S\), we have calculated \((B_{1,t}^*, B_{2,t}^*)\). Using item 2 in Proposition 3 again, we can compute the \(\tilde{b}_{1,t+1}\) that corresponds to the optimal future debt \((B_{1,t+1}^*, B_{2,t+1}^*)\). We then can find the \(Y_{t+1}\), which further allows us to use Proposition 3 to solve for \(Y_t = \Upsilon(X_t)\).

4. Repeat step 1 to 3 for all periods \(t < T\) until \(t = 1\), we thus solve the general model. The infinite horizon case can then be approximated using the solution at a very large \(T\).

REFERENCES


