Labor Policy in a Dynamic Search-Matching Model with Heterogeneous Workers and Firms

Jeremy Lise† Julien Pascal‡ Jean-Marc Robin§

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Abstract

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†University of Minnesota. Email jeremy.lise@gmail.com
‡Sciences-Po, Paris. Email julien.pascal@gmail.com
§Sciences-Po, Paris and University College of London. Email jmarc.robin@gmail.com
1 Introduction

We introduce a minimum wage in Lise, Robin. We could also introduce a labor tax on output \( p \) and use is to finance unemployment. Budget balanced in the long run.

Questions: 1) Optimal minimum wage – starting from our estimates in Lise, Robin, which could be interpreted as corresponding to an economy without a legal minimum wage. 2) Optimal conduct of labor policy across the business cycle.

2 The model

2.1 Heterogeneous agents and aggregate shocks

The economy is populated by a continuum of infinitely-lived workers indexed by ability \( x \), and a continuum of firms indexed by technology \( y \). The total measures of workers and firms are fixed and normalized to one. The distribution of \( x \) across workers is denoted \( \ell(x) \) and is exogenous. The distribution of \( y \) across firms is uniform on \([0, 1]\). So \( y \) is just a way of ranking firms. The distribution of workers per job type is endogenous, and determined by firms’ recruiting decisions and workers’ mobility decisions. The cost of posting \( v \) job opportunities \( c(v) \) is exogenous. The aggregate state of the economy is indexed by \( z_t \). At the beginning of each period the aggregate state changes from \( z \) to \( z' \) according to the Markov transition probability \( \pi(z, z') \).

2.2 Worker distributions and meetings

At the beginning of period \( t \), a measure \( u_t(x) \) of unemployed workers of type \( x \) and a measure \( h_t(x, y) \) of workers of type \( x \) employed at firms of type \( y \) are inherited from period \( t - 1 \), with

\[
 u_t(x) + \int h_t(x, y) \, dy = \ell(x). \tag{1}
\]

Then, the aggregate state changes from \( z_{t-1} \) to \( z_t \). For simplicity, we assume that separations and meetings occur sequentially after the realization of the aggregate productivity shock: separations first, then the unemployed and the surviving employees get a chance to draw a new offer. Throughout we assume that match formation and separation is efficient.

Let \( B_t(x) \) denote the present value of unemployment at the beginning of period,
just after realization of shock $z_t$. Let $P_t(x,y)$ denote the present value of a match $(x,y)$. We assume that some legal constraints (e.g. minimum wage) imposes a minimal value on a match, say $P_t(x,y)$. Obviously, no match can either remain active if unemployment yields a higher value than $P_t(x,y)$. Assuming zero fixed investment in job creation, any vacancy generated by job destruction is lost and has zero continuation value; there is no severance payment or experience rating. Hence, after revelation of the new aggregate shock $z_t$, a match $(x,y)$ that was still active at the end of period $t-1$ is terminated if $P_t(x,y) < B_t(x) \wedge P_t(x,y)$, where we use the notation $a \wedge b = \max\{a,b\}$. In addition, we allow for a source of idiosyncratic job destruction shocks $\delta$. Let $u_{t+}(x)$ denote the stock of unemployed workers of type $x$ immediately after the realization of $z_t$ (at time $t+$) and the ensuing job destructions, and let $h_{t+}(x,y)$ be the stock of matches of type $(x,y)$ that survive the destruction shocks. We thus have

$$h_{t+}(x,y) = (1 - \delta) \mathbf{1} \{P_t(x,y) < B_t(x) \wedge P_t(x,y)\} h_t(x,y), \quad (2)$$

and $u_{t+}(x) = \ell(x) - \int h_{t+}(x,y) \, dy$.

Together, all surviving employed workers and unemployed produce effective search effort

$$L_t = \int u_{t+}(x) \, dx + s \iint h_{t+}(x,y) \, dx \, dy, \quad (3)$$

where the search effort of unemployed workers has been normalized to one and $s$ is thus the relative search effort of employed workers.

Let $v_t(y)$ denote the measure of type-$y$ job opportunities chosen by firm $y$, with $V_t = \int v_t(y) \, dy$ (see Section 2.7 for details). Together, worker stocks and job vacancies are brought together through the following meeting technology:

$$M_t = M(L_t,V_t), \quad \lambda_t = M_t/L_t, \quad q_t = M_t/V_t. \quad (4)$$

$M_t$ is the total measure of meetings at time $t$ given $L_t$ and $V_t$; $\lambda_t$ is the probability an unemployed searcher contacts a vacancy, and $s\lambda_t$ is the probability an employed searcher contacts a vacancy in period $t$; $q_t$ be the probability per unit of recruiting effort $v_t(y)$ that a firm contacts any searching worker.
2.3 The value of unemployment

Consider a worker of type $x$ who is unemployed for the whole period $t$. During that period she earns $b(x, z_t)$, which depends on her own type and the current aggregate productivity of the economy. She also anticipates that at the beginning of period $t + 1$, after revelation of the new aggregate state, she will meet a vacancy of type $y$ with probability $\lambda_{t+1} \frac{v_{t+1}(y)}{V_{t+1}}$. Let $P_{t+1}(x, y)$ be the value of a match $(x, y)$ at time $t + 1$ and $\bar{P}_{t+1}(x, y)$ be the corresponding minimum job value for any acceptable contract. We assume that unemployed workers have zero bargaining power and are offered $B_{t+1}(x) \land \bar{P}_{t+1}(x, y)$. The value to this unemployed worker is therefore

$$B_t(x) = b(x, z_t) + \frac{1}{1 + r} \mathbb{E}_t \left[ (1 - \lambda_{t+1}) B_{t+1}(x) ight]$$

$$+ \lambda_{t+1} \int [B_{t+1}(x) \land 1 \{ P_{t+1}(x, y) \geq \bar{P}_{t+1}(x, y) \} \bar{P}_{t+1}(x, y)] \frac{v_{t+1}(y)}{V_{t+1}} dy,$$

where $r$ is the discount rate and $\mathbb{E}_t$ is the expectation operator with respect to future aggregate states given the information set at time $t$. Using the notations $S_t(x, y) = P_t(x, y) - B_t(x)$ and $\bar{S}_t(x, y) = \bar{P}_t(x, y) - B_t(x)$ for all $t$, we can rewrite this equation as

$$B_t(x) = b(x, z_t) + \frac{1}{1 + r} \mathbb{E}_t \left[ B_{t+1}(x) \right]$$

$$+ \lambda_{t+1} \int 1 \{ S_{t+1}(x, y) \geq \bar{S}_{t+1}(x, y)^+ \} \bar{S}_{t+1}(x, y)^+ \frac{v_{t+1}(y)}{V_{t+1}} dy] \right], \quad (5)$$

with the additional notation: $a^+ = 0 \land a$.

2.4 The value and surplus of a match

Firms have access to a production technology, defined at the match level, that combines the skills of a worker, $x$, and the technology of a firm, $y$, with aggregate productivity to create value added $p(x, y, z_t)$ in period $t$. The continuation value depends on whether the match is destroyed at the beginning of period $t + 1$ or not. The current match continues in period $t + 1$ with probability $(1 - \delta) 1 \{ P_{t+1}(x, y) \geq B_{t+1}(x) \land \bar{P}_{t+1}(x, y) \}$. It is otherwise destroyed and the continuation value of the match is the value of unemployment $B_{t+1}(x)$.
If the match continues, then the worker draws an alternative offer from a firm of type $y'$ with probability $s \lambda t+1 \frac{\nu t+1(y')}{V t+1}$. We adopt the sequential auction framework of Postel-Vinay and Robin (2002). Incumbent and poaching firms engage in Bertrand competition which grants the worker a value equal to the second highest bid. Specifically, either $P_{t+1}(x, y') > P_{t+1}(x, y)$ and the worker moves to firm $y'$ and receives the incumbent employer’s reservation value $P_{t+1}(x, y)$ as continuation value; or $P_{t+1}(x, y') \leq P_{t+1}(x, y)$ and the worker stays with her current employer with continuation value the minimum of $P_{t+1}(x, y')$ and the worker’s current contract. Hence, Bertrand competition makes the continuation value of the match independent of whether the employee is poached or not:

$$P_t(x, y) = p(x, y, z_t) + \frac{1}{1 + r} \mathbb{E}_t \left[ (1 - (1 - \delta) 1 \{ P_{t+1}(x, y) \geq B_{t+1}(x) \land P_{t+1}(x, y) \} ) B_{t+1}(x) (1 - \delta) 1 \{ P_{t+1}(x, y) \geq B_{t+1}(x) \land P_{t+1}(x, y) \} P_{t+1}(x, y) \right].$$

Finally, defining match surplus as $S_t(x, y) = P_t(x, y) - B_t(x)$, and making use of equation (5), the preceding equation simplifies to

$$S_t(x, y) = p(x, y, z_t) - b(x, z_t) - \frac{1}{1 + r} \mathbb{E}_t \left[ \lambda t+1 \int 1 \{ S_{t+1}(x, y') \geq S_{t+1}(x, y')^+ \} S_{t+1}(x, y')^+ \frac{\nu t+1(y')}{V t+1} dy' \right] + \frac{1 - \delta}{1 + r} \mathbb{E}_t \left[ 1 \{ S_{t+1}(x, y) \geq S_{t+1}(x, y)^+ \} S_{t+1}(x, y) \right]. \quad (6)$$

We can also re-express the stocks $h_{t+}(x, y)$ as

$$h_{t+}(x, y) = (1 - \delta) 1 \{ S_t(x, y) \geq S_t(x, y)^+ \} h_t(x, y). \quad (7)$$

and $u_{t+}(x) = \ell(x) - \int h_{t+}(x, y) dy$.

### 2.5 Labor contracts

A labor contract is a way of sharing the surplus. For tractability, we assume that labor contracts take the form of a constant piece rate $\sigma \in [0, 1]$ such that, at all future
times, the employer commits to pay a wage \( w_t(\sigma, x, y) \) to the worker delivering value

\[
W_t(\sigma, x, y) = B_t(x) + P_t(x, y) + \sigma \left( P_t(x, y) - B_t(x) \right)
= B_t(x) + S_t(x, y)^+ + \sigma \left[ S_t(x, y) - S_t(x, y)^+ \right].
\]

Contracts are renegotiated by mutual agreement. Hiring from unemployment requires setting \( \sigma = 0 \), and competition between two firms \( y \) and \( y' \) at time \( t_0 \), with \( S_{t_0}(x, y) \leq S_{t_0}(x, y') \), yields

\[
\sigma = \frac{S_{t_0}(x, y) - S_{t_0}(x, y)^+}{S_{t_0}(x, y') - S_{t_0}(x, y)^+}.
\]

For any \( \sigma, x, y \), the wage \( w_t(\sigma, x, y) \) is such that

\[
B_t(x) + S_t(x, y)^+ + \sigma \left[ S_t(x, y) - S_t(x, y)^+ \right] = w_t(x, y, \sigma) + \frac{1}{1 + r} E_t B_{t+1}(x)
+ \frac{1 - \delta}{1 + r} E_t \left[ 1 \left\{ S_{t+1}(x, y) \geq S_{t+1}(x, y)^+ \right\} \left( s \lambda_{t+1} \int I_{t+1}(\sigma, x, y, y') \frac{v_{t+1}(y')}{V_{t+1}} dy' \right.ight.
\]

\[
\left. \left. + (1 - s \lambda_{t+1}) \left[ (1 - \sigma) S_{t+1}(x, y)^+ + \sigma S_{t+1}(x, y) \right] \right) \right].
\]

where \( I_{t+1}(\sigma, x, y, y') \) is the second best of the three values: \( S_{t+1}(x, y') \), \( S_{t+1}(x, y) \) and the reservation value \( (1 - \sigma) S_{t+1}(x, y)^+ + \sigma S_{t+1}(x, y) \), that is

\[
I_t(\sigma, x, y, y') = \begin{cases} 
S_t(x, y) & \text{if } S_t(x, y') > S_t(x, y), \\
S_t(x, y') & \text{if } (1 - \sigma) S_t(x, y)^+ + \sigma S_t(x, y) \leq S_t(x, y') \leq S_t(x, y), \\
(1 - \sigma) S_t(x, y)^+ + \sigma S_t(x, y) & \text{if } S_t(x, y') < (1 - \sigma) S_t(x, y)^+ + \sigma S_t(x, y).
\end{cases}
\]

That is, after eliminating unemployment values using equation (5),

\[
\sigma S_t(x, y) + (1 - \sigma) S_t(x, y)^+ = w_t(x, y, \sigma) - b(x, z_t)
\]

\[
- \frac{1}{1 + r} E_t \left[ \lambda_{t+1} \int 1 \left\{ S_{t+1}(x, y') \geq S_{t+1}(x, y)^+ \right\} S_{t+1}(x, y)^+ \frac{v_{t+1}(y')}{V_{t+1}} dy' \right]
\]

\[
+ \frac{1 - \delta}{1 + r} E_t \left[ 1 \left\{ S_{t+1}(x, y) \geq S_{t+1}(x, y)^+ \right\} \left( s \lambda_{t+1} \int I_{t+1}(\sigma, x, y, y') \frac{v_{t+1}(y')}{V_{t+1}} dy' \right.ight.
\]

\[
\left. \left. + (1 - s \lambda_{t+1}) \left[ (1 - \sigma) S_{t+1}(x, y)^+ + \sigma S_{t+1}(x, y) \right] \right) \right].
\]
And finally, using equation (6),

\[
\begin{align*}
    w_t(x, y, \sigma) &= (1 - \sigma)S_t(x, y)^+ + \sigma p(x, y, z_t) + (1 - \sigma) b(x, z_t) \\
    &+ \frac{1 - \sigma}{1 + r} \mathbb{E}_t \left[ \lambda_{t+1} \int 1 \{ S_{t+1}(x, y') \geq S_{t+1}(x, y')^+ \} S_{t+1}(x, y')^+ \frac{v_{t+1}(y')}{V_{t+1}} \, dy' \right] \\
    &- \frac{1 - \delta}{1 + r} \mathbb{E}_t \left[ 1 \{ S_{t+1}(x, y) \geq S_{t+1}(x, y)^+ \} \right] \\
    &\times \left( s \lambda_{t+1} \int [I_{t+1}(\sigma, x, y, y') - \sigma S_{t+1}(x, y)] \frac{v_{t+1}(y')}{V_{t+1}} \, dy' \\
    &+ (1 - s \lambda_{t+1}) (1 - \sigma) S_{t+1}(x, y)^+ \right].
\end{align*}
\]

(8)

2.6 Minimum wage

Let \( w_t \) denote legal minimum wage. We can use equation (8) with \( \sigma = 0 \) and \( w_t(x, y, 0) = w_t \) to determine the minimum surplus:

\[
\begin{align*}
    S_t(x, y) &= w_t - b(x, z_t) \\
    &- \frac{1}{1 + r} \mathbb{E}_t \left[ \lambda_{t+1} \int 1 \{ S_{t+1}(x, y') \geq S_{t+1}(x, y')^+ \} S_{t+1}(x, y')^+ \frac{v_{t+1}(y')}{V_{t+1}} \, dy' \right] \\
    &+ \frac{1 - \delta}{1 + r} \mathbb{E}_t \left[ 1 \{ S_{t+1}(x, y) \geq S_{t+1}(x, y)^+ \} \right] \\
    &\times \left( s \lambda_{t+1} \int [I_{t+1}(0, x, y, y') - \sigma S_{t+1}(x, y)] \frac{v_{t+1}(y')}{V_{t+1}} \, dy' + (1 - s \lambda_{t+1}) S_{t+1}(x, y)^+ \right].
\end{align*}
\]

(9)

where

\[
I_{t+1}(0, x, y, y') = \begin{cases} 
    S_{t+1}(x, y) & \text{if } S_{t+1}(x, y') > S_{t+1}(x, y), \\
    S_{t+1}(x, y') & \text{if } S_{t+1}(x, y)^+ \leq S_{t+1}(x, y') \leq S_{t+1}(x, y), \\
    S_{t+1}(x, y)^+ & \text{if } S_{t+1}(x, y') < S_{t+1}(x, y)^+.
\end{cases}
\]

Note that the minimum match surplus is also the worker’s surplus at the minimum wage. If the latter is lower than the value of unemployment, then the worker is paid above the minimum wage or the match is not feasible. This is why we have \((1 - s \lambda_{t+1}) S_{t+1}(x, y)^+ \) at the end the right-hand side: if \( S_{t+1}(x, y) < 0 \), then the match is destroyed.
2.7 Vacancy creation

Each period firms can buy the advertising of \( v \) job opportunities from job placement agencies at a price \( c(v) \geq 0 \) that is assumed independent of the firm’s type, increasing and convex. In equilibrium, the number of advertised job opportunities is determined by equating the marginal cost to the expected value of a job opening,

\[
c'[v_t(y)] = q_t J_t(y),
\]

where \( J_t(y) \) denotes the expected value of a contact by a vacancy of type \( y \), and \( q_t \) is the probability, per unit of recruiting effort, that a firm contacts a searching worker. The assumption that \( c(\cdot) \) is increasing and convex guarantees a non-degenerate distribution of vacancies \( v_t(y) \).

Any job opportunity that does not deliver a contact with a worker in the period is lost and generates no continuation value. Any contact that does not end up in an employment contract is lost and has zero value.\(^1\)

The expected value of a contact is calculated as

\[
J_t(y) = \int \frac{u_{t+}(x)}{L_t} \left[ S_t(x, y) - S_t(x, y)^+ \right]^+ \, dx \\
+ \iint \frac{s h_{t+}(x, y')}{L_t} \left[ S_t(x, y) - S_t(x, y') \right]^+ \, dx \, dy'.
\]

The contact is with an unemployed worker of type \( x \) with probability \( \frac{u_{t+}(x)}{L_t} \) and a match is formed if the net match surplus \( S_t(x, y) - S_t(x, y)^+ \) is positive, in which case it is entirely appropriated by the employer. The contact is with a worker of type \( x \) that is currently employed at a firm of type \( y' \) with complementary probability \( \frac{s h_{t+}(x, y')}{L_t} \). Poaching is successful if \( S_t(x, y) > S_t(x, y') \) and Bertrand competition grants the poacher a value \( S_t(x, y) - S_t(x, y') = P_t(x, y) - P_t(x, y') \).

\(^1\)A more sophisticated job creation process could be envisioned, in which fixed initial investments would give value to job creation above and beyond the service provided by placement agencies.
2.8 Labor market flows

The law of motion for employment is therefore

\[
h_{t+1}(x, y) = h_t(x, y) \left[ 1 - \int s \lambda_t \frac{v_t(y')}{V_t} \mathbf{1}\{S_t(x, y') > S_t(x, y)\} \, dy' \right]
+ \int h_t(x, y') s \lambda_t \frac{v_t(y)}{V_t} \mathbf{1}\{S_t(x, y) > S_t(x, y')\} \, dy'
+ u_{t+1}(x) \lambda_t \frac{v_t(y)}{V_t} \mathbf{1}\{S_t(x, y) \geq S_t(x, y)^+\},
\]

subtracting those lost to more productive poachers, and adding the \((x, y)\)-jobs created by poaching from less productive firms and hiring from unemployment. Unemployment follows as

\[
u_{t+1}(x) = \ell(x) - \int h_{t+1}(x, y) \, dy.
\]

2.9 Steady state

If the aggregate state of the economy remains for ever the same \(z = 1\), the equations become.

Values and wages.

\[
rB(x) = (1 + r)b(x) + \lambda \int \mathbf{1}\{S(x, y) \geq S(x, y)^+\} S(x, y)^+ \frac{v(y)}{V} \, dy,
\]

\[
S(x, y) = p(x, y) - b(x) - \frac{1}{1 + r} \left[ \lambda \int \mathbf{1}\{S(x, y') \geq S(x, y')^+\} S(x, y')^+ \frac{v(y')}{V} \, dy' \right]
+ \frac{1 - \delta}{1 + r} \left[ \mathbf{1}\{S(x, y) \geq S(x, y)^+\} S(x, y) \right],
\]

For \(S(x, y) \geq S(x, y)^+\).
\[ w(x, y, \sigma) = (1 - \sigma)S(x, y)^+ + \sigma p(x, y) + (1 - \sigma)b(x) \]
\[ + \frac{1 - \sigma}{1 + r} \left[ \lambda \int 1 \{ S(x, y') \geq S(x, y')^+ \} S(x, y')^+ \frac{v(y')}{V} \ dy' \right] \]
\[ - \frac{1 - \delta}{1 + r} \left[ S(x, y) \geq S(x, y) \} \right] \]
\[ \times \left( s\lambda \int [I(\sigma, x, y, y') - \sigma S(x, y)] \frac{v(y')}{V} \ dy' + (1 - s\lambda)(1 - \sigma) S(x, y)^+ \right), \quad (15) \]
where
\[
I(\sigma, x, y, y') = \begin{cases} 
S(x, y) & \text{if } S(x, y') > S(x, y), \\
S(x, y') & \text{if } (1 - \sigma)S(x, y)^+ + \sigma S(x, y) \leq S(x, y') \leq S(x, y), \\
(1 - \sigma)S(x, y)^+ + \sigma S(x, y) & \text{if } S(x, y') < (1 - \sigma)S(x, y)^+ + \sigma S(x, y). 
\end{cases}
\]
\[
S(x, y) = w - b(x, z) - \frac{\lambda}{1 + r} \int 1 \{ S(x, y') \geq S(x, y')^+ \} S(x, y')^+ \frac{v(y')}{V} \ dy' 
+ \frac{1 - \delta}{1 + r} \left[ 1 \{ S(x, y) \geq S(x, y)^+ \} \left( s\lambda \int I(0, x, y, y') \frac{v(y')}{V} \ dy' + (1 - s\lambda) S(x, y)^+ \right) \right], \quad (16) 
\]
\textbf{Distributions.}
\[
h_+(x, y) = (1 - \delta)h(x, y), \quad u_+(x) = \ell(x) - \int h_+(x, y) \ dy 
\]
\[
\left\{ 1 - (1 - \delta) \left[ 1 - \int s\lambda \frac{v(y')}{V} 1\{ S(x, y') > S(x, y) \} \ dy' \right] \right\} h(x, y) 
= \int h_+(x, y') s\lambda \frac{v(y)}{V} 1\{ S(x, y) > S(x, y') \} \ dy' 
+ u_+(x) \lambda \frac{v(y)}{V} 1\{ S(x, y) \geq S(x, y)^+ \}. \quad (17) 
\]
Vacancies.

\[ c'(v(y)) = \frac{M(L,V)}{V}J(y), \quad (18) \]

\[ V = \int v(y) \, dy = \int (c')^{-1} \left( \frac{M(L,V)}{V}J(y) \right) \quad (19) \]

\[ L = \int u_+(x) \, dx + s \int \int h_+(x,y) \, dx \, dy \quad (20) \]

\[ \lambda = \frac{M}{L}, \quad q = \frac{M}{V} \]

\[ J(y) = \int \frac{u_+(x)}{L} \left[ S(x,y) - S(x,y)^+ \right]^+ \, dx + \int \int \frac{sh_+(x,y')}{L} \left[ S(x,y) - S(x,y') \right]^+ \, dx \, dy'. \quad (21) \]

2.10 Computation of the stochastic search equilibrium

**Approximation of the surplus function**  The surplus functions \( S_t \) and \( S_t \) depend on \( t \) via \( h_t \) and \( z_t \). The difficulty is that \( h_t \) is an infinite-dimensional parameter. An approximation is therefore necessary. However, experience drawn from past work (e.g. Lise and Robin, 2017) teaches us that the matching distribution may not be very smooth, because of the positive-surplus selection, which makes it difficult to approximate with smooth polynomial expansions. Fortunately, a closer look to the equations for the surplus and minimum surplus shows that we can replace \( h_t \) by \((\lambda_t, v_t/V_t)\) in the state space.

Secondly, we can further reduce the state space’s dimensionality assuming that \( v_t/V_t \) can be approximated by a simple parametric distribution with finite-dimensional parameter \( \beta_t \),

\[ v_t(y)/V_t = g(y; \beta_t), \]

for example a beta distribution.

Thirdly, we assume that the dynamics of \((\lambda_t, \beta_t)\) is approximately autoregressive,

\[ (\lambda_{t+1}, \beta_{t+1}) = \theta^0(z_{t+1}) + \theta^1(z_{t+1}) (\lambda_t, \beta_t), \]

where \( \theta(z) = [\theta^0(z), \theta^1(z)] \) is a polynomial matrix that we need to learn. If needed, the link between \((\lambda_{t+1}, \beta_{t+1})\) and \((\lambda_t, \beta_t)\) can be made polynomial and the density \( g \) can be made more flexible by using a finite mixture of parametric density functions.
The state variable is therefore reduced to the finite-dimensional vector $Z_t = (z_t, \lambda_t, \beta_t)$. Specifically,

$$S_t(x, y) = S(x, y|z_t, \lambda_t, \beta_t; \theta), \quad \overline{S}_t(x, y) = \overline{S}(x, y|z_t, \lambda_t, \beta_t; \theta).$$

Finally, we approximate $S$ and $\overline{S}$, for a given $\theta$, using polynomial expansions of the form

$$(x, y, Z) \mapsto S(x, y|Z; \theta) = \sum_{k=0}^{K} \alpha_k(Z; \theta) T_k(x, y),$$

$$(x, y, Z) \mapsto \overline{S}(x, y|Z; \theta) = \sum_{k=0}^{K} \alpha_k(Z; \theta) T_k(x, y),$$

where $T_k(x, y)$ is a family of orthogonal polynomials and spectral coefficients $\alpha_k(Z; \theta)$, $\overline{\alpha}_k(Z; \theta)$ are linear or quadratic functions of $Z$. The idea is that we need to approximate well the way $S_t$ and $\overline{S}_t$ depend on $(x, y)$, but somewhat less precisely their dependency to $Z_t$.

Finally, coefficients $(\alpha, \overline{\alpha})$ are determined by solving:

$$S(x, y|Z; \theta) = p(x, y, z) - b(x, z) + \frac{1}{1+r} \int \pi(z'|z) dz' \left[ -\lambda' \int 1 \{ S(x, y'|Z'; \theta) \geq \overline{S}(x, y'|Z'; \theta)^+ \} \overline{S}(x, y'|Z'; \theta)^+ g(y'; \beta') dy' 
 + (1 - \delta) 1 \{ S(x, y|Z'; \theta) \geq \overline{S}(x, y|Z'; \theta)^+ \} S(x, y|Z'; \theta) \right],$$

and

$$\overline{S}(x, y|Z; \theta) = w - b(x, z) + \frac{1}{1+r} \int \lambda' \pi(z'|z) dz' \left[ -\lambda' \int 1 \{ S(x, y'|Z'; \theta) \geq \overline{S}(x, y'|Z'; \theta)^+ \} \overline{S}(x, y'|Z'; \theta)^+ g(y'; \beta') dy' 
 + (1 - \delta) 1 \{ S(x, y|Z'; \theta) \geq \overline{S}(x, y|Z'; \theta)^+ \} 
 \times \left( s\lambda' \int I(x, y, y', Z'; \theta) g(y'; \beta') dy' + (1 - s\lambda') \overline{S}(x, y|Z'; \theta)^+ \right) \right].$$
where
\[
I(x, y, y', Z'; \theta) = \begin{cases} 
S(x, y|Z'; \theta) & \text{if } S(x, y'|Z'; \theta) > S(x, y|Z'; \theta), \\
S(x, y'|Z'; \theta) & \text{if } S(x, y'|Z'; \theta)^+ \leq S(x, y'|Z'; \theta) \leq S(x, y|Z'; \theta), \\
S(x, y|Z'; \theta)^+ & \text{if } S(x, y'|Z'; \theta) < S(x, y|Z'; \theta)^+.
\end{cases}
\]

and
\[
Z' = (z', \lambda', \beta'), \quad (\lambda', \beta') = \theta^0(z') + \theta^1(z') (\lambda, \beta).
\]

To do that, first project both sides of these two equations on orthogonal polynomials \(T_k(x, y)\), given \(Z\). This yields \(a_k(Z; \theta), a_k(Z; \theta)\) on the left hand side, and integrals with respect to \(z', x, y\) on the right-hand side. Then, construct a grid of values of \(Z\), for example using a simulation of the steady state model. Lastly, minimize the Euclidean norm of the difference between LHS and RHS calculated for that grid with respect to the parameters of \(Z\) in \(a_k(Z; \theta), a_k(Z; \theta)\).

**Simulation of an economy’s trajectory**  We simulate an economy and learn on \(\theta\) as follows.

1. Given \(\theta_{t-1}\), calculate surplus functions \(S(x, y|Z; \theta_{t-1}), S(x, y|Z; \theta_{t-1})\) for any \((x, y, Z)\).

2. Given \(z_{t-1}\), draw \(z_t\).

3. Given \(h_t, z_t\) and \(\theta_{t-1}\), solve for \((V_t, \beta_t, \lambda_t)\) such that, with \(Z_t = (z_t, \lambda_t, \beta_t)\),

\[
\begin{align*}
h_{t+}(x, y) &= (1 - \delta) \mathbf{1} \{S(x, y|Z_t; \theta_{t-1}) \geq S(x, y|Z_t; \theta_{t-1})^+\} h_t(x, y), \\
u_{t+}(x) &= \ell(x) - \int h_{t+}(x, y) \, dy, \\
L_t &= \int u_{t+}(x) \, dx + s \int\int h_{t+}(x, y) \, dx \, dy, \\
\lambda_t &= M(L_t, V_t) / L_t, \\
q_t &= M(L_t, V_t) / V_t, \\
c' [V_t g(y; \beta_t)] &= q_t J_t(y),
\end{align*}
\]
and

\[ J_t(y) = \int \frac{u_{t+}(x)}{L_t} [S(x, y|Z_t; \theta_{t-1}) - S(x, y|Z_t; \theta_{t-1}^{-})]^+ \, dx \\
+ \int \int \frac{sh_{t+}(x, y')}{L_t} [S(x, y|Z_t; \theta_{t-1}) - S(x, y'|Z_t; \theta_{t-1}^{-})]^+ \, dx \, dy'. \]

4. Update

\[ h_{t+1}(x, y) = h_{t+}(x, y) \left[ 1 - \int s \lambda_t g(y'; \beta_t) \mathbf{1}\{S(x, y'|Z_t; \theta_{t-1}) > S(x, y|Z_t; \theta_{t-1})\} \, dy' \right] \]
\[ + \int h_{t+}(x, y') s \lambda_t g(y; \beta_t) \mathbf{1}\{S(x, y|Z_t; \theta_{t-1}) > S(x, y'|Z_t; \theta_{t-1})\} \, dy' \]
\[ + u_{t+}(x) \lambda_t g(y; \beta_t) \mathbf{1}\{S(x, y|Z_t; \theta_{t-1}) \geq S(x, y|Z_t; \theta_{t-1}^{-})\}. \]

5. Update \( \theta_t \) by fitting \( (\lambda_s, \beta_s) \) with \( \theta_t^0(z_t) + \theta_t^1(z_t) (\lambda_{s-1}, \beta_{s-1}) \), keeping only the last \( s = t, ..., t-T+1 \) periods (say \( T = 100 \), short enough to remove dependency to initial condition, long enough to remove dependency to draws of aggregate shocks).

This algorithm can be initialized with the identity matrix, \( \theta_t = \text{Id} \), for the first \( T \) periods.

**References**
