A Quantitative Model of Bubble-Driven Business Cycles*

– preliminary and incomplete draft –

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Abstract: The 2007-2008 financial crisis highlighted that a turmoil in the financial sector including bursting asset price bubbles can cause pronounced and persistent fluctuations in real economic activity. This motivates the consideration of evolving and bursting asset price bubbles as another source of fluctuations in a business cycle model. In this paper rational asset price bubbles are therefore incorporated into a life-cycle RBC model as first developed by Ríos-Rull (1996). The calibration of the model to the post-war US economy and the numerical solution show that the model is able to generate plausible bubble-driven business cycles – economic fluctuations caused by evolving and bursting asset price bubbles.

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1 Introduction

The 2007-2008 financial crisis highlighted that a turmoil in financial markets and bursting asset price bubbles can cause pronounced and persistent fluctuations in real economic activity. This motivates the consideration of evolving and bursting asset price bubbles as another source of fluctuations in a business cycle model. Building on the contribution by Martin and Ventura (2012) I therefore set up and numerically solve a life-cycle RBC model as first developed by Ríos-Rull (1996) with asset price bubbles. The calibration of the model to the post-war US economy and the numerical solution show that the model is able to depict plausible bubble-driven business cycles. In particular, the model generates i) a higher and empirically more plausible volatility of consumption at the cost of ii) a slightly lower and empirically less plausible contemporaneous correlation of consumption with output than the life-cycle RBC model without bubbles.

Aggregate asset prices and real aggregate output, consumption, and investment comove over the US business cycle as shown in fig. 1. The two upper graphs represent the financial sector while the bottom graph represents the real macroeconomy. The first graph depicts the detrended aggregate share price index and the second graph depicts the detrended Case-Shiller house price index for the US from 1961:Q1 to 2010:Q4. It is difficult to explain the pronounced negative deviations of these two variable from their respective trends in the periods 2001 - 2003 and 2008 - 2010 by resorting to fundamentals like negative TFP shocks, demographic shocks, natural disasters, or policy shocks. It proved therefore hard to reject the hypothesis that these fluctuations in aggregate asset prices were driven by bubbles.1 This is underlined by evidence of two recent empirical studies from Brunnermeier and Schnabel (2015) and Jordà et al. (2015) who show that bursting bubbles cause recessions. Given their empirical relevance for business cycles it is important to integrate asset price bubbles into DSGE models.

The model presented in this paper bridges the gap between the literature on quantitative DSGE models that include financial shocks and the literature on rational bubbles modeled mostly within simple but elegant two-period overlapping generations (OLG) models.

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1 Existing empirical tests are not able provide clear-cut evidence for the existence of bubbles. See, for instance, Gürkaynak (2008) for a survey on empirical test for rational bubbles. Testing aggregate asset price bubbles is even more challenging.
First, many recent contributions within the DSGE literature consider financial shocks as impulses to real economic fluctuations. Two examples are “valuation shocks” in Gertler and Karadi (2011) and “liquidity shocks” in Kiyotaki and Moore (2012). These shocks to the value of certain assets, balance sheets, or the liquidity of some assets are all exogenous. I contribute to this literature by modeling explicitly the origin of these shocks to asset prices.

Second, since the seminal contribution by Tirole (1985) rational asset price bubbles have been incorporated in general equilibrium models. In the Tirole (1985) model bubbles always crowd-out capital. Empirically it is not plausible that the capital stock and output decline during episodes of existing asset price bubbles, and increase when bubbles burst. The correlation should be the reverse as can be inferred from fig. 1. Therefore, recent models extend the Tirole (1985) model – mostly by financial frictions – in order to derive equilibria where bubbles create a crowding-in of capital such that capital and output increase during periods of existing bubbles (see, among others, Galí (2014), Martin and Ventura (2012), and Farhi and Tirole (2012)). Most of these contributions consider two-period OLG models and derive explicit conditions for the existence of bubbles and the

Source: All time series are expressed in logarithms and have been detrended with the HP filter. The share price index is from the OECD’s Monthly Monetary and Financial Statistics, deflated by the CPI. The house price index is from the Federal Reserve Economic Database, deflated by the CPI. Real output, consumption, and investment are from the U.S. Bureau of Economic Analysis (BEA), tables 1.1.6 and 5.3.3. All three series are seasonally adjusted.
characteristics of bubbles in general equilibrium.\textsuperscript{2} Two-period OLG models are well suited for deriving insightful analytical results for the economic mechanisms at work, but their drawback is that they are not capable of replicating the short and medium run behavior of empirical time series by numerical solutions. The reason is that calibrating two-period OLG models implies that one period in the model corresponds to approximately 30 years in real time. Accordingly, the length of a recession would be at least 30 years and bubbles would exist also at least for 30 years – both implausibly long. I contribute to this literature by considering rational bubbles within a \textit{large-scale} OLG model. As a result, one period in the model corresponds to one quarter in real time and the model with bubbles can be brought to the data.

A few contributions within the infinitely-lived agent DSGE literature do also consider asset price bubbles. \textit{Bernanke and Gertler (1999)} consider \textit{irrational} asset price bubbles within a RBC model with financial frictions. As the authors state, they ”use the term “bubble” [here] loosely to denote temporary deviations of asset prices from fundamental values” (p. 19). Similarly, \textit{Luik and Wesselbaum (2014)} considers “near-rational” asset price bubbles. \textit{Miao et al. (2015)} numerically solve a model with rational bubbles relying on local perturbation methods. By log-linearizing the dynamic systems of equations only small shocks to bubbles can be considered. It is hard to justify, however, that bursting bubbles presents a small shock and that local solution techniques are an accurate approximation. I therefore apply a global solution technique for solving the model in this paper. To the best of my knowledge, this paper is the first to set up and numerically solve a life-cycle RBC model featuring rational bubbles. By implementing rational bubbles in a life-cycle RBC model the effect of evolving and bursting bubbles as an impulse for aggregate fluctuations can be analyzed and compared to business cycles resulting from TFP shocks.

\textsuperscript{2}One exception is \textit{Gali (2017)}. In a basic New-Keynesian model with bubbles and a large number of overlapping generations \textit{Gali (2017)} analyzes under which conditions bubbly equilibria can exist and how monetary policy affects asset price bubbles. The mechanism through which bubbles affect the real economy is, however, very different from the one I consider in this paper.
2 The Model

2.1 Set Up

Demographics. Time is discrete and the economy exists up to infinity. The economy consists of individuals of different age groups. Each period some individuals enter and some individuals leave the economy. The age of an individual is denoted by the subscript $s$. Young individuals enter the economy at a real life age of 21 years which is denoted by $s = 1$. After entering the economy individuals supply inelastically one unit of labor per period for $T \geq 1$ periods. Then individuals live for further $T_R \geq 1$ periods in retirement before dying and leaving the economy. Hence, the life-span for every individual in this economy is $T + T_R$ periods. In the classic OLG-model developed by Allais (1947), Samuelson (1958), and Diamond (1965) it is assumed that $T = T_R = 1$. In the vein of Auerbach and Kotlikoff (1987) I will calibrate the parameters $T$ and $T_R$ such that one period in the model corresponds to one quarter in real time.

Total population is assumed to grow at a constant rate $n \geq 0$. Let $N^s_t$ denote the size of the cohort consisting of individuals of age $s$ in period $t$. Total population is then given by $N_t = \sum_{s=1}^{T+T_R} N^s_t$. Without loss of generality total population in period $t = 0$ is set equal to one. It follows that total population is given by

$$N_t = (1 + n)^t,$$

and the cohort size is

$$N^s_t = \begin{cases} 
(1 + n)^{t-s+1} N_0^1, & \text{if } 0 < s \leq T + T_R \\
0, & \text{else}
\end{cases}$$

where

$$N_0^1 = \frac{1}{\sum_{s=1}^{T+T_R} (1 + n)^{1-s}}$$
is the size of the cohort consisting of individuals that enter the economy in period $t = 0$. 


**Firms.** Firms of mass one employ two production factors – capital $K_t$ and labor $L_t$ – to produce a homogeneous final output good according to the Cobb-Douglas technology

$$Y_t = A_t K_t^\alpha (E_t L_t)^{1-\alpha},$$

where $0 < \alpha < 1$ is the capital income share. Technological progress is labor augmenting, deterministic and expressed as $E_{t+1} = (1 + g)E_t$. The variable $A_t$ depicts exogenous and stochastic total factor productivity (TFP) shocks evolving according to an AR(1) process:

$$\ln A_{t+1} = \rho \ln A_t + z_{a_{t+1}},$$

where $z_{a_t} \sim N(0, \sigma^a)$ is the innovation term and $0 \leq \rho < 1$ measures the persistence of TFP-shocks.

Competitive profit maximization yields wages $w_t$ and interest rates $r_t$

$$w_t = (1 - \alpha) \frac{Y_t}{L_t}, \quad (1)$$

$$r_t = \alpha \frac{Y_t}{K_t} - \delta, \quad (2)$$

where $\delta$ is the capital depreciation rate.

**Utility.** After entering the economy individuals supply $n_{st}$ units of labor in each period $s \leq T$, earning wages $w_t$. For $s > T$ retirement is mandatory, households do not work ($n_{st} = 0$) and live from wealth accumulated in earlier periods of their lives. Preferences of an individual of age $s = 1$ at period $t$ are described by the intertemporal von Neumann-Morgenstern expected utility function

$$E_t U_t^s = E_t \sum_{s=1}^{T+T_R} \beta^{s-1} u(c_{t+s-1}^s, n_{t+s-1}^s),$$

where $0 < \beta$ is the subjective discount factor, $u(c_{t+s-1}^s, n_{t+s-1}^s)$ is the instantaneous utility function, $c_t^s$ denotes consumption of an individual of age $s$ in period $t$, and $n_t^s$ denotes the units of time an individual of age $s$ in period $t$ spends for working.
Instantaneous utility is described by GHH-preferences\(^3\) (Greenwood \textit{et al.}, 1988)

\[ u(c_{t+s-1}^s, n_{t+s-1}^s) = \left[ \frac{c_{t+s-1}^s - \theta_t \left( \frac{n_{t+s-1}^s}{1 + \chi} \right)^{1+\chi}}{1 - \epsilon} \right]^{1-\epsilon} - 1, \]

where \(\epsilon > 0\) is the constant coefficient of relative risk aversion, \(\chi\) is the inverse of the Frisch elasticity of labor supply and \(\theta_t\) is a time-varying coefficient measuring the weight of disutility from labor. The last parameter grows at the growth rate of the economy, i.e. \(\theta_t \equiv (1 + g)^t \theta\), where \(\theta\) denotes a constant parameter. This is necessary for a balanced growth path to exist.

\section*{Elasticities of the utility function.} The intertemporal elasticity in consumption is given by (I suppress the superscripts)

\[
\frac{d^2 c_{t+1}}{c_t^2} \cdot \frac{c_{t+1}}{c_t} = \frac{c_{t+1} - \theta_{t+1} \left( \frac{n_{t+1}}{1 + \chi} \right)^{1+\chi}}{\epsilon c_{t+1}} \leq \frac{1}{\epsilon}.
\]

For retirees it holds that \(n_{t+1} = 0\) such that the intertemporal elasticity of substitution is constant and equal to \(\frac{1}{\epsilon}\).

The Arrow-Pratt measure of relative risk aversion in consumption is given by

\[
\frac{-u_c(c_n)c}{u''(c)} = \epsilon \frac{c}{c - \theta_t \left( \frac{n}{1 + \chi} \right)^{1+\chi}}.
\]

The intertemporal elasticity in leisure \(l \equiv 1 - n\) is given by

\[
\frac{d^2 l_{t+1}}{l_t^2} \cdot \frac{l_{t+1}}{l_t} = \frac{1 - l_{t+1}}{\chi l_{t+1}},
\]

while the intertemporal elasticity in labor-supply is

\[
\frac{d^2 n_{t+1}}{n_t^2} \cdot \frac{n_{t+1}}{n_t} = -\frac{1}{\chi}.
\]

\(^3\)Note, Schmitt-Grohé and Uribe (2012) apply the more general instantaneous utility function from Jaimovich and Rebelo (2009) which nests the GHH-form. In Bayesian and maximum-likelihood estimations they show that the relevant parameter is close or equal to zero, implying that the special case of GHH-preferences is the empirically more relevant.
**Financial friction.** Besides differences in age individuals further differ with respect to their investment efficiency. A small share of the population is comparably productive at investing in capital used for production. Think of these productive individuals as the entrepreneurs in the economy or individuals with higher financial literacy. The population share of productive individuals is constant and given by $0 \leq \eta \leq 1$. These productive individuals are able to transform one unit of the final output good one-to-one into productive capital used in production. The remainder of the population, $1 - \eta$, is unproductive at investing and has to incur costs of $1 - \sigma$ when investing one unit of the final output good, where $0 \leq \sigma \leq 1$. The gross return to investment in capital faced by productive individuals is then equal to $1 + r_t$ while it is $\sigma(1 + r_t)$ for unproductive individuals. Individuals are born as either productive or unproductive investors and stay productive or unproductive investors throughout their whole life.

If financial markets were frictionless, unproductive individuals would lend all their wealth to productive individuals such that both groups would be better off, the allocation of investment would be efficient, and the capital stock in the economy would be larger than in an economy without lending between unproductive and productive individuals. However, it is assumed – as in Martin and Ventura (2012) – that unproductive individuals lending resources to productive individuals face a default probability greater than $1 - \sigma$ such that it is never profitable for unproductive individuals to lend resources to productive individuals. The result is a very strong financial friction: no borrowing or lending takes place between productive and unproductive individuals, investments are inefficiently allocated, and a wedge is drawn between the returns to capital faced by productive and unproductive individuals.

**Bubbles.** Individuals can save labor income in order to smooth consumption and finance consumption in retirement when no labor income is earned. Therefore individuals can either supply capital $a_t^{j,s}$ for the production of a final output good, or purchase a zero-dividend asset $b_t^{j,s}$, where $a_t^{j,s}$ ($b_t^{j,s}$) denotes the capital stock (zero-dividend assets) held by an individual of productivity type $j \in \{U, P\}$ and age $s$ at the beginning of period $t$.\(^4\) Note, $b_t^{j,s}$ denotes the value of the zero-dividend asset in terms of the final output.

\(^4\)Tirole (1985) first introduced rational asset price bubble in a simple two-period OLG model.
An asset price can be decomposed into a fundamental and a bubble component, where the fundamental component is the sum of the discounted stream of expected future dividends or rents.\textsuperscript{5} The zero-dividend asset’s fundamental is by definition zero such that its price contains a bubble when \( b_t \) is greater than zero. Following Tirole (1985) I refer to the zero-dividend asset just as “bubble” in the remainder.

The aggregate bubble \( B_t \) is obtained by summing up all the bubbles held by individuals in the economy
\[
B_t = \sum_{s=1}^{T+T^R} N_t^s \left[ \eta b_t^{P,s} + (1 - \eta) b_t^{U,s} \right],
\]
where \( b_{t+1}^{j,s} \) is the quantity of bubbles purchased by an individual of productivity type \( j \in \{ U, P \} \) and of age \( s \) at the end of period \( t \). The size of the bubbles held by all productive (unproductive) individuals of age \( s \) in period \( t \) is \( \eta N_t^s b_t^{P,s} \left( (1 - \eta) N_t^s b_t^{U,s} \right) \). Summing up the bubbles of all cohorts and productivity types gives \( B_t \) as described by (3).

In a given period \( t \) bubbles can be initiated and existing bubbles can burst.\textsuperscript{6} Following Martin and Ventura (2012) the creation and destruction of bubbles is governed by investors’ sentiment \( z_t^b \in \{ 0, 1 \} \). When investors are optimistic \( z_t^b = 1 \), they believe that the zero-dividend asset has a positive value, new bubbles can be initiated, and \( b_t \geq 0 \). When investors are pessimistic \( z_t^b = 0 \), they want to sell all their zero-dividend assets, they will not purchase further zero-dividend assets, and \( b_t = 0 \). Investors’ sentiment \( z_t^b \) follows a simple Markov-process with the transition matrix \( P \), as depicted in table 1.

When investors are optimistic, \( z_t = 1 \), working productive individuals create new bubbles \( \hat{b}_t^N \) and sell them to other individuals in the same period. For the sake of simplicity it is assumed that each working productive individual creates the same amount of new bubbles. The creation of new bubbles is exogenous and given by
\[
\hat{b}_t^N = \theta^E E_t + \theta^E \frac{B_t}{N_t^s},
\]
where \( \theta^E \geq 0 \) is the size of newly created bubbles when technological growth is absent \( (E_t = 1) \) and investors have just become optimistic \( (z_t^b = 1 \text{ and } z_{t-1}^b = 0) \). The parameter

\textsuperscript{5}See, for instance, Blanchard and Watson (1983, 2f.), or, more recently, Brunnermeier (2008, 3).

\textsuperscript{6}Weil (1987) first added a stochastic probability of bursting bubbles to the Tirole (1985) model.
Table 1: Transition matrix $P$

<table>
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<tr>
<th>$z_{t+1}^b$</th>
<th>$z_{t+1}^b$</th>
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<tbody>
<tr>
<td>$z_t^b = 1$</td>
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<tr>
<td>$z_t^b = 1$</td>
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<tr>
<td>$z_t^b = 0$</td>
<td>$1 - p$</td>
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$\theta^E$ has to be multiplied by labor productivity $E_t$ in order for a balanced-growth path to exist. The second parameter $\theta^b > 0$ reflects the ability of productive individuals to create more new bubbles when the already existing bubble is larger.

The aggregate value of newly created bubbles is then

$$B_t^N \equiv \hat{b}_t^N \sum_{s=1}^T \eta_t^s = \frac{\eta_t}{1 + \phi} N_t \hat{b}_t^N,$$

where $\phi \equiv \frac{\sum_{s=T+1}^{T+R} N_t^s}{\sum_{s=1}^T N_t^s}$ is the constant old-age dependency ratio, $\frac{1}{1 + \phi}$ is the constant population share of all working individuals, and $\frac{n_t}{1 + \phi} N_t$ is the size of all productive working individuals.

The ex-post realized return to bubbles is given by

$$1 + q_t = \begin{cases} 1 + q_t^e = \frac{B_{t+1}}{B_t + B_t^N} & \text{if } z_t^b = 1, \\ 0 & \text{if } z_t^b = 0, \end{cases} \quad (4)$$

or equivalently as $(1 + q_t) = z_t^b (1 + q_t^e)$, where $q_t^e$ is the return to bubbles under the realization $z_t^b = 1$.

**Household optimization.** For $j \in \{P, U\}$ the households’ budget constraints read

$$c_t^j + a_{t+1}^j + b_{t+1}^j = n_t^j w_t^j + R_t^j a_t^j + Q_t^j b_t^j$$

$$s = 1, \ldots, T$$

$$c_t^j + a_{t+1}^j + b_{t+1}^j = R_t^j a_t^j + Q_t^j b_t^j$$

$$s = T + 1, \ldots, T + T^R$$

$$0 = a_t^{j,1} = b_t^{j,1} = a_t^{j,T+T^R+1} = b_t^{j,T+T^R+1}, \quad (5)$$
where \( Q_t \equiv 1 + q_t \) is the (ex-post) gross-return to bubbles, \( R_t^P \equiv 1 + r_t \) is the (ex-post) gross-return to investment faced by productive individuals, and \( R_t^U \equiv \sigma(1 + r_t) \) is the (ex-post) gross-return to investment faced by unproductive individuals. The income component \( w_t^j \) is defined as follows: \( w_t^U \equiv w_t \) and \( w_t^P \equiv w_t + \frac{Q_t}{n_t} \hat{b}_t \).

The households optimization problem is given by

\[
\max \left\{ \sum_{t=1}^{T+T_R} \mathbb{E}_t \left[ \beta^{s-1} u(c_{t+s}^{j,s}, n_{t+s}^{j,s}) \right] \right\} \text{ s.t. } b_{t+s}^{j,s+1} \geq 0, \quad (5)
\]

The first order conditions consist of the budget constraints (5), a complementary slackness condition

\[
\omega_{t+1}^{j,s+1} b_{t+1}^{j,s+1} = 0, \quad \omega_{t+1}^{j,s+1} \geq 0,
\]

an intratemporal optimality condition

\[
-u(c_t^{j,s}, n_t^{j,s}) = w_t u(c_t^{j,s}, n_t^{j,s}) \quad s = 1, \ldots, T \quad (6)
\]

and two intertemporal optimality conditions

\[
u_c(c_t^{j,s}, n_t^{j,s}) = \begin{cases} \beta \mathbb{E}_t \left[ R_{t+1}^{j} u_c(c_{t+1}^{j,s+1}, n_{t+1}^{j,s+1}) \right] & s = 1, \ldots, T + T_R - 1 \\ \beta \mathbb{E}_t \left[ Q_{t+1}^{j} u_c(c_{t+1}^{j,s+1}, n_{t+1}^{j,s+1}) \right] + \omega_{t+1}^{j,s+1} & s = 1, \ldots, T + T_R - 1. \end{cases} \quad (7)
\]

Note, \( n_t^{s,j} \equiv 0 \) for \( s > T \). The first equation is the Euler equations for capital \( a \) and the second equation is the Euler equations for bubbles \( b \). The Euler equations show that the expected marginal cost of consuming one unit less in the present – the left hand side – has to be equal to the expected marginal benefit from saving this additional unit and consuming in the subsequent period – the right hand side.

### 2.2 Equilibrium

**Definition.** A competitive equilibrium consists of a sequence of individual consumption, capital, and bubbles \( \left\{ \left\{ c_t^{P,s}, c_t^{U,s}, a_t^{P,s}, a_t^{U,s}, b_t^{P,s}, b_t^{U,s} \right\}_{s=1}^{T+T_R+1} \right\}_{t=0}^{\infty} \) satisfying the FOC’s of the household optimization problem as given by (6) and (7), a sequence of prices.
\{w_t, r_t, q_t^c\}_{t=0}^\infty \mbox{ satisfying (1), (2) and (4), a sequence of shocks } \{z_t^a, z_t^b\}_{t=1}^\infty \mbox{ drawn from their respective distributions and initial values } \{a_0^P, a_0^U, b_0^P, b_0^U\}_{s=1}^{T+TR-1}, z_0^a, z_0^b, q_0^c \mbox{ such that }

- the labor market clears

\[ L_t = \sum_{s=1}^T \left( \eta n_t^{P,s} + (1 - \eta) n_t^{U,s} \right) N_s^t, \]

- the capital market clears

\[ K_{t+1} = \sum_{s=1}^{T+TR} N_{t+1}^{s+1} \left[ \eta a_{t+1}^{P,s+1} + \sigma (1 - \eta) a_{t+1}^{U,s+1} \right], \]

- the market for bubbles clears

\[ (1 + q_t)(B_t + B_t^N) = \sum_{s=1}^{T+TR} N_{t+1}^{s+1} \left[ \eta b_{t+1}^{P,s+1} + (1 - \eta) b_{t+1}^{U,s+1} \right], \]

- the goods market clears\(^7\)

\[ Y_t = C_t + \Delta K_{t+1} + \delta K_t + \sum_{s=1}^{T+TR} (1 - \eta)(1 - \sigma) N_s^t a_{t+1}^{U,s+1}, \]

\[ \text{output-loss due to the financial friction} \]

- capital does not become negative (or, equivalently, the bubble does not become too large)\(^8\)

\[ \sum_{s=1}^{T+TR} \eta N_s^t a_{t+1}^{P,s+1} \geq 0 \quad \text{and} \quad \sum_{s=1}^{T+TR} (1 - \eta) N_s^t a_{t+1}^{U,s+1} \geq 0 \]

- and bubbles are freely disposable

\[ B_t \geq 0. \]

\(^7\)Although it is not necessary to state this equation – it is implied by the budget constraints – it is useful as a consistency check in the numerical solution. Note further that aggregate consumption is defined as

\[ C_t \equiv \sum_{s=1}^{T+TR} N_t^s \left( \eta c_t^{P,s} + (1 - \eta) c_t^{U,s} \right). \]

\(^8\)Due to the financial friction the total capital of unproductive or productive individuals cannot be negative. If, for example, total capital of unproductive individuals would be negative that would imply that unproductive individuals borrow resources from productive individuals at the return \(\sigma(1 + r_t)\) contradicting the imposed financial friction.
Labor supply. Due to the GHH-preferences the *intratemporal* optimality conditions yields the following analytical solution for the individual supply of labor

\[ n_{t}^{j,s} = \left( \frac{w_{t}}{\theta_{t}} \right)^{\frac{1}{\chi}} \equiv n_t. \]

Labor supply of a working-age individual is independent of age and the productivity-type. Along the balanced growth path \( n_{t}^{j,s} \) shall not grow, because it is bounded between zero and one. However, since the wage \( w_t \) grows at a strictly positive rate along the balanced growth path \( \theta_t \) has to grow at the same rate as the wage rate, as has been assumed above.

Aggregating labor supply yields

\[ L_t = \sum_{s=1}^{T} n_{t}^{s} = n_t \sum_{s=1}^{T} N_t^{s} = n_t \frac{1}{1 + \phi} N_t = \left( \frac{w_{t}}{\theta_{t}} \right)^{\frac{1}{\chi}} \frac{1}{1 + \phi} N_t \]

Inserting the competitive wage given by (1) and rearranging yields \( L_t \) as

\[ L_t = \left( \frac{1 - \alpha}{\theta} A_t \left[ \frac{K_t}{E_t} \right]^{\alpha} \right)^{\frac{1}{\alpha + \chi}} \left( \frac{N_t}{1 + \phi} \right)^{\frac{1}{\alpha + \chi}}. \]

Euler equations. The Euler equations can be simplified to

\[ \begin{cases} 
\beta E_t \left[ R_{t+1}^{j,s+1} - \theta_{t+1} \left( \frac{n_{t+1}^{j,s+1}}{1 + \chi} \right)^{1+\chi} \right]^{\epsilon} \\
\beta E_t \left[ Q_{t+1}^{j,s+1} - \theta_{t+1} - \left( \frac{n_{t+1}^{j,s+1}}{1 + \chi} \right)^{1+\chi} \right]^{\epsilon} + \omega_{t+1}^{j,s+1} \quad s = 1, \ldots, T - 1
\end{cases} \]

Making use of the optimal labor supply yields marginal utility of consumption as

\[ u_c(c_{t}^{j,s}, n_t^{j,s}) = \left[ c_{t}^{j,s} - \theta_t \left( \frac{w_{t}}{1 + \chi} \right) \right]^{-\epsilon} \]

\[ \left[ c_{t}^{j,s} - \theta_t \left( \frac{w_{t}}{1 + \chi} \right) \right]^{\frac{1+\chi}{\chi}} \]
and the complete set of Euler equations then reads

\[
\left( c_{t}^{j,s} - \theta_{t} \frac{1}{1 + \chi} \frac{w_{t+1}^{1+\chi}}{1+\chi} \right)^{-\epsilon} = \begin{cases} 
\beta E_t \left[ R_{t+1}^j \left( c_{t+1}^{j,s+1} - \theta_{t+1} \frac{1}{1+\chi} \frac{w_{t+1}^{1+\chi}}{1+\chi} \right) \right] & s = 1, \ldots, T-1 \cr
\beta E_t \left[ Q_{t+1}^j \left( c_{t+1}^{j,s+1} - \theta_{t+1} \frac{1}{1+\chi} \frac{w_{t+1}^{1+\chi}}{1+\chi} \right) \right] + \omega_{t+1}^j s = 1, \ldots, T-1 \cr
\beta E_t \left[ R_{t+1}^j \left( \frac{c_{t+1}^{j,s+1}}{1+\chi} \right)^{-\epsilon} \right] & s = T \cr
\beta E_t \left[ Q_{t+1}^j \left( \frac{c_{t+1}^{j,s+1}}{1+\chi} \right)^{-\epsilon} \right] + \omega_{t+1}^j s = T \cr
\end{cases}
\]

\[
\left( c_{t}^{j,s} \right)^{-\epsilon} = \begin{cases} 
\beta E_t \left[ R_{t+1}^j \left( \frac{c_{t+1}^{j,s+1}}{1+\chi} \right)^{-\epsilon} \right] & s = T + 1, \ldots, T + T^R - 1 \cr
\beta E_t \left[ Q_{t+1}^j \left( \frac{c_{t+1}^{j,s+1}}{1+\chi} \right)^{-\epsilon} \right] + \omega_{t+1}^j s = T + 1, \ldots, T + T^R - 1 \cr
\end{cases}
\]

### 3 Numerical Solution

The model economy grows asymptotically at a rate \( g + n + gn \). Some variables have therefore to be normalized in order to yield a stationary system. The notation of normalized variables is as follows: A normalized aggregate variable \( X_t \) is defined by \( x_t \equiv \frac{X_t}{E_t N_t} \). A normalized individual-level variable \( x_{t}^{j,s} \) is defined by \( \tilde{x}_{t}^{j,s} \equiv \frac{x_{t}^{j,s}}{E_t} \). The normalized system of equations is derived in section 4.0.1 of the appendix.

#### 3.1 Calibration

The model is calibrated with respect to the postwar US economy at a quarterly frequency. Whenever possible I use values that are standard in the literature. The calibration of the model is summarized in table 2.

**Demographics.** The values for \( T \) and \( T^R \) are taken from Heer and Maußner (2012), who calibrate a life-cycle RBC model to quarters and set \( T = 160 \) and \( T^R = 80 \), corresponding to a working life of 40 years and retirement of 20 years. In 1950 the US were inhabited by 159 million individuals and in 2013 by 230 million (UN, 2013, 61). The resulting annual population growth rate is 0.58 percent. The corresponding quarterly population growth rate \( n \) is 0.15 percent.
Preferences. The subjective discount factor is set below, but close to unity as common in the literature, i.e. $\beta = 0.9975$. Microeconometric estimates suggest values of the Frisch elasticity of labor supply between 0 and 0.5 while macroeconometric estimates suggest values between 2 and 4, as highlighted by Peterman (2016). A low Frisch elasticity of labor supply at the micro-level and a high elasticity at the macro-level can be reconciled by allowing for fluctuations in hours at the extensive margin. In the present model I abstract from fluctuations of labor at the extensive margin such that the micro Frisch elasticity of labor supply is equal to the macro Frisch elasticity of labor supply. In order to obtain plausible fluctuations in hours I therefore match the larger macro Frisch elasticity of labor supply. By employing a more consistent microeconometric estimation strategy Peterman (2016) suggests a value of 3 for the macro-Frisch elasticity of labor supply. I set $\chi$ therefore equal to $1/3$. The parameter $\epsilon$ determines the intertemporal elasticity of substitution, but its inverse is not equal to the IES of all individuals. To be precise, the IES of retired individuals is constant and equal to $1/\epsilon$. According to a recent empirical meta-analysis by Havránek (2015) it should be far below unity, plausibly between 0.2 and 0.4. I therefore calibrate the IES (of retirees) to $1/3$, i.e. $\epsilon = 3$. Lastly, $\theta$ is calibrated such that the resulting share of time spent working, $n^*$, is equal to one third (Cooley and Prescott, 1995), yielding $\theta = 1.7$.

Production. The capital income share and the capital depreciation rate are set in accordance with standard values in the literature – see, for instance, King and Rebelo (1999) – implying $\alpha = 0.3$ and $\delta = 0.026$, where the latter corresponds to an annual depreciation rate of 10%. The growth rate $g$ is determined by the population growth rate $n$ and the growth rate of aggregate output. In the period 1960-2014 real US-GDP grew on average by 3.07 percent per annum based on data from US Bureau of Economic Analysis (BEA). The implied quarterly growth rate is 0.758 percent. Aggregate output in the model economy grows at the rate $n + g + ng$. The growth rate of $E_t$ is then given by $g = (0.00758 - n)/(1 + n) \approx 0.006$. The parameters for the AR(1) process for technology are set equal to $\rho = 0.979$ and $\sigma^a = 0.0072$ as given in King and Rebelo (1999).

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9See, for instance, Heer and Maussner (2009).
### Table 2: Set of parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Note/Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>Quarters working</td>
<td>160</td>
<td>Heer and Maußner (2012)</td>
</tr>
<tr>
<td>( T^R )</td>
<td>Quarters in retirement</td>
<td>80</td>
<td>Heer and Maußner (2012)</td>
</tr>
<tr>
<td>( n )</td>
<td>Population growth rate</td>
<td>0.0015</td>
<td>0.58% per year (UN, 2013)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Time preference</td>
<td>0.9975</td>
<td>standard value</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>inverse of IES (retirees)</td>
<td>3</td>
<td>IES=1/3 (Havránek, 2015)</td>
</tr>
<tr>
<td>( \chi )</td>
<td>Inverse of Frisch elast.</td>
<td>1/3</td>
<td>Frisch elasticity 3 (Peterman, 2016)</td>
</tr>
<tr>
<td>( \theta )</td>
<td>weight of labor in ( u )</td>
<td>1.7</td>
<td>ratio of time working=1/3 (Cooley and Prescott, 1995)</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Capital share parameter</td>
<td>0.3</td>
<td>standard value</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Depreciation rate</td>
<td>0.026</td>
<td>10% per year (King and Rebelo, 1999)</td>
</tr>
<tr>
<td>( \varrho )</td>
<td>Growth rate of labor prod.</td>
<td>0.006</td>
<td>real GDP growth of 3.07% per year (BEA)</td>
</tr>
<tr>
<td>( \rho )</td>
<td>TFP shock autocorrelation</td>
<td>0.979</td>
<td>Cooley and Prescott (1995)</td>
</tr>
<tr>
<td>( \sigma^a )</td>
<td>TFP shock volatility</td>
<td>0.0072</td>
<td>Cooley and Prescott (1995)</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Share of entrepreneurs</td>
<td>0.05</td>
<td>“plausible” bubbles</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Financial friction parameter</td>
<td>0.6</td>
<td>“plausible” bubbles</td>
</tr>
<tr>
<td>( o )</td>
<td>Prob. of staying optimistic</td>
<td>40/41</td>
<td>financial cycle: 10 years upswing</td>
</tr>
<tr>
<td>( p )</td>
<td>Prob. of staying pessimistic</td>
<td>24/25</td>
<td>financial cycle: 6 years downswing</td>
</tr>
<tr>
<td>( \theta^c )</td>
<td>Bubble creation</td>
<td>0.1</td>
<td>“plausible” bubbles</td>
</tr>
<tr>
<td>( \theta^b )</td>
<td>Bubble creation</td>
<td>14</td>
<td>“plausible” bubbles</td>
</tr>
</tbody>
</table>

**Financial friction.** The parameters associated with the financial friction and the bubble are not common in the literature. Further, depending on the choice of these parameters i) bubbles either can or cannot exist, and ii) bubbles can be procyclical or countercyclical. Empirically plausible bubbles should be procyclical. In the calibration of these parameters I therefore chose parameter constellations that yield solutions where plausible – i.e. procyclical – bubbles do exist. I assume that 5 percent of the population is much more efficient in investing than the rest of the population. Further, \( \sigma \) is chosen such that the wedge between the interest rate of productive and unproductive individuals is sufficiently large for expansionary bubbles to exist. Therefore, \( \eta = 0.05 \) and \( \sigma = 0.6 \).

**Bubbles.** A growing body of literature investigates the characteristics of financial cycles. The studies differ, amongst others, with respect to the definition of financial cycles, the method of measurement, the relevant variables for measuring the financial cycle, as well as the length of the financial cycle, economic upturns and downturns. According to Drehmann *et al.* (2012) the average length of the financial cycle for the post-world war II period in industrialized countries is 16 years. Galati *et al.* (2016) and Rünstler and Vlekke (2015) derive very similar values (14.4 years and 12 to 16.5 years). Further, Drehmann

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et al. (2012) provide a measure for the average length of aggregate asset prices’ up- and downswings. According to this measure the share of time spent in an upswing (downswing) within one financial cycle is approximately 60 (40) percent. Hence, the upswing’s (downswing’s) duration is on average 10 (6) years or, equivalently, 40 (24) quarters. In the model the average duration of an upswing is given by the Markov process governing the behavior of $z^b_t$ and is equal to\footnote{The sequence corresponds to an infinite arithmetico-geometric sequence which converges since $0 < o < 1$.}

$$o(1-o) + 2o^2(1-o) + \ldots = \frac{o}{1-o}.$$  

Hence, $o$ is given by $o = \frac{40}{40+1}$, and, similarly, $p = \frac{24}{24+1}$. The parameters governing the creation of new bubbles, $\theta^E$ and $\theta^b$, are set such that the aggregate bubble-to-output ratio is equal to 1 percent in the deterministic steady state. Large values of $\theta^b$ and $\theta^E$ can result in bubbles becoming too large and violating the equilibrium conditions. I therefore set $\theta^E$ equal to 0.1 and $\theta^b$ to 14.

### 3.2 Deterministic steady state

The deterministic steady state is a hypothetical trajectory where the stochastic exogenous state-variables $z^a_t$ and $z^b_t$ are equal to their expected values and all other variables are growing at a constant rate, i.e the variables of the normalized system in section 4.0.1 are constant. The computation of the deterministic steady state is necessary for the numerical solution of the dynamic system in the next section.

The limiting distribution of the Markov process of investors’ sentiment, $\pi$, is given by

$$\pi = \begin{bmatrix} \frac{1-p}{(1-o)+(1-p)} \\ \frac{1-o}{(1-o)+(1-p)} \end{bmatrix}.$$

The expected value of $z^b_t$ is then given by $\bar{z} \equiv \lim_{T \to \infty} \mathbb{E}_t[z^b_T] = \frac{1-p}{(1-o)+(1-p)}$. In the deterministic steady state $z^b_t = \bar{z}$, $z^a_t = 0$, and $A_t = 1$ for all $t$.

The stochastic Euler equations given by (12) are now deterministic, i.e. the expectations operator $\mathbb{E}_t$ disappears.
In what follows I assume $\theta^E > 0$, as in the calibration, such that new bubbles are created in the deterministic steady state. What is the relation between the certain return to bubbles $q$ and the return to capital $r$?

First, when $(1 + q) = R^j$ individuals are indifferent with respect to the allocation of their wealth between productive capital $a^{j,s}$ and bubbles $b^{j,s}$, otherwise individuals hold only productive capital or only bubbles. The equilibrium condition for bubbles (8) necessitates that $q$ is large enough such that some individuals in the economy choose to purchase bubbles, because new bubbles are created in each period. This rules out $1 + q < \sigma(1 + r)$, because otherwise the newly created bubbles would not be purchased and the equilibrium condition (8) would be violated.

Second, assume $(1 + q) = R^U = \sigma(1 + r)$. Unproductive individuals hold bubbles and their non-negativity restriction does not bind, i.e. $\omega^U,s = 0$. Productive individuals would not purchase bubbles because $1 + q < R^P = (1 + r)$ and their non-negativity restriction would bind, i.e. $\omega^P,s > 0$, and $\bar{b}^P,s = 0$. Unproductive individuals are indifferent with respect to the amount of bubbles purchased. Any distribution of the existing bubble across all unproductive individuals would satisfy the household optimality and market equilibrium conditions. I assume that each individual holds the same amount of bubbles, i.e. $b^{U,s} = \bar{b}^U$.

Third, only if $(1 + q) \geq (1 + r)$ all individuals in the economy would hold bubbles, i.e. $\omega^{j,s} = 0$. The discussion is summarized by

$$1 + q = \begin{cases} \sigma(1 + r) & \text{if } b < \sum_{s=1}^{T+TR} (1 - \eta)N_0^{s+1}\bar{s}^{U,s} \\ \in [\sigma(1 + r), (1 + r)] & \text{if } b = \sum_{s=1}^{T+TR} (1 - \eta)N_0^{s+1}\bar{s}^{U,s} \\ (1 + r) & \text{if } b > \sum_{s=1}^{T+TR} (1 - \eta)N_0^{s+1}\bar{s}^{U,s}, \end{cases}$$

where total individual wealth $\bar{s}^{j,s}$ is defined as $\bar{s}^{j,s} \equiv \tilde{a}^{j,s} + \tilde{b}^{j,s}$ for $j \in \{U, P\}$.

In the simulations I consider only the case where $1 + q = \sigma(1 + r)$, because the model would otherwise yield empirically implausible bubbles where the capital stock declines when bubbles emerge and increases when bubble burst. Therefore it is always checked that the wealth of the unproductive individuals is smaller or equal than the existing bubble such that (9) is satisfied. The size of individual bubbles is then given by $\bar{b}^P,s = 0$.
Figure 2: Age profiles

and

\[ \tilde{b}^U = \frac{1}{(1 - \eta)(1 - N_0)} b. \]

The numerical procedure that has been employed to solve for the steady state is related to Auerbach and Kotlikoff (1987) and the “direct computation method” as described in Heer and Maussner (2009) – see section 4.1.1 for the details.

Figure 2 depicts the individual wealth and consumption profiles. The upper graph shows that the productive \( s^{P,s} \) and undproductive individuals’ wealth profiles \( s^{U,s} \) are mostly increasing up to retirement, but diverging afterwards. Young unproductive individual borrow in the very first quarters, i.e. \( s^{U,s} \) is negative, before accumulating wealth. After retirement productive individuals are still accumulating wealth before consuming it all while unproductive individuals’ wealth declines immediately. The dashed green line depicts the bubbles held by unproductive individuals \( b^{U,s} \). Unproductive individuals hold only a small amount of their total wealth in the form of bubbles because the aggregate bubble-to-output-ratio is calibrated to be one percent. The consumption profiles of productive and unproductive individuals – shown in the lower graph in fig. 2 – are also
very different. That difference is driven by the severeness of the financial friction, which implies that unproductive and productive individuals face different returns to savings, directly affecting relative consumption and savings through the Euler equation.

3.3 Dynamics

The model is solved with the deterministic extended path (DEP) method. This procedure was first applied to DSGE models by Gagnon (1990). As shown by Heer and Maußner (2008) the DEP method yields the highest accuracy in the computation of the standard business cycle model at the cost of longer computational time when compared with log-linear solution methods, second-order approximations, the parameterized expectations approach, Galerkin projections, and value function iterations. In contrast to local methods like the log-linear or higher-order approximations as well as linear-quadratic approximations the DEP method is a global method and therefore better suited for the present model, because the bubble-based shocks can shift the economy far away from its steady state.

The DEP method works as follows: For given endogenous and exogenous (stochastic) state variables in period $t$ the conditional expectations of the stochastic state variables for the entire time span of the simulation is calculated and plugged into the stochastic Euler equations. The result is a deterministic dynamic system of equations. This deterministic system is solved for a time-span large enough for the economy to be very close to its deterministic steady state. From the solution the endogenous state variables at $t + 1$ and the control variables at $t$ are stored. In $t + 1$ a new realization of the shocks is drawn and the whole procedure is repeated. Proceeding in this manner one obtains an approximation of a realized time path of the true stochastic model which – as shown by Heer and Maußner (2008) – can be very accurate. The solution algorithm and the deterministic counterpart of the true dynamic stochastic system of equations are described in section 4.1.2.

I will compare three different numerical solutions in the following: i) an economy without bubbles, i.e. $\theta^E$ and $\theta^b$ are set equal to zero, ii) an economy with bubbles but without TFP shocks, and iii) an economy with both types of shocks.

Figure 4 in the appendix depicts the impulse response functions for the model economy
without bubbles. For all $t = 2,\ldots,30$ $z_t^a = 0$, except for $t = 1$, where $z_1^a = \sigma^a$. When the positive TFP shock materializes in the first period, output, wages, interest rates, consumption, labor, and investment increase immediately, whereas capital reacts with a lag of one period. The pronounced increase in the capital stock overcompensates the decline in TFP, such that labor and output increase for some periods before converging back to their steady state level. The impulse response function for consumption is also hump-shaped reaching its maximum 25 quarters after the initial shock.

Figure 3 depicts a simulation for an economy with bubbles and without TFP shocks for 32 quarters. During the first 12 quarters a bubbles exists, leading to increasing output, capital, labor and consumption through the relaxation of the financial friction and the increase in investment efficiency. In quarter 13 investor become pessimistic and the bubble bursts. In the remaining 20 quarters no new bubble evolves. After the bursting of the bubble the capital stock, labor, output, and consumption start to decline and the economy
Table 3: Business cycles statistics of the US-economy

<table>
<thead>
<tr>
<th>Variable</th>
<th>Standard deviation</th>
<th>Relative standard deviation</th>
<th>First-order auto-correlation</th>
<th>Contemporaneous correlation with output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>1.81</td>
<td>1.00</td>
<td>0.84</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption</td>
<td>1.35</td>
<td>0.74</td>
<td>0.80</td>
<td>0.88</td>
</tr>
<tr>
<td>Investment</td>
<td>5.3</td>
<td>2.93</td>
<td>0.87</td>
<td>0.80</td>
</tr>
<tr>
<td>Hours</td>
<td>1.79</td>
<td>0.99</td>
<td>0.88</td>
<td>0.88</td>
</tr>
<tr>
<td>Wage</td>
<td>0.68</td>
<td>0.38</td>
<td>0.66</td>
<td>0.12</td>
</tr>
<tr>
<td>Interest rate</td>
<td>0.30</td>
<td>0.16</td>
<td>0.60</td>
<td>-0.35</td>
</tr>
</tbody>
</table>

Table 4: Business cycles statistics of the model

<table>
<thead>
<tr>
<th>Variable</th>
<th>Standard deviation</th>
<th>Relative standard deviation</th>
<th>First-order auto-correlation</th>
<th>Contemporaneous correlation with output</th>
</tr>
</thead>
<tbody>
<tr>
<td>TFP-shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output</td>
<td>2.17</td>
<td>1.00</td>
<td>0.78</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption</td>
<td>1.85</td>
<td>0.85</td>
<td>0.78</td>
<td>1.00</td>
</tr>
<tr>
<td>Investment</td>
<td>7.32</td>
<td>3.38</td>
<td>0.66</td>
<td>0.92</td>
</tr>
<tr>
<td>Hours</td>
<td>1.62</td>
<td>0.75</td>
<td>0.78</td>
<td>1.00</td>
</tr>
<tr>
<td>Wage</td>
<td>0.54</td>
<td>0.25</td>
<td>0.78</td>
<td>1.00</td>
</tr>
<tr>
<td>Interest rate</td>
<td>0.22</td>
<td>0.10</td>
<td>0.74</td>
<td>0.95</td>
</tr>
<tr>
<td>Bubble-shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output</td>
<td>0.21</td>
<td>1.00</td>
<td>0.96</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.25</td>
<td>1.17</td>
<td>0.92</td>
<td>0.82</td>
</tr>
<tr>
<td>Investment</td>
<td>3.27</td>
<td>15.32</td>
<td>0.76</td>
<td>-0.02</td>
</tr>
<tr>
<td>Hours</td>
<td>0.16</td>
<td>0.75</td>
<td>0.96</td>
<td>1.00</td>
</tr>
<tr>
<td>Wage</td>
<td>0.05</td>
<td>0.25</td>
<td>0.96</td>
<td>1.00</td>
</tr>
<tr>
<td>Interest rate</td>
<td>0.01</td>
<td>0.06</td>
<td>0.96</td>
<td>-1.00</td>
</tr>
<tr>
<td>Both shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output</td>
<td>1.93</td>
<td>1.00</td>
<td>0.72</td>
<td>1.00</td>
</tr>
<tr>
<td>Consumption</td>
<td>1.64</td>
<td>0.85</td>
<td>0.72</td>
<td>1.00</td>
</tr>
<tr>
<td>Investment</td>
<td>7.13</td>
<td>3.70</td>
<td>0.56</td>
<td>0.85</td>
</tr>
<tr>
<td>Hours</td>
<td>1.44</td>
<td>0.75</td>
<td>0.72</td>
<td>1.00</td>
</tr>
<tr>
<td>Wage</td>
<td>0.48</td>
<td>0.25</td>
<td>0.72</td>
<td>1.00</td>
</tr>
<tr>
<td>Interest rate</td>
<td>0.20</td>
<td>0.11</td>
<td>0.68</td>
<td>0.94</td>
</tr>
</tbody>
</table>

experiences a recession. Consumption experiences a very sudden drop which is due to the destruction of some part of the households’ wealth. In total, fig. 3 depicts one bubble-driven business cycle.

How does the model fare with respect to second moments? To answer this question second moments of empirical time series of aggregate output, consumption, investment, labor, wages, and interest rates for the US are reported in table 3 and second moments of the time series generated by the model are reported in table 4. All variables except the interest rate are expressed in natural logarithms and have been HP-filtered with an smoothing parameter of 1600. The time series generated by the model are obtained from

\[12\] The empirical moments are taken from King and Rebelo (1999).
simulating the model for 300 periods and dropping the first 50 observations in order to make sure that initial conditions do not effect the calculated second moments.

The results for the benchmark life-cycle RBC model without bubbles are depicted in the first five rows of table 4. Output, consumption and investment are too volatile and hours, wage and the interest rate are not sufficiently volatile in comparison to the empirical moments. Relative volatilities are matched reasonably well, while the first-order autocorrelations are slightly too small for output, consumption, investment and hours, and slightly too high for the wage and the interest rate. The correlation of the wage and interest rate with output is considerably too large, and the interest rate’s contemporaneous correlation with output deviates also qualitatively from its empirical counterpart. The positive correlation of the interest rate with output is a well-known, implausible implication of the standard real business cycle model (King and Rebelo, 1999).

Considering the model economy with shocks from evolving and bursting bubbles and without TFP-shocks in table 4 reveals rather implausible business cycle statistics. Due to the rather small bubble (1 percent of output in the steady state) bubble-driven business cycles imply rather small deviations of all variables from their respective trends. The main mechanism through which bubbles affect the other variables is investment, which is much more volatile than all other variables. As a result, the interest rate is almost perfectly negatively correlated with output.

Adding asset price bubbles to the model economy with TFP-shocks, however, results in second moments that match their empirical counterparts partly better. The model with both TFP shocks and bubbles yields performs better with regards to the volatility of output, the autocorrelations of the wage and the interest rate, and the contemporaneous correlation of investment and output. That comes, however, at the cost of too much relative volatility in investment.

4 Conclusion

Given their empirical relevance for business cycles it is important to integrate asset price bubbles into DSGE models. This paper presents an important step towards this goal. The simulated time paths show that the model is able to depict plausible bubble-
driven business cycles, i.e. The computation of a life-cycle RBC model featuring rational asset price bubbles shows that under an empirically reasonable parameter constellation asset price bubbles lead to strong fluctuations in output, consumption, capital stock and investment. i) evolving asset price bubbles together with increasing capital, investment, consumption and output and ii) bursting asset price bubbles leading to decreasing capital, investment, consumption, and output.

The model abstracts from the government sector including taxes, pensions and monetary policy reacting to evolving and bursting bubbles. Adding a social security system with pay-as-you-go pensions to the model would result in lower aggregate savings and a lower capital stock, increasing the interest rate and possibly making the existence of rational bubble impossible. However, only a part of the US-pension system is based on governmental pay-as-you-go transfers, while the major part of pension is either occupational or private. Further, since World War II investment in the US has been heavily driven by capital imports as the US run current account deficits most of the time. The model economy is a closed economy abstracting from international capital flows. Considering international capital flows would increase the capital stock and make bubbles more likely. Including further relevant macroeconomic aspects into the model to see how bubbles can exist and conducting a welfare analyses remains therefore of future interest.
References


4.0.1 Normalization

**Firms.** Normalized output, wages, and interest rates are given by

\[ y_t = A_t k_t^{\alpha} l_t^{1-\alpha} \]

\[ \bar{w}_t \equiv \frac{w_t}{E_t} = (1 - \alpha) \frac{y_t}{l_t} \]

\[ r_t = \alpha \frac{y_t}{k_t} - \delta. \]  

(10)

Note, \( l_t \equiv \frac{L_t}{N_t} \).

**Bubbles.** The equations for bubbles in normalized variables change as follows

\[ b_t = \sum_{s=1}^{T+R} N_0^s \left[ \eta \bar{b}_t^{P,s} + (1 - \eta) \bar{b}_t^{U,s} \right] \]

\[ \bar{b}_t^N \equiv \frac{\hat{b}_t^N}{E_t} = \theta^E + \theta^b b_t \]

\[ \bar{b}_t^{N} = \frac{\eta}{1 + \phi} \bar{b}_t^N \]

\[ Q_t = 1 + q_t = z_t (1 + q_t^{\alpha}). \]

Note, \( \frac{N_t^s}{N_t} = \frac{N_t^s}{N_0} = N_0^s \) is the time-constant population share of cohort \( s \).
Household optimum. The budget constraints now read

\[ \tilde{c}_t^{j,s} + (1 + g) \left( \tilde{a}_{t+1}^{j,s+1} + \tilde{b}_{t+1}^{j,s+1} \right) = n_t^{j,s} \tilde{w}_t^{j} + R_t^{j} \tilde{a}_t^{j,s} + Q_t^{j} \tilde{b}_t^{j,s} \quad s = 1, ..., T \]

\[ \tilde{c}_t^{j,s} + (1 + g) \left( \tilde{a}_{t+1}^{j,s+1} + \tilde{b}_{t+1}^{j,s+1} \right) = R_t^{j} \tilde{a}_t^{j,s} + Q_t^{j} \tilde{b}_t^{j,s} \quad s = T + 1, ..., T + T^R \]

\[ 0 = \tilde{a}_t^{j,1} = \tilde{b}_t^{j,1} = \tilde{a}_t^{j,T+T^R+1} = \tilde{b}_t^{j,T+T^R+1}, \quad (11) \]

with \( \tilde{w}_t^P \equiv \tilde{w}_t + \frac{Q_t^{j} \tilde{b}_t^{j,N}}{n_t} \) and \( \tilde{w}_t^{U} \equiv \tilde{w}_t \).

The individual labor supply is given by

\[ n_t = \left( \frac{\tilde{w}_t}{\theta} \right)^{\frac{1}{\chi}} \]

and the set of Euler equations read

\[ (1+g)^{\epsilon} \left( \tilde{c}_t^{j,s} - \theta \frac{1}{\chi} \tilde{w}_t^{j,s} \frac{1+\chi}{1+\chi} \right)^{-\epsilon} = \begin{cases} 
\beta \mathbb{E}_t \left[ R_t^{j} \left( \tilde{c}_{t+1}^{j,s+1} - \theta \frac{1}{\chi} \tilde{w}_{t+1}^{j,s+1} \frac{1+\chi}{1+\chi} \right)^{-\epsilon} \right] & s = 1, ..., T - 1 \\
\beta \mathbb{E}_t \left[ Q_t^{j} \left( \tilde{c}_{t+1}^{j,s+1} - \theta \frac{1}{\chi} \tilde{w}_{t+1}^{j,s+1} \frac{1+\chi}{1+\chi} \right)^{-\epsilon} \right] & s = T \\
\beta \mathbb{E}_t \left[ R_t^{j} (\tilde{c}_{t+1}^{j,s+1})^{-\epsilon} \right] + \tilde{\omega}_{t+1}^{j,s+1} & s = T + 1, ..., T + T^R - 1 \\
\beta \mathbb{E}_t \left[ Q_t^{j} (\tilde{c}_{t+1}^{j,s+1})^{-\epsilon} \right] + \tilde{\omega}_{t+1}^{j,s+1} & s = T + 1, ..., T + T^R - 1 
\end{cases} \quad (12) \]

with \( \tilde{\omega}_{t+1}^{j,s+1} \equiv E_{t+1}^{j} \omega_{t+1}^{j,s+1} \). Further, the complementary slackness condition reads

\[ \tilde{\omega}_{t+1}^{j,s+1} \tilde{b}_{t+1}^{j,s+1} = 0. \]

Equilibrium. The equilibrium conditions in normalized variables are given by

\[ l_t = \left[ \frac{1 - \alpha}{\theta} A_t k_t^{\alpha} \right]^{\frac{1}{\alpha + \chi}} (1 + \phi)^{\frac{\chi}{\alpha + \chi}} \]

\[ k_{t+1} = \sum_{s=1}^{T+T^R} N_{t+1}^{s+1} \left[ \eta \tilde{a}_{t+1}^{P,s+1} + \sigma (1 - \eta) \tilde{a}_{t+1}^{U,s+1} \right] \]
\[(1 + q_t)(b_t + b_t^N) = (1 + g)(1 + n) \sum_{s=1}^{T+TR} N_{s+1}^0 \left[ \eta \tilde{b}_{t+1}^{P,s+1} + (1 - \eta) \tilde{b}_{t+1}^{U,s+1} \right] \]

\[y_t + (1 - \delta)k_t = c_t + (1 + n)(1 + g)k_{t+1} + (1 + g)(1 - \eta)(1 - \sigma) \sum_{s=1}^{T+TR} N_{s+1}^0 a_{t+1}^{s\sim U,s+1} \]

\[\sum_{s=1}^{T+TR} \eta N_{s+1}^{s\sim P,s+1} \geq 0 \quad \text{and} \quad \sum_{s=1}^{T+TR} (1 - \eta) N_{s+1}^{s\sim U,s+1} \geq 0.\]
4.1 Computation methods

4.1.1 Solution algorithm for the steady state

Define total wealth as \( \tilde{s}^{j,s} \equiv \tilde{a}^{j,s} + \tilde{b}^{j,s} \). When the individual return to capital and the return to bubbles are equal the portfolio shares of both assets are undetermined. I assume that the portfolio share of bubbles, \( x^{j,s+1} \equiv \frac{b^{j,s+1}}{s^{j,s+1}} \), is equal across individuals of a group \( j \).

The budget constraints can be simplified as

\[
\tilde{c}^{j,s} + (1 + g)\tilde{s}^{j,s+1} = \tilde{n}\tilde{w}^{j} + R^{j}\tilde{s}^{P,s} \quad s = 1, ..., T
\]
\[
\tilde{c}^{j,s} + (1 + g)\tilde{s}^{j,s+1} = R^{j}\tilde{s}^{P,s} \quad s = T + 1, ..., T + T^{R}
\]
\[
0 = \tilde{s}^{j,1} = \tilde{s}^{j,T+T^{R}+1}
\]

where \( \tilde{n} \) is the individual steady state labor supply and \( \tilde{w}^{U} = \tilde{w} \) and \( \tilde{w}^{P} = \tilde{w} + \frac{Q}{\tilde{n}}b^{N} \).

Savers and entrepreneurs are either indifferent between capital and bubbles, hold only bubbles, or hold only capital. In all cases each group’s two Euler equations can be summarized by one Euler equation since returns are equal when bubbles exist. The computation makes use of the recursive nature of the Euler equation by forward-iteration.

The reformulated Euler equation reads

\[
\tilde{c}^{j,s+1} = \begin{cases} 
\frac{(\beta R^{j})^{\frac{1}{2}}}{1 + g} \tilde{c}^{j,s} - (\beta R^{j})^{\frac{1}{2}} \frac{\theta^{1}}{(1 + g)(1 + \chi)} \tilde{w}^{1+x} + \frac{\theta^{1}}{1 + \chi} \tilde{w}^{1+x} & s = 1, ..., T - 1 \\
\frac{(\beta R^{j})^{\frac{1}{2}}}{1 + g} \tilde{c}^{j,s} - (\beta R^{j})^{\frac{1}{2}} \frac{\theta^{1}}{(1 + g)(1 + \chi)} \tilde{w}^{1+x} & s = T \\
(\beta R^{j})^{\frac{1}{2}} \tilde{c}^{j,s} + \frac{\theta^{1}}{(1 + g)(1 + \chi)} \tilde{w}^{1+x} & s = T + 1, ..., T + T^{R} - 1
\end{cases}
\]

The solution algorithm works as follows\(^{13}\)

1. Define all parameter values, set \( A \) equal to one, and compute \( \tilde{z} \) as well as \( N^{0} \).

2. Provide an initial guess for \( k \).

\(^{13}\) I consider only the solution where the bubble is so small that productive individuals do not find it profitable to purchase it.
3. Given \( k \) obtain the values for (obey the order) \( l, \bar{w}, \bar{n}, r, R^U, R^P, q, q^1, Q \) and

\[
b = \frac{\eta \theta^E Q}{(1 + g)(1 + n)(1 + \phi) - (1 + \phi + \eta \theta^b)Q},
\]

\[
b^N = \frac{\eta}{1 + \phi} (\theta^E + \vartheta^b).
\]

4. Solve the individual problem, i.e. for each \( j \in \{P, U\} \):

a) Guess \( \tilde{c}_j^1 \).

b) Obtain the sequence of \( \{\tilde{c}_j^s\}^{T+T_R}_{s=1} \) by iterating over the Euler equation (13)

c) Given \( \{\tilde{c}_j^s\}^{T+T_R}_{s=1} \) obtain all \( \{\tilde{s}_j^{s+1}\}^{T+T_R}_{s=1} \) from the budget constraints, where \( s_j^1 \) is set equal to zero, but \( s_j^{1,T+T_R+1} \) is not predetermined.

d) If \( \tilde{s}_j^{1,T+T_R+1} \) is not close enough to zero, as defined by a chosen tolerance criterion, repeat the last three steps with an updated \( \tilde{c}_j^1 \) according to the Secant Method (Newton’s Method does also work, but it is slower):

\[
\tilde{c}_{i+2}^j = \tilde{c}_{i+1}^j - \frac{\tilde{c}_{i+1}^j - \tilde{c}_{i}^j}{\tilde{s}_{i+1}^{j,T+T_R+1} - \tilde{s}_{i}^{j,T+T_R+1}} \tilde{s}_{i+1}^{j,T+T_R+1}
\]

5. Compute the portfolio share of bubbles according to

\[
x^U = \frac{b}{\sum_{s=2}^{T+T_R} (1 - \eta)N_0^U \tilde{s}_{U,s}}.
\]

6. Compute the individual bubble of savers by \( \tilde{b}^{U,s} = x^U \tilde{s}_{U,s} \), the individual capital held by savers by \( \tilde{a}^{U,s} = (1 - x^U) \tilde{s}_{U,s} \), and individual capital held by entrepreneurs by \( \tilde{a}^{P,s} = \tilde{s}_{P,s} \).

7. Compute \( k' \) by making use of the capital market equilibrium condition

\[
k' = \frac{1}{1 + n} \sum_{s=1}^{T+T_R} N_0^s [\eta \tilde{a}^{P,s+1} + \sigma (1 - \eta) \tilde{a}^{U,s+1}]
\]

8. Compare \( k' \) with \( k \). If the difference is close to zero, as defined by a chosen tolerance criterion, stop. Otherwise update the initial guess of \( k \) and go back to step 3.
In the last step of the solution algorithm the updating of the initial guess for \( k \) is linear, i.e. a linear combination of \( k' \) and the previous initial guess \( k \).

### 4.1.2 Solution of the dynamic stochastic system

The conditional expectation of the bubble shocks is equal to the condition probability that \( z^b_{t+k} = 1 \) (the other possibility, \( z^b_{t+k} = 0 \), drops out due to the zero):

\[
\tilde{z}^b_{t+k} = \mathbb{E}_t[z^b_{t+k}] = (1 - z^b_t) P_{21}^{(k)} + z^b_t P_{11}^{(k)},
\]

where \( P_{ij}^{(k)} \) is the entry in the \( i \)-th row and \( j \)-th column of \( P^k \).

The conditional expectation of the TFP-shock is equal to zero, i.e. \( \tilde{z}^a_{t+k} \equiv 0 \) for all \( k > 0 \). The conditional expectation of \( A \) from period \( t \) on is given by

\[
\tilde{A}_{t+k} = A^0_t.
\]

The deterministic dynamic system is now obtained by replacing all \( z^b_{t+k} \) by \( \tilde{z}_{t+k} \) and all \( A_{t+k} \) by \( \tilde{A}_{t+k} \) for \( k > 0 \). In the firm sector \( A_{t+k} \) is replaced by \( \tilde{A}_{t+k} \) for all \( k > 1 \) in the equations for output, wages, and interest rates (10).

The households’ optimization problem is now a problem under certainty with the Euler equations:

\[
(1 + g)^\epsilon \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} - \theta \frac{1+\chi}{\chi} \tilde{w}_t^{\frac{1+\chi}{\chi}} \right)^{\epsilon} = \begin{cases} 
\beta \left[ R^j_{t+1} \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} - \theta \frac{1+\chi}{\chi} \tilde{w}_t^{\frac{1+\chi}{\chi}} \right)^{-\epsilon} \right] & s = 1, ..., T - 1 \\
\beta \left[ Q^j_{t+1} \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} - \theta \frac{1+\chi}{\chi} \tilde{w}_t^{\frac{1+\chi}{\chi}} \right)^{-\epsilon} \right] + \tilde{\omega}_t^{s+1} & s = 1, ..., T - 1 \\
\beta \left[ R^j_{t+1} \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} \right)^{-\epsilon} \right] & s = T \\
\beta \left[ Q^j_{t+1} \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} \right)^{-\epsilon} \right] + \tilde{\omega}_t^{s+1} & s = T 
\end{cases}
\]

\[
(1 + g)^\epsilon \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} \right)^{\epsilon} = \begin{cases} 
\beta \left[ R^j_{t+1} \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} \right)^{-\epsilon} \right] & s = T + 1, ..., T + T^R - 1 \\
\beta \left[ Q^j_{t+1} \left( \tilde{c}_t^{s, \frac{1+\chi}{\chi}} \right)^{-\epsilon} \right] + \tilde{\omega}_t^{s+1} & s = T + 1, ..., T + T^R - 1 
\end{cases}
\]
with \( \tilde{\omega}^{j,s+1}_{t+1} = E^{s}_{t+1} \omega^{j,s+1}_{t+1} \). Further, the complementary slackness condition reads

\[
\tilde{\omega}^{j,s+1}_{t+1} \tilde{b}^{j,s+1}_{t+1} = 0.
\]

The budget constraints given by (11) as well as the optimal labor supply remain unchanged.

The investment-efficiency-specific returns \( R^{P}_{t+1} = (1 + r_{t+1}) \) and \( R^{U}_{t+1} = \sigma (1 + r_{t+1}) \) are deterministic, as well as the return to bubbles as given by

\[
Q_{t+1} = \tilde{z}_{t+1} (1 + q^{b}_{t+1}^{o}).
\]

The same arguments as in the deterministic steady state apply with respect to the return to bubbles (see (9)). I consider only bubbles that potentially crowd-in capital such that the return to bubbles is given by

\[
\tilde{Q}_{t+k} = \tilde{z}_{t+k} (1 + q^{b}_{t+k}^{o}) = \sigma (1 + r_{t+k})
\]

for all \( k > 0 \). As a result, only savers find it profitable to purchase bubbles, i.e. \( \tilde{b}^{P,s}_{t} = 0 \), \( \tilde{\omega}^{U,s}_{t} = 0 \), and the Euler equations for each group become

\[
\begin{align*}
\tilde{\omega}^{P,s+1}_{c_{t+1}} &= \begin{cases}
\left[ \beta (1 + r_{t+1}) \right]^{\frac{1}{2}} (1 + g)^{-1} \left[ \frac{c^{P,s}_{t}}{c^{s}_{t}} - \theta \frac{1}{\chi} \frac{\tilde{w}^{1+\chi}_{t}}{1 + \chi} \right] + \theta \frac{1}{\chi} \frac{\tilde{w}^{1+\chi}_{t+1}}{1 + \chi} & s = 1, \ldots, T - 1 \\
\left[ \beta (1 + r_{t+1}) \right]^{\frac{1}{2}} (1 + g)^{-1} \left[ \frac{c^{P,s}_{t}}{c^{s}_{t}} - \theta \frac{1}{\chi} \frac{\tilde{w}^{1+\chi}_{t}}{1 + \chi} \right] & s = T \\
\left[ \beta (1 + r_{t+1}) \right]^{\frac{1}{2}} (1 + g)^{-1} \frac{c^{P,s}_{t}}{c^{s}_{t}} & s = T + 1, \ldots, T + T^{R} - 1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\tilde{\omega}^{U,s+1}_{c_{t+1}} &= \begin{cases}
\left[ \beta (1 + r_{t+1}) \right]^{\frac{1}{2}} (1 + g)^{-1} \left[ \frac{c^{U,s}_{t}}{c^{s}_{t}} - \theta \frac{1}{\chi} \frac{\tilde{w}^{1+\chi}_{t}}{1 + \chi} \right] + \theta \frac{1}{\chi} \frac{\tilde{w}^{1+\chi}_{t+1}}{1 + \chi} & s = 1, \ldots, T - 1 \\
\left[ \beta (1 + r_{t+1}) \right]^{\frac{1}{2}} (1 + g)^{-1} \left[ \frac{c^{U,s}_{t}}{c^{s}_{t}} - \theta \frac{1}{\chi} \frac{\tilde{w}^{1+\chi}_{t}}{1 + \chi} \right] & s = T \\
\left[ \beta (1 + r_{t+1}) \right]^{\frac{1}{2}} (1 + g)^{-1} \frac{c^{U,s}_{t}}{c^{s}_{t}} & s = T + 1, \ldots, T + T^{R} - 1
\end{cases}
\end{align*}
\]
Note further that savers are indifferent with respect to the portfolio shares of bubbles and capital. Under this indeterminacy I have to pick one distribution of bubbles: I (arbitrarily) assume that all savers have the same portfolio shares, i.e. \( x_t = \frac{\tilde{b}_{U,s}^t}{\tilde{b}_{s}^t} = \frac{\sum_{s=1}^{T+TR} (1-\eta)N_b^s \tilde{b}_{U,s}^t}{\sum_{s=1}^{T+TR} (1-\eta)N_b^s \tilde{b}_{s}^t} \).

**Solution Algorithm**

1. Initialization:
   - Define parameter values and initial values \( \{\tilde{a}_{P,s}^t, \tilde{a}_{U,s}^t\}_{s=1}^{T+TR}, b_1, x_1, q_1^0 \)
   - Choose \( nn \) numbers of periods for the simulation
   - Draw a random sequence of the shocks \( \{z_a^t, z_b^t\}_{t=1}^{nn} \)
   - Choose a number of transitional periods \( N \) large enough for the (deterministic) system to be close to its steady state
   - Calculate the deterministic steady state as explained in section 4.1.1

2. At each point in time \( t = 1, ..., nn \):
   - Calculate the conditional expectations \( \tilde{z}_{t+k}^a \) and \( \tilde{z}_{t+k}^b \) \( \forall k = 1, ..., N \) as described by (15) and (14)
   - Transform the whole dynamic stochastic equation system into a deterministic system by replacing \( (z_{a_{t+k}}^a, z_{b_{t+k}}^b) \) with their expected values \( (\tilde{z}_{a_{t+k}}^a, \tilde{z}_{b_{t+k}}^b) \) and solve this deterministic model for the \( N \) periods by applying the direct computation method, i.e. guess \( \{k_{t+k}\}_{k=1}^T \) (that gives also \( \tilde{w}_t, \tilde{l}_t, n_t, r_t, q_t, q_1^t, b_t, \tilde{b}_{U,t}, \) and \( \tilde{b}_{N,t}^N \)), solve for the individual problem, and update the initial guess \( \{k_{t+k}\}_{k=1}^T \) until convergence.\(^{14}\)
   - From the solution store \( \{\tilde{a}_{P,s}^t, \tilde{a}_{U,s}^t, \tilde{c}_{P,s}^t, \tilde{c}_{U,s}^t\}_{s=1}^{T+TR}, x_{t+1}, b_{t+1} \) and \( q_{t+1}^0 \).
   - Use \( \{\tilde{a}_{P,s}^t, \tilde{a}_{U,s}^t\}_{s=1}^{T+TR}, b_{t+1}, x_{t+1} \) and \( q_{t+1}^0 \) as initial values for period \( t + 1 \).

When calculating the transition one has to take care of the equilibrium condition that total wealth of unproductive individuals does not turn negative. In the numerical solution that condition is checked in each period. Let \( W_t^U \) denote the aggregate wealth of

\(^{14}\) The household problem is solved by guessing consumption in period \( t \), i.e. \( \tilde{c}_{j,s}^t \) for all \( j \) and all \( s \), computing all other \( c's \) by iterating over the deterministic Euler equations, computing then the sequences of \( s \), and updating the initial guess of \( \tilde{c}_{j,s}^t \) until each individual’s \( c_{j,s}^{T+TR+1} = 0 \) for \( k \geq t \).
unproductive individuals. If at one point $t$, for a given $\{k_{t+k}\}_{k=1}^T$ at some periods total unproductive individuals’ wealth, $W_t^U$, turns negative, say at periods $\mathcal{T} \subset \{t, t + 1, \ldots, T\}$, the following is done:

- Find the period $x \in \mathcal{T}$ such that $W_x^U = \min \{W_t^U \}_{t \in \mathcal{T}}$.

- Find the corresponding return that unproductive individuals face, $q_x$, such that $W_x^U = 0$. In this step the whole problem of unproductive individuals has to be solved again, with the previously given sequence of interest rates, except that for period $t = x$ the interest rate has to be obtained.

- At the end we get a new sequence of optimal unproductive wealth levels at each point in time.

The smart thin here is that it probably won’t happen that unproductive individuals’ wealth is exactly equal to zero in several periods, but it should be sufficient to change the return in one period, because that already lifts the neighboring periods $W_t^U$. That needs not be the case, but we will check the solution, and if it works out that is an solution to the individual problem and it satisfies the condition set out for an general equilibrium.
4.2 Figures

**Figure 4:** Impulse response functions for a TFP shock in a bubbleless economy

- Panel A: \( \%\text{-deviation from trend} \) for a TFP shock in a bubbleless economy.
- Panel B: \( \%\text{-deviation from trend} \) for a TFP shock in a bubbleless economy.
- Panel C: \( \%\text{-deviation from trend} \) for a TFP shock in a bubbleless economy.
- Panel D: \( \%\text{-deviation from trend} \) for a TFP shock in a bubbleless economy.

The figures show the impulse response functions for various economic variables, including output \( Y_t \), labor \( L_t \), capital \( K_t \), and real wages \( w_t \), after a TFP shock in a bubbleless economy.