Home Production with Time to Consume

Bill Bednar¹ and Nick Pretnar^{*1}

¹*Carnegie Mellon University, Tepper School of Business*

February 10, 2019[†]

Abstract

We construct a general equilibrium model with home production where consumers choose how much time to spend using market purchases. With aggregate United States consumption data and the American Time Use Survey, we observe that relative consumption of goods to services has fallen while the relative time households' spending using goods to services has risen. Our estimates reveal that using services has become relatively more productive than using goods in home production. This is because the outputs of the home production process are complementary, so price elasticities of relative goods/services consumption and relative goods/services time use have opposite sign. Our framework explains the observed negative co-movement of relative consumption to relative off-market time use since the early 2000s. In counterfactual analyses, we discuss the implications of our findings for the structural transformation of the U.S. economy from a goods dominated one to one where services play a larger role.

Keywords: home production, time use, structural change, services, goods

JEL Classification: D13, E21, O33

^{*}Corresponding author. Carnegie Mellon University – Tepper School of Business, 5000 Forbes Ave., Pittsburgh, PA 15218; npretnar@cmu.edu. Nick acknowledges support from the National Science Foundation Graduate Research Fellowship under Grant No. DGE1252522.

[†]We are grateful for comments from Finn Kydland, Peter Rupert, Javier Birchenall, Ali Shourideh, Laurence Ales, Alan Montgomery, and Steve Spear, as well as other seminar participants at the University of California - Santa Barbara and workshop participants at Carnegie Mellon University.

1 Introduction

All consumption takes time. One does not simultaneously purchase and consume a product. It takes time to procure the product and then to subsequently mold it into its final consumable form. While Aguiar and Hurst (2007) deal with shopping time and consumption trade offs, few in the literature have considered the trade offs resulting from the fact that consumers combine both market goods and time to produce a final consumable commodity.¹ We argue that time is always required in order to derive utility from market purchases. Resurrecting the ideas originally laid out in Gary Becker’s 1965 paper, “A Theory of the Allocation of Time,” we develop and estimate a novel, general equilibrium model with home production with multiple market goods each associated with different degrees of off-market time use. The model reconciles a new observation in aggregate data, namely that relative consumption of goods to services has fallen at the same time that relative in-home time devoted to using goods verses services has risen. This is a result of the novel way we model time use within the home: consumers choose how to allocate their off-market time toward using different market commodities. Further, we allow for the ability of consumers to engage in home production activities to evolve exogenously, along with firm total factor productivities. We show that, within the household, using market services has become relatively more productive over time than using market goods. Complementarities between the outputs of home production lead consumers to substitute market commodities for time use in home processes with relatively higher efficiencies, leaving them to spend more time on processes for which market inputs are less efficient at producing output.

We can thus explain why relative consumption of goods to services has fallen at the same time that relative off-market time use has risen, as seen in Figure 1. Using in-home time use data from the American Time Use Survey (ATUS), we can compute the relative amount of off-market time households spend using goods $n_{g,t}$ to the time they spend using services $n_{s,t}$. We then compare the ratio $n_{g,t}/n_{s,t}$ to the ratio of aggregate consumption (\$2014 = 1) of goods and services $q_{g,t}/q_{s,t}$, which features on the right hand vertical axis of Figure 1. Notice that the two series move in opposite directions. As individuals consume relatively more services than goods, they spend relatively less time using services. From 2003 to 2014, relative consumption of goods to services fell by 14.1% while relative time spent using goods verses services rose by 16.4%. Why might this be the case? We suspect that services are becoming more efficient to consume due to relative quality improvements, so demand for services relative to goods has increased, freeing up household time to use goods more. Consider for example the proliferation of mobile apps for smart phones which make everything from air travel to ordering takeout easier and more time efficient. We show in our model that this only happens if the final consumption commodities, each of which are separately produced by goods and services with intermediate home production processes, are themselves complements.

¹Goolsbee and Klenow (2006) provide a notable exception. The authors model time complementarities with broadband services consumption, showing that high wage earners should spend less time using the internet for leisure than low earners.

Figure 1: The ratio of time use devoted towards goods relative to services rose from 2003-2014 at the same time that the ratio of final consumption of goods relative to services fell (\$2014 = 1). Time use source: ATUS (Bureau of Labor Statistics 2017). Consumption quantity source: U.S. KLEMS Data (Jorgenson, Ho, and Samuels 2016). Data details are discussed further in Section 3.2.

In Section 2 we present other anecdotes that provide intuition for this fundamental empirical observation, then proceed to outline a general equilibrium model of home production where households combine market goods with time to produce ultimate, final consumption goods in the spirit of Becker (1965). The model is general enough to allow for both relative time use and relative consumption to move in the same or opposite directions. In Section 3 we describe the features of our estimation procedure for the general equilibrium model which we perform using Hamiltonian Monte Carlo (HMC) integration on the Bayesian posterior distribution of parameters given data. In this section we also detail features of the model that allow us to estimate structural elasticities and back out productivity time series for both firms and within-home activities. We conclude our discussion of the quantitative results with a counterfactual exercise placing our home production model within the literature examining the structural transformation of the United States economy from one dominated by goods manufacturing to one more heavily based on services production today. In Section 4 we conclude.

2 The Economy

Consider a closed economy where market prices are determined endogenously by general equilibrium market clearing conditions. The economy consists of firms and households. Firms make

market goods in some stylized production process which utilizes capital and labor leased from households. Households purchase market goods and use them along with their time spent away from work to produce final consumption goods in the home. Together, all households comprise a single sector of the economy. Within this sector, a household engages in off-market activities within the home, the values of which are not directly quantified by market prices. Ultimately, households consume and derive final utility from the non-traded output of these off-market activities. In Section 2.1 we provide anecdotal and theoretical motivation for this characterization of household behavior. In Section 2.2 we construct a stylized model of household decisions which captures fundamental aspects of the theory described in Section 2.1. In our set up, the activities of firms follow fairly standard conventions in aggregate economics, so we only briefly touch on them in Section 2.3 where we also outline the general equilibrium market clearing conditions which must hold. We conclude formal specification of the model by defining the equilibrium concept in Section 2.4.

2.1 Theory of the Household

For illustration, consider a consumer who purchases a real quantity of groceries q at price p which he intends to use to make dinner for his family. The household derives final utility in the sense of Becker (1965) from consuming this dinner prepared and cooked by our master homemaker using recently purchased market goods — the groceries — as inputs. Perhaps he also uses knives, mixing bowls, cutting boards, the stove or microwave. Denote these durable items owned by the household which are used to achieve the production of dinner as a . Finally, the master homemaker uses his time n both to shop for q and to prepare the dinner. If our homemaker is indeed a gastronomic maestro, the amount of time he uses will likely exceed that of the microwaving novice. Along both extremes, time is required nonetheless. Let c denote dinner, the final consumption commodity, produced using market goods q , durable possessions a , and time n according to some production process f , which is such that $c = f(q, a, n)$. This formulation differs from those of Gronau (1977), Graham and Green (1984), Benhabib, Rogerson, and Wright (1991), Rupert, Rogerson, and Wright (1995), Ingram, Kocherlakota, and Savin (1997), Aguiar and Hurst (2007), and Aguiar, Hurst, and Karabarbounis (2013) who all, for the most part, dispense with the idea that market goods *and* time are combined together in some home production function to produce a final commodity. Notable exceptions are McGrattan, Rogerson, and Wright (1993) and Gomme, Kydland, and Rupert (2001) which allow for durable capital to be combined with time toward the production of home consumption commodities, but additional non-durable market goods are still absent from their formulations. Clearly however, when thinking about how households use market goods, it is anecdotally evident that complementarities exist between the consumption of these purchased goods and non-work time. Most importantly, such complementarities exist because households derive final utility not simply from purchasing a product at a store, but rather as the outcome of what they do with that product after it is purchased. This fundamental observation lies at the core of Gary Becker’s original treatise — that time and market purchases are combined

to produce the final commodity from which the household derives its final utility.

Such observations are readily apparent when thinking about the inputs required in such a process as making a meal. Meanwhile, this model of household behavior challenges the convention in modern macroeconomics to classify as “leisure” all time spent outside of a formal, income-producing job. Perhaps the preparer of dinner does indeed enjoy preparing dinner, though he may also only enjoy the meal being produced, but the distinction is irrelevant as long as he enjoys c . Truly, the degree to which he enjoys the process as opposed to just the final product is encoded in the level of c itself via f . For the expert gastronome who enjoys the process of meal preparation, for each level of final consumption c , the returns to time input n are higher than returns for the microwaving novice. As a result, expert output is higher for each n than that of the novice. Thus, distinguishing between enjoying one’s time versus using one’s time to make something one enjoys is unnecessary. Consider for further illustration what one might classify as a standard “leisurely” activity, such as reading for pleasure. This requires a book or a magazine which themselves are market goods. If we define “reading for pleasure” as the final consumption commodity, this commodity is again produced using both some quantities of time n and market goods q via some abstract production process f . Taking the argument to an extreme, there is no “leisure” activity enjoyed by a consumer which *only* requires time. Even merely sitting in a chair and staring at a wall, if something as banal as this so pleases you, requires a chair and a wall. Indeed, any activity undertaken by an individual consumer can be thought to somehow involve combinations of time and other stuff which was at some point sold on some market somewhere.²

In fact consider someone who purchases a market good, like say a boat, and does not spend any time using it. Can we truly say that the boat is providing value to this consumer? He works overtime nights and weekends to pay for a boat that he never uses. It is well-established common knowledge that boats depreciate in value very quickly, so surely this consumer does not view the purchase as a capital investment. He must make time in his schedule to use the boat and derive utility from it. He thus faces an implicit tradeoff between working more to earn more money to pay for the boat and all the maintenance it requires versus spending time actually using the boat and deriving utility from it. If he knows he does not have the time to both supply labor to pay for the boat and enjoy the boat once he has it, he will not make the decision to purchase the boat.³

Finally, the implicit returns a household derives from using a product one period may be different than the returns derived from using the same product in another period. Perhaps the household’s time is more valuable due to wage increases, so the time they once spent cooking an intricate meal requiring one hour of preparation may be more valuably spent on another task and so the household is more willing to order take out. But wages do not just have to change to affect the implicit returns of home production. Holding wages and all other prices fixed, suppose our master homemaker takes a cooking class and learns some new techniques in the kitchen he is now itching to try out. The same meal he cooked using the same ingredients in a previous

²If a prudent reader can indeed think of an activity which requires time *and only* time, we would welcome the counterexample.

³Or at least he shouldn’t.

period now tastes better due to the culinary knowledge he acquired. The productivity of his meal preparation has thus improved because he is now able to produce a higher quality product within the home in the same amount of time, purchasing the same ingredients at the same market price. Modeling explicit complementarities between time use and market goods purchases affirms three important anecdotal facts about consumer behavior: 1) a consumer cannot enjoy a market good and derive final utility from its consumption if he cannot spend time using it; 2) there appear to exist no market goods for which the former is not true; 3) the return consumers derive from using market goods in home production processes depends both on market prices and the exogenous, time-varying ability of the consumer to engage in transformation of market goods to final goods.

2.2 General Model of the Household

Time is discrete and indexed by t . There is a single household in the economy. Each period the household derives utility from the consumption of I final goods $c_{i,t}$ each produced under the home production process $f_{i,t}$, which takes as inputs a J_i dimensional vector of market goods $\mathbf{q}_{i,t}$ and time $n_{i,t}$.⁴ Final consumption is such that

$$c_{i,t} = f_{i,t}(\mathbf{q}_{i,t}, n_{i,t}) \quad \forall i \quad (1)$$

We dispense with modeling fixed durable assets as inputs. Instead, we will assume that all capital is used by firms to produce market goods. In our quantitative exercises we will construct the data series so that the value of service flows from durable goods held by households is included in the value of market goods, as if the household rents durables from some firm “producing” new service flows each period.

Assumption 1. $f_{i,t}(\mathbf{q}_{i,t}, n_{i,t})$ is strictly increasing and strictly concave.

Let $\mathbf{q}_t$ be a J dimensional vector whose components are each of the commodities on the market place. Let $\mathbf{p}_t$ be a J dimensional vector of market prices and $\mathbf{x}_t$ the associated expenditure vector, the components of which are such that $p_{j,t}q_{j,t} = x_{j,t}$.

Assumption 2. There is no joint production using market goods. That is if $q_{j,t}$ is a component of $\mathbf{q}_{i,t}$ then $q_{j,t}$ cannot be a component of $\mathbf{q}_{i',t}$ for any $i' \neq i$.

Assumption 3. All input resources are used up in the production of final consumption commodities. That is $\forall j \in \{1, \dots, J\}, \exists i \in \{1, \dots, I\}$ such that $q_{j,t}$ is a component of $\mathbf{q}_{i,t}$.

Assumptions 2 and 3 together imply that $J = \sum_{i=1}^I J_i$.⁵

Each period the household derives flow utility $u(\mathbf{c}_t)$ from consumption of the I dimensional vector of final goods $\mathbf{c}_t$. We assume the utility function is time independent and of the constant

⁴All vectors are column vectors and denoted using bold font.

⁵Assumption 2 allows us to avoid the pitfalls of home production, described in a back and forth between Bill Barnett, Robert Pollak, and Michael Wachter in the 1970s (Pollak and Wachter 1975; Barnett 1977).

elasticity of substitution (CES) variety

$$u(c_t) = \left(\sum_{i=1}^I \theta_i c_{i,t}^\rho \right)^{\frac{1}{\rho}} \quad (2)$$

ρ indirectly parameterizes the intratemporal elasticity of substitution which is $\frac{1}{1-\rho}$. This structure will allow us to invoke an aggregation theorem to collapse the commodity space into indices such that the production process for each final good takes one and only one market good as input. This is described in Lemma 1 and its proof, which requires specification of equilibrium conditions. Before moving on to that, let us finish characterizing the household's choices.

Savings is accomplished by investing s_t in storable, non-consumable capital assets k_t .⁶ The stock of household capital assets evolves according to

$$k_{t+1} \leq (1 - \delta)k_t + s_t \quad (3)$$

δ is the net depreciation rate of capital. Let r_t be the net rental rate of capital service flows. The household receives gross return on capital assets of R_t with $R_t = r_t + 1 - \delta$ as usual.

Let $\bar{n}_t$ denote the total time available to the household in period t . $\bar{n}_t$ is time dependent, because the number of individual agents in the household sector is changing over time. This detail warrants further discussion. First, our household is not a single representative household in the Gorman (1961) sense. Rather, the household is a sector in the broader economy, the decisions of which are the aggregates of the decisions of every individual household in the economy. Thus all individual households in the traditional sense comprise our collective household. In this manner, one can think of individual, specific households as atomic with household aggregates equal to the integral of all decisions by specific households. Then, $\bar{n}_t$ changes over time because the size of the mass of individual households changes over time as both population levels and the number of individual households in the economy grows. Nonetheless, throughout this exposition when we refer to "the household," just think of it as the integrated, collective "household sector." The household sector earns wages w_t from supplying labor on the open market place l_t . Given this information, we can write the period t household budget constraint:

$$p_t^\top \cdot q_t + k_{t+1} \leq w_t l_t + R_t k_t \quad (4)$$

Denote the starting period of our analysis as $t = 0$. The household geometrically discounts the future with time preferences governed by β . Let $\mathbb{E}_0$ represent the agent's expectations in period zero. Given an initial stock of capital k_0 , the household chooses sequences $\{q_t, n_t, l_t, k_{t+1}\}_{t=0}^\infty$ so as to solve

$$\max_{\{q_t, n_t, l_t, k_{t+1}\}_{t=0}^\infty} \mathbb{E}_0 \left\{ \sum_{t=0}^\infty \beta^t \left(\sum_{i=1}^I \theta_i c_{i,t}^\rho \right)^{\frac{1}{\rho}} \right\} \quad (5)$$

⁶Household choices will be distinguished from firm choices by using lower case letters.

$$\text{subject to } \mathbf{p}_t^\top \cdot \mathbf{q}_t + k_{t+1} \leq w_t l_t + R_t k_t \quad \forall t \quad (6)$$

$$c_{i,t} = f_{i,t}(\mathbf{q}_{i,t}, n_{i,t}) \quad \forall t \quad (7)$$

$$l_t + \sum_{i=1}^I n_{i,t} \leq \bar{n}_t \quad \forall t \quad (8)$$

Expectations in (5) are taken over $f_{i,t}$, which follows some known stochastic process, and prices which depend on firm productivities.

We do not explicitly specify a parameterization for $f_{i,t}(\mathbf{q}_{i,t}, n_{i,t})$. The reason for this is that our modeling assumptions allow us to invoke an aggregation theorem over commodities to make the analysis more compact. This is described in Lemma 1.

Lemma 1. Assume the household is a utility maximizer. Under Assumptions 1 and 2, and under Theorem 1 of Green (1964) attributed to Leontief (1947) we can restrict our analysis to

$$\tilde{u}_t(q_{1,t}, \dots, q_{i,t}, \dots, q_{I,t}, n_{1,t}, \dots, n_{I,t}) \quad (9)$$

where $q_{i,t}$ is some index that describes the grouping of market goods $\{q_{i,1,t}, \dots, q_{i,j_i,t}, \dots, q_{i,J_i,t}\}$.

Proof. See Appendix A. □

The intuition behind Lemma 1 is that the CES form of $u(c_t)$ and Assumption 2 guarantee that the intratemporal marginal rate of substitution for two goods used in the same home production process is independent of other goods not used in that process. The econometrician can then use an index of his choice to form a single composite good $q_{i,t}$ which describes the market value of the entire vector of goods $\mathbf{q}_{i,t}$ used in production process $f_{i,t}$.⁷ Thus rather than specifying a functional form for $f_{i,t}(\mathbf{q}_{i,t}, n_{i,t})$, from here on our analysis operates on $\tilde{f}_{i,t}(q_{i,t}, n_{i,t})$, the aggregated home production function for final good $c_{i,t}$. Let $z_{i,t}$ describe the total factor productivity of process $\tilde{f}_{i,t}$. This term captures two things: 1) the household's exogenously evolving ability to spend time using market commodity i in order to produce the final consumption commodity; 2) the intrinsic value to the household of quality gains to $q_{i,t}$ above and beyond those captured by the market price. We choose a Cobb-Douglas form for $\tilde{f}_{i,t}(q_{i,t}, n_{i,t})$:

$$\tilde{f}_{i,t}(q_{i,t}, n_{i,t}) = z_{i,t} q_{i,t}^{\omega_i} n_{i,t}^{1-\omega_i} \quad (10)$$

ω_i is the output elasticity of market goods in process i .

Assumption 4. $\omega_i \in (0, 1), \forall i$.

⁷Scalar $q_{i,t}$ is the composite good. Again, bold font is reserved for vectors.

Composing $u(c_t)$ with each $\tilde{f}_{i,t}(q_{i,t}, n_{i,t})$, we get

$$u\left(\tilde{f}_{1,t}(q_{1,t}, n_{1,t}), \dots, \tilde{f}_{i,t}(q_{i,t}, n_{i,t}), \dots, \tilde{f}_{I,t}(q_{I,t}, n_{I,t})\right) = \tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t) = \left(\sum_{i=1}^I \theta_i z_{i,t}^\rho q_{i,t}^{\rho\omega_i} n_{i,t}^{\rho-\rho\omega_i}\right)^{\frac{1}{\rho}} \quad (11)$$

Unlike $u(c_t)$, composed utility depends also on exogenous home productivities, so we index $\tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)$ with t to account for this. Each home production process is associated with time and one market input, which implies that there are the same number of market goods as final goods, i.e. $J = I$, in the aggregated problem.

2.2.1 Household Equilibrium Conditions

From here on, our analysis operates using the composite, aggregated flow utility function $\tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)$. Let $\frac{\mu_t}{\beta^t}$ denote the period t marginal utility of wealth, i.e. the Lagrange multiplier on (6), weighted by time preference. Substitute the time constraint (8) into the budget constraint (6) for l_t . Assuming households take $\{\bar{n}_t\}_{t=0}^\infty$ as given, we can restrict our attention each period to choices of $\mathbf{q}_t, \mathbf{n}_t$, and k_{t+1} , which along with the budget constraint must satisfy

$$\frac{\partial \tilde{u}_t}{\partial q_{i,t}} = p_{i,t} \mu_t \quad (12)$$

$$\frac{\partial \tilde{u}_t}{\partial n_{i,t}} = w_t \mu_t \quad (13)$$

$$\mu_t = \beta \mathbb{E}_t \{ R_{t+1} \mu_{t+1} \} \quad (14)$$

Marginal utilities are such that

$$\frac{\partial \tilde{u}_t}{\partial q_{i,t}} = \tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)^{1-\rho} \omega_i \theta_i z_{i,t}^\rho q_{i,t}^{\rho\omega_i} n_{i,t}^{\rho-\rho\omega_i} \left(\frac{1}{q_{i,t}}\right) \quad (15)$$

$$\frac{\partial \tilde{u}_t}{\partial n_{i,t}} = \tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)^{1-\rho} (1 - \omega_i) \theta_i z_{i,t}^\rho q_{i,t}^{\rho\omega_i} n_{i,t}^{\rho-\rho\omega_i} \left(\frac{1}{n_{i,t}}\right) \quad (16)$$

We can derive conditions describing the equilibrium relationship between time use and market inputs for a single production process. Dividing (15) by (16) we get the infra-marginal rate of substitution between time and market goods for process i :

$$\frac{n_{i,t} \omega_i}{q_{i,t} (1 - \omega_i)} = \frac{p_{i,t}}{w_t} \quad (17)$$

Using (17), we can write the equilibrium choice of $n_{i,t}$ as an implicit function of $q_{i,t}$ and vice-versa:

$$n_{i,t}(q_{i,t}) = \left(\frac{1 - \omega_i}{\omega_i}\right) \left(\frac{p_{i,t}}{w_t}\right) q_{i,t} \quad (18)$$

$$q_{i,t}(n_{i,t}) = \left(\frac{\omega_i}{1 - \omega_i} \right) \left(\frac{w_t}{p_{i,t}} \right) n_{i,t} \quad (19)$$

Proposition 1 provides us with an equilibrium characterization of relative demand for market inputs j and i that is independent of time use.

Proposition 1. Under Lemma 1, CES utility for final consumption, and Cobb-Douglas aggregated home production, the relative demand for market good j to market good i can be written

$$\left(\frac{q_{j,t}}{q_{i,t}} \right) = \left[\frac{\theta_i \omega_i [(1 - \omega_i)/\omega_i]^{(1-\omega_i)\rho}}{\theta_j \omega_j [(1 - \omega_j)/\omega_j]^{(1-\omega_j)\rho}} \right]^{\frac{1}{\rho-1}} p_{j,t}^{\frac{1-\rho+\rho\omega_j}{\rho-1}} p_{i,t}^{\frac{1-\rho+\rho\omega_i}{1-\rho}} w_t^{\frac{\rho(\omega_i-\omega_j)}{\rho-1}} \left[\frac{z_{i,t}}{z_{j,t}} \right]^{\frac{\rho}{\rho-1}} \quad (20)$$

Proof. See Appendix A. □

Relative market good consumption is thus a power function of prices, wages, and unobserved relative productivities. Proposition 1 is the starting point toward deriving a system of linear estimating equations. In Section 3.1 we will dive into further detail by assuming a process for relative productivities that allows us to estimate structural utility parameters.

Corollary 1. The same procedure can be applied as in Proposition 1 to express the time devoted toward production process j relative to process i as follows:

$$\left(\frac{n_{j,t}}{n_{i,t}} \right) = \left[\frac{\theta_i (1 - \omega_i) [\omega_i / (1 - \omega_i)]^{\rho\omega_i}}{\theta_j (1 - \omega_j) [\omega_j / (1 - \omega_j)]^{\rho\omega_j}} \right]^{\frac{1}{\rho-1}} p_{j,t}^{\frac{\rho\omega_j}{\rho-1}} p_{i,t}^{\frac{\rho\omega_i}{1-\rho}} w_t^{\frac{\rho(\omega_i-\omega_j)}{\rho-1}} \left[\frac{z_{i,t}}{z_{j,t}} \right]^{\frac{\rho}{\rho-1}} \quad (21)$$

Proof. See Appendix A. □

If indeed the researcher possesses information about time use in home production, Corollary 1 provides additional degrees of freedom for estimation or calibration.

2.3 Firms and Market Clearing

The supply side of the market consists of $I + 1$ firms with I of them producing the I aggregated market goods and an additional firm producing an investment good. Firms that produce market goods have idiosyncratic Cobb-Douglas production technology but must pay the same final wages w_t and final capital rental rate r_t to households. We assume that the firm which produces the investment good costlessly transforms capital one-to-one to investment.

Firms are characterized by idiosyncratic Cobb-Douglas production technology which features firm-specific, exogenous time-varying total factor productivity $A_{i,t}$ and firm-specific, exogenous time-varying labor productivity $\zeta_{i,t}$. α_i is the output elasticity of capital and $Y_{i,t}$ describes the totality of market good i produced in period t .⁸ Output in sector i thus satisfies

$$Y_{i,t}(K_{i,t}, L_{i,t}) = A_{i,t} K_{i,t}^{\alpha_i} (\zeta_{i,t} L_{i,t})^{1-\alpha_i} \quad (22)$$

⁸We use capital letters to denote a firm's choice variables.

Each period, firm $i \in \{1, \dots, I\}$ chooses inputs of capital $K_{i,t}$ and labor $L_{i,t}$ to produce market good $Y_{i,t}$ by solving a static profit maximization problem:

$$\max_{L_{i,t}, K_{i,t}} p_{i,t} A_{i,t} K_{i,t}^{\alpha_i} (\zeta_{i,t} L_{i,t})^{1-\alpha_i} - r_t K_{i,t} - w_t L_{i,t} \quad (23)$$

Profit maximization implies

$$\frac{r_t}{p_{i,t}} = A_{i,t} \alpha_i \left(\frac{K_{i,t}}{\zeta_{i,t} L_{i,t}} \right)^{\alpha_i - 1} \quad \forall i \quad (24)$$

$$\frac{w_t}{p_{i,t}} = A_{i,t} \zeta_{i,t} (1 - \alpha_i) \left(\frac{K_{i,t}}{\zeta_{i,t} L_{i,t}} \right)^{\alpha_i} \quad \forall i \quad (25)$$

By assuming a single capital rental rate and wage rate, in general equilibrium the marginal products of capital and labor must equate across all firms.

As in all competitive market economies, equilibrium prices ensure that markets clear. Market clearing for goods implies

$$q_{i,t} = Y_{i,t}(K_{i,t}, L_{i,t}) \quad (26)$$

The investment good is such that

$$s_t = K_{I+1,t} \quad (27)$$

Further, we normalize prices so that the price of the investment good is unity every period. Capital markets clear when

$$k_t = \sum_{i=1}^{I+1} K_{i,t} \quad (28)$$

Labor markets clear when

$$l_t = \sum_{i=1}^I L_{i,t} \quad (29)$$

2.4 Equilibrium Definition

Given an initial stock of capital k_0 , a known sequence of total available time $\{\bar{n}_t\}_{t=0}^{\infty}$, and known stochastic processes for productivities $\{z_t, A_t, \zeta_t\}_{t=0}^{\infty}$, a general equilibrium economy under Beck-erian home production consists of⁹

1. Sequences of household policies for market purchases, time use, and capital investment

$$\{q_t^*, n_t^*, l_t^*, k_{t+1}^*\}_{t=0}^{\infty} \quad (30)$$

⁹We use superscript stars to denote equilibrium objects.

2. Sequences of firm policies for capital and labor inputs

$$\{K_{i,t}^*, L_{i,t}^*\}_{t=0}^{\infty} \quad \forall i \in \{1, \dots, I\} \quad (31)$$

$$\{K_{I+1,t}^*\}_{t=0}^{\infty} \quad (32)$$

3. Sequences of prices

$$\{p_t^*, r_t^*, w_t^*\}_{t=0}^{\infty} \quad (33)$$

such that

- i. Each period the household budget constraint holds with equality and choices satisfy (12) thru (14).
- ii. Each period the choices of firms satisfy (24) thru (25).
- iii. Each period market clearing conditions (26) thru (29) are satisfied.

3 Quantitative Model

Home production is not usually modeled with explicit time complementarities as we are doing, so there is little precedent for work that makes and subsequently justifies explicit, formal assumptions that generate the equilibrium conditions of our theoretical model. While our formulation captures anecdotal regularities, the beauty of our specification is that it can be easily linearized to tractably estimate the structural parameters and subsequently back out both home and firm productivity series. The estimation operates on household relative demands for market goods and off-market time described in Proposition 1 and Corollary 1 as well as firm marginal product conditions for the United States economy with consumers choosing to purchase two broad consumption aggregates — “goods” (g) and “services” (s). We exploit these equilibrium conditions in a Hamiltonian Monte Carlo (HMC) routine to estimate the posterior distribution of reduced-form parameters conditional upon the data observations. From the reduced-form parameters, distributions of structural utility parameters and productivity series can be recovered. Below, we first outline assumptions and features of the estimation procedure in Section 3.1, then we describe the data we use to estimate the model in Section 3.2, and finally in Section 3.3 we present the results of our estimation routine including some tests of model performance.

3.1 Estimation Procedure

From here on, we consider a two good economy where $i \in \{g, s\}$.¹⁰ Return now to Proposition 1 and Corollary 1. Taking logs of the relative demands for market goods and off-market time from

¹⁰Choosing to estimate a home production model where consumers buy just “goods” and “services” is a matter of convenience: in the broader marketplace, inputs and outputs by sector are already classified in such a manner. Further, this places our work in conversation with other papers dealing with the structural evolution of the United

(20) and (21), and the log for the marginal product of labor for each firm from (25) we can write

$$\underbrace{\ln \left(\frac{q_{g,t}}{q_{s,t}} \right)}_{\ln \tilde{q}_t} = \beta_0 + \beta_1 \ln p_{g,t} + \beta_2 \ln p_{s,t} + \beta_3 \ln w_t + \xi_t^1 \quad (34)$$

$$\underbrace{\ln \left(\frac{n_{g,t}}{n_{s,t}} \right)}_{\ln \tilde{n}_t} = \beta_0 + (\beta_1 + 1) \ln p_{g,t} + (\beta_2 - 1) \ln p_{s,t} + \beta_3 \ln w_t + \beta_4 + \xi_t^1 \quad (35)$$

$$\underbrace{\ln \left(\frac{r_t}{p_{i,t}} \right)}_{\ln \tilde{r}_{i,t}} = \ln \alpha_i + (\alpha_i - 1) \underbrace{\ln \left(\frac{K_{i,t}}{L_{i,t}} \right)}_{\ln \tilde{K}_{i,t}} + \xi_{i,t}^2 \quad \forall i \in \{g, s\} \quad (36)$$

$$\underbrace{\ln \left(\frac{w_t}{p_{i,t}} \right)}_{\ln \tilde{w}_{i,t}} = \ln(1 - \alpha_i) + \alpha_i \underbrace{\ln \left(\frac{K_{i,t}}{L_{i,t}} \right)}_{\ln \tilde{K}_{i,t}} + \xi_{i,t}^2 \quad \forall i \in \{g, s\} \quad (37)$$

$$\text{where } \beta_0 = \left(\frac{1}{\rho - 1} \right) \ln \left[\frac{\theta_s \omega_s [(1 - \omega_s)/\omega_s]^{(1-\omega_s)\rho}}{\omega_g [(1 - \omega_g)/\omega_g]^{(1-\omega_g)\rho}} \right] \quad (38)$$

$$\beta_1 = \frac{1 - \rho + \rho \omega_g}{\rho - 1} \quad (39)$$

$$\beta_2 = \frac{1 - \rho + \rho \omega_s}{1 - \rho} \quad (40)$$

$$\beta_3 = \frac{\rho(\omega_s - \omega_g)}{\rho - 1} \quad (41)$$

$$\beta_4 = \frac{(1 - \omega_s)/\omega_s}{(1 - \omega_g)/\omega_g} \quad (42)$$

$$\xi_t^1 = \left(\frac{\rho}{\rho - 1} \right) \ln \left[\frac{z_{s,t}}{z_{g,t}} \right] \quad (43)$$

$$\xi_{i,t}^2 = \ln A_{i,t} - (\alpha_i - 1) \ln \zeta_{i,t} \quad (44)$$

To simplify notation, we refer to the ratios of quantities and time use with tildes. In the above ξ_t^1 and $\xi_{i,t}^2$ will be parameterized as unit root random walks, as described in Assumption 5, to facilitate a regression on first differences. Also, we cannot separately identify θ_g and θ_s , the share parameters in the CES utility function, so we normalize $\theta_g = 1$ and parameterize the estimation in terms of the ratio, which we simply call θ_s . The values of β_1 thru β_3 are the elasticities of relative demand $q_{g,t}/q_{s,t}$ with respect to $p_{g,t}$, $p_{s,t}$, and w_t . In Proposition 2, below, we specify restrictions on the support of these values which depend on the underlying values of ρ , ω_g , ω_s , and θ_s . The restrictions are a result of the model structure and confirm classical economic intuition regarding the relationships between quantity demanded and price.

States economy over the latter half of the 20th century (see Ngai and Pissarides (2007), Buera and Kaboski (2012), Herrendorf, Rogerson, and Valentinyi (2013), and Boppart (2014), for example).

Assumption 5. Let ξ_t be the stacked column vector of productivity trend components, and let Ω be the $2I + 1$ dimensional square, positive semi-definite variance/covariance matrix. Then ξ_t follows a normal random walk:

$$\xi_t = \begin{pmatrix} \xi_t^1 \\ \xi_{g,t}^2 \\ \xi_{s,t}^2 \end{pmatrix} \sim \mathcal{N}(\xi_{t-1}, \Omega) \quad (45)$$

Let Δ be the backwards first-difference operator so that $\Delta \ln \tilde{q}_t = \ln \tilde{q}_t - \ln \tilde{q}_{t-1}$ for example. Under Assumption 5, we can estimate $\beta_1, \beta_2, \beta_3, \alpha_g$, and α_s with a regression on first differences:

$$\Delta \ln \tilde{q}_t = \beta_1 \Delta \ln p_{g,t} + \beta_2 \Delta \ln p_{s,t} + \beta_3 \Delta \ln w_t + \epsilon_t^1 + \nu_t^q \quad (46)$$

$$\Delta \ln \tilde{n}_t = (\beta_1 + 1) \Delta \ln p_{g,t} + (\beta_2 - 1) \Delta \ln p_{s,t} + \beta_3 \Delta \ln w_t + \epsilon_t^1 + \nu_t^n \quad (47)$$

$$\Delta \ln \tilde{r}_{i,t} = (\alpha_i - 1) \Delta \ln \tilde{K}_{i,t} + \epsilon_{i,t}^2 + \nu_{i,t}^r \quad \forall i \in \{G, S\} \quad (48)$$

$$\Delta \ln \tilde{w}_{i,t} = \alpha_i \Delta \ln \tilde{K}_{i,t} + \epsilon_{i,t}^2 + \nu_{i,t}^w \quad \forall i \in \{G, S\} \quad (49)$$

Note that $\epsilon_t^1 = \xi_t^1 - \xi_{t-1}^1$,¹¹ which is the idiosyncratic structural error with autocorrelation differenced out. $\nu_t^q, \nu_t^n, \nu_{i,t}^r$, and $\nu_{i,t}^w$ are model specification errors that must be incorporated to leverage information from both market purchases and time use and both capital and wage rates. Theoretically, if the specification describes the true data generating process, then these terms would be unnecessary because the data would reflect the fact that $\Delta \ln \tilde{n}_t$ and $\Delta \ln \tilde{q}_t$ are deterministically related according to the equilibrium conditions of the model and similarly for $\Delta \ln \tilde{r}_{i,t}$ and $\Delta \ln \tilde{w}_{i,t}$. This is almost certainly not true in the data, however. The idea then is to understand to what degree this is true, which we can accomplish by estimating the equations jointly.

All covariates in (46) thru (49) are endogenous, including the firms' choices of the capital/labor ratio. But information about capital and labor supply decisions by the household are fully encoded in prices and quantities. Thus, $\tilde{K}_{i,t}$ is correlated with household decision outcomes only through $\tilde{r}_{i,t}$ and $\tilde{w}_{i,t}$, since households respond to prices when choosing how much to work and invest. Similarly, quantities of market purchases chosen by the household are linked with firm output only through prices, which contain information about firm productivity. Thus, in (46) and (47) prices are endogenous. One could instrument for firms' choices of capital/labor ratio using exogenous supply-side variables. For now, lacking such instruments, we accept the potential endogeneity bias that results from the current specification. Instead, we argue that the underlying decisions of firms and households are unrelated except through prices: firms do not think about how much households will adjust their labor supply or capital supply in response to their own decisions, nor do households think about how much firms will raise or lower output in response to the effect of their quantity purchases on prices. Instrumenting for supply shocks on (48) and (49), but that would require an argument that the shocks chosen for instruments are themselves

¹¹Obviously, $\epsilon_{i,t}^2$ has the same property.

somehow unrelated to broader productivity trends captured by the error process. Under such a specification, we would be making a stronger statement than we are now, arbitrarily distinguishing between supply shocks which themselves drive productivity changes and those we choose as instruments which by assumption would not. We choose to tolerate the inevitable criticism of I.V. hegemonists, noting that this paper is not fundamentally about an estimation procedure, but rather reimagining the theory behind household consumption/savings decisions and the more general quantitative implications of such a construct.

Let Σ be the $2I + 2$ dimensional square, positive semi-definite variance/covariance matrix associated with the first-difference error process.¹² With a slight abuse of notation, let ϵ_t be the stacked vector of error terms in (46) thru (49), including the additional model specification errors. After first-differencing, the error process is

$$\epsilon_t = \begin{pmatrix} \epsilon_t^1 + \nu_t^q \\ \epsilon_t^1 + \nu_t^n \\ \epsilon_{g,t}^2 + \nu_{g,t}^r \\ \epsilon_{s,t}^2 + \nu_{s,t}^r \\ \epsilon_{g,t}^2 + \nu_{g,t}^w \\ \epsilon_{s,t}^2 + \nu_{s,t}^w \end{pmatrix} = \begin{pmatrix} \xi_t^1 - \xi_{t-1}^1 + \nu_t^q \\ \xi_t^1 - \xi_{t-1}^1 + \nu_t^n \\ \xi_{g,t}^2 - \xi_{g,t-1}^2 + \nu_{g,t}^r \\ \xi_{s,t}^2 - \xi_{s,t-1}^2 + \nu_{s,t}^r \\ \xi_{g,t}^2 - \xi_{g,t-1}^2 + \nu_{g,t}^w \\ \xi_{s,t}^2 - \xi_{s,t-1}^2 + \nu_{s,t}^w \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \Sigma) \quad (50)$$

We impose an LKJ prior distribution on the unit diagonal Cholesky factor, Ξ , of the variance/covariance matrix Σ to ensure computational tractability in the Hamiltonian Monte Carlo routine implemented with the Stan software package.¹³ In Stan, LKJ priors are implemented with a unit diagonal, so we allow the diagonal terms χ_j to take independent half Cauchy priors:

$$\text{chol}(\Sigma) = \text{diag}(\chi) \cdot \Xi \quad (51)$$

$$\chi_j \sim \text{Cauchy}_{(0,\infty)}(0, 2) \quad \forall j \in \{1, \dots, 2I + 2\} \quad (52)$$

$$\Xi \sim \text{LKJ}(2) \quad (53)$$

Note that our utility function is strictly concave only for $\rho < 1$. Further, if $\rho = 0$, then utility is Cobb-Douglas in final goods. In the Cobb-Douglas scenario, estimation would be simple because the ratio of goods to services expenditure would be constant over time, as Lemma 2 explains.

Lemma 2. Suppose $\rho = 0$ so that $u(c_t)$ is Cobb-Douglas. Then

$$\frac{p_{g,t}q_{g,t}}{p_{s,t}q_{s,t}} = \frac{\theta_g\omega_g}{\theta_s\omega_s} \in \mathbb{R}_+ \quad (54)$$

¹²This matrix is 6×6 when dealing with goods and services only.

¹³For more information on the LKJ distribution see Lewandowski, Kurowicka, and Joe (2009). For more information on Hamiltonian Monte Carlo see the brief summary in Gelman et al. (2013b) or more technical treatments in Neal (2011) and Betancourt and Stein (2011). We also provide a brief overview of how HMC works in Appendix B.1. For more information on the Stan software we use to estimate the model, see Stan Development Team (2016) and Gelman et al. (2013a).

That is, expenditure ratios are constant over time.

Proof. See Appendix A. □

If $\rho = 0$ then we cannot identify home productivity changes from first order conditions. However, if $\rho \neq 0$ then reduced-form parameters are further restricted as Proposition 2 states.

Proposition 2. Assume $\omega_i \in (0, 1)$ for all $i \in \{g, s\}$ and suppose $\rho \neq 0$. Then one of the following reduced-form restrictions must hold:

- i. Final goods are substitutes so that $\rho \in (0, 1)$ and:

$$\begin{aligned}\beta_1 &\in (-\infty, -1) \\ \beta_2 &\in (1, \infty) \\ \beta_3 &\in (-\infty, \infty)\end{aligned}\tag{55}$$

- ii. Final goods are complements so that $\rho \in (-\infty, 0)$ and:

$$\begin{aligned}\beta_1 &\in (-1, 0) \\ \beta_2 &\in (0, 1) \\ \beta_3 &\in (-1, 1)\end{aligned}\tag{56}$$

For completeness, $\beta_0 \in (-\infty, \infty)$ and $\beta_4 \in (0, \infty)$ always.

Proof. See Appendix A. □

Using expenditure series data we can test if $\rho = 0$ and, if not, we can run two separate restricted regressions for both the substitutes and complements cases, then use model selection techniques to determine which restriction fits best. Results of these tests are discussed in Section 3.3.

The sign restrictions on β_1 thru β_3 are natural. Recall from earlier discussion that β_1 is the elasticity of relative demand with respect to the goods price $p_{g,t}$, β_2 is the elasticity with respect to the services price $p_{s,t}$, and β_3 is the elasticity with respect to wages w_t . When the final consumption commodities produced using goods and services are substitutes, relative demand elastically adjusts as one would expect: a 1% increase in the price of services, for example, yields a $> 1\%$ increase in the relative demand of goods to services and vice-versa. When final consumption commodities are complements, price responses are inelastic. The fact that β_3 can be either negative or positive reflects the complex time use tradeoff consumers face between using goods and services they buy verses working more to earn more and buy more goods and services. When β_3 is negative, two things could be happening. First, an increase in wages is associated with a relative increase in services consumption, consistent with previous work documenting the importance of preference-based income effects driving relative consumption changes (Echevarria 1997; Caselli and Coleman 2001; Kongsamut, Rebelo, and Xie 2001; Herrendorf, Rogerson, and Valentinyi 2013;

Boppart 2014). Second, relatively more time is spent using services, which can happen both as a consequence of total consumption increasing, with services consumption increasing relatively faster, and if the absolute level of goods consumption falls. When β_3 is positive, income effects counter to those documented over the long-run by Kuznets (1966) are occurring, which would be due to consumers supplementing goods consumption with off-market time at such a high rate that they purchase relatively more goods than services in equilibrium to maximize utility.

Given our motivating empirical observation, the most interesting elasticity values are those for which relative consumption $q_{g,t}/q_{s,t}$ and relative time use $n_{g,t}/n_{s,t}$ move in opposite directions. Note that the elasticity of relative time use with respect to goods price $p_{g,t}$ is $\beta_1 + 1$. Thus for $\beta_1 \in (-1, 0)$, relative consumption falls in prices while relative time use rises. A similar phenomenon occurs for responses to $p_{s,t}$. $\beta_2 - 1$ is the elasticity of relative time use with respect to the services price $p_{s,t}$. If $\beta_2 \in (0, 1)$, then relative consumption rises in prices while relative time use falls. Since both of these constitute the observed case when looking at time use and consumption data from 2003-2014, our elasticity estimates should reflect this. Proposition 3 summarizes the details of this phenomenon.

Proposition 3. The elasticities of relative consumption with respect to prices have opposite sign from the elasticities of relative time use with respect to prices if and only if final goods are complements, so that $\rho \in (-\infty, 0)$.

Proof. See Appendix A. □

Depending on whether the complements or the substitutes model is estimated, Assumption 6 describes the priors we impose on the reduced-form regression coefficients:

Assumption 6. Let $\mathcal{U}$ denote the uniform distribution, and $\mathcal{N}$ a Gaussian normal distribution parameterized in terms of mean and variance.

- i. Final goods are substitutes so that $\rho \in (0, 1)$. The priors we impose are

$$\begin{aligned}\beta_1 &\sim \mathcal{U}(-\infty, -1] \\ \beta_2 &\sim \mathcal{U}[1, \infty) \\ \beta_3 &\sim \mathcal{N}(0, 5^2)\end{aligned}\tag{57}$$

- ii. Final goods are complements so that $\rho \in (-\infty, 0)$. The priors we impose are

$$\begin{aligned}\beta_1 &\sim \mathcal{U}[-1, 0] \\ \beta_2 &\sim \mathcal{U}[0, 1] \\ \beta_3 &\sim \mathcal{N}_{[-1, 1]}(0, 5^2)\end{aligned}\tag{58}$$

For the substitutes case, β_1 and β_2 have flat, improper priors, yet we find no performance issues in the HMC associated with this specification. We find that for the complements case, the

unit-interval uniform prior provides the best computational performance, leading to minimal divergent Hamiltonian transitions. For firm output elasticities, we impose a flat uniform prior as well

$$\alpha_i \sim \mathcal{U}[0, 1] \quad \forall i \in \{g, s\} \quad (59)$$

We have now defined all of the objects needed to estimate the Bayesian posterior distribution of parameters conditional on data using a Hamiltonian Monte Carlo regression on first differences. Let $\mathcal{P}$ be the set of structural parameters and $\mathcal{D}$ the data. Let $\pi(\cdot)$ denote an arbitrary density function. We seek to estimate the posterior distribution:

$$\mathcal{P} = \{\boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}\} \quad (60)$$

$$\mathcal{D} = \left\{ \{\Delta \ln \tilde{q}_t, \Delta \ln \tilde{n}_t, \Delta \ln \tilde{r}_t, \Delta \ln \tilde{w}_t, \Delta \tilde{K}_t, \Delta \ln p_{g,t}, \Delta \ln p_{s,t}, \Delta \ln w_t\}_{t=2}^T \right\} \quad (61)$$

$$\underbrace{\pi(\mathcal{P} | \mathcal{D})}_{\text{Posterior}} \propto \underbrace{\pi(\mathcal{D} | \mathcal{P})}_{\text{Data Given Parameters}} \underbrace{\pi(\mathcal{P})}_{\text{Prior}} \quad (62)$$

$\pi(\mathcal{D} | \mathcal{P})$ represents the probability of realizing the data given the parameterization, i.e. the “likelihood” function:

$$\pi(\mathcal{D} | \mathcal{P}) \propto \prod_{t=2}^T \exp \left\{ -\frac{1}{2} \boldsymbol{\epsilon}_t' \boldsymbol{\Sigma}^{-1} \boldsymbol{\epsilon}_t \right\} \quad (63)$$

The full prior representation depends on the whether we initially assume the final goods are complements or substitutes.

Further identification assumptions are required to back out structural productivities after unravelling the first-difference regression equations. First, we cannot identify β_0 in the first-difference regression. Thus, we make the following assumption.

Assumption 7. Let $t = 1$ be the first model period for which we have a complete data observation. Normalize reduced-form trend productivities so that $\boldsymbol{\xi}_1 = \mathbf{0}$. Then

$$\beta_0 = \ln \tilde{q}_1 - \beta_1 \ln p_{g,1} - \beta_2 \ln p_{s,1} - \beta_3 \ln w_1 \quad (64)$$

$$\beta_4 = \ln \tilde{n}_1 - \beta_0 - (\beta_1 + 1) \ln p_{g,1} - (\beta_2 - 1) \ln p_{s,1} - \beta_3 \ln w_1 \quad (65)$$

Assumption 7 allows us to unravel the first-difference equations and get reduced-form productivities back:

$$\xi_t^1 = \xi_{t-1}^1 + \frac{1}{2} [2\epsilon_t^1 + \mathbf{v}_t^q + \mathbf{v}_t^n] \quad (66)$$

$$\xi_{i,t}^2 = \xi_{i,t-1}^2 + \frac{1}{2} [2\epsilon_{i,t}^2 + \mathbf{v}_{i,t}^r + \mathbf{v}_{i,t}^w] \quad \forall i \in \{g, s\} \quad (67)$$

Given our introduction of model error terms, we update ξ_t using the midpoints of the reduced-form, first-difference errors.

3.2 Data

For prices, output quantities, capital and labor supply, we use the United States input/output tables constructed for Jorgenson, Ho, and Samuels (2016).¹⁴ The input/output tables contain both nominal and constant dollar input values for capital, labor, and intermediate goods in 65 industries. We classify these industries into 3 broad sectors — “durables,” “non-durables,” and “services” — roughly following the classification of final goods in NIPA Tables 2.4.3 and 2.4.4. (Bureau of Economic Analysis 2017).¹⁵

The data series spans from 1947-2014. We normalize output prices in each broad sector so that the price in 2014 is unity. We do the same for the aggregate wage and aggregate supply of capital, which we compute by adding the given capital column with the intermediate goods column. Thus, “capital” in this sense encompasses all non-labor inputs to production. Since our primary left hand side variable is the ratio of final consumption of goods to services, we need to construct this ratio accounting for the fact that some share of observed firm output is invested in capital which is then used again for production next period. Turning to BEA Fixed Asset Table 1.1, we define nominal capital used in private residences as equal to total private residential capital plus consumer durables and compute the economy-wide shares of total capital used by private residences and businesses and government. As was previously mentioned, we account for the fact that consumers derive utility from durable assets that do not fully depreciate in one model period by constructing the data so that the value of “goods” includes both non-durable purchases and the current value of service flows for durable assets held. We thus use the shares computed from the BEA data to redefine “durables” in our KLEMS data to be equal to the total nominal stock of private residential capital in the economy. Market purchases of “goods” in our final analysis is equal to “durables” plus “non-durables” after this redefinition. Figure 2 presents the shares of final nominal market expenditure for the “goods” and “services” categories we construct. As has been well-documented in the structural change literature, services comprise an ever increasing share of final market purchases since mid-century.

For off-market time use, we turn to the American Time Use Survey (ATUS) which makes data available from 2003-2016 (Bureau of Labor Statistics 2017).¹⁶ We truncate the time series so that it terminates in 2014 for consistency with our input/output tables. Ultimately, we will run two separate estimations. First, we ignore the within-home time use condition and simply estimate the model on the entire series 1947-2014 using relative demand and factors prices on the KLEMS data we construct. Second, we use relative within-home time use to estimate the full model on the

¹⁴The raw data is available for download at the World KLEMS website: U.S. KLEMS Data. The R scripts and supplemental files we use for construction of the data featured in the estimation is available upon request.

¹⁵The spreadsheet detailing the exact broad sector into which we file each industry in the World KLEMS data is available upon request.

¹⁶Stata ‘do’ files for constructing the ATUS sample are available upon request.

Figure 2: Composition of total household market expenditure (current dollar value), 1947-2014. KLEMS Data, (Jorgenson, Ho, and Samuels 2016) U.S. KLEMS. Durable goods shares computed using BEA Fixed Asset Table 1.1, BEA.

series 2003-2014. The ATUS dataset provides an incredibly convenient way to distinguish between activities associated with using “goods” verses “services.” For example, the dataset contains both a variable that presents the time an individual respondent spent “Interior cleaning” and a separate variable that presents the time that same individual respondent spent “Using interior cleaning services.” In fact, almost all tasks within the survey are classified in this manner. Continuing with the interior cleaning example, a researcher can reasonably assume that an individual engaged in “Interior cleaning” is using his time along with goods like soaps, brushes, vacuums, dusters, etc. to accomplish the task of cleaning, while one engaged in “Using interior cleaning services” could reasonably be thought to be spending time monitoring a maid or housekeeper for whom he pays to perform cleaning services. This is just one example. almost every household task features in ATUS is classified in this manner. Thus, the survey structure makes it easy to distinguish between tasks that are complementary to using market goods and those which are complementary to using services.

3.3 Estimation Results

Our estimation strategy falls into two camps. First, we estimate the system containing just equations (46), (48), and (49) on the KLEMS data series for years 1947-2014, ignoring the time use equation since we do not possess time use data for that entire time span. This initial estimation is thus over a system of 5 simultaneous equations. Second, we estimate the full system (46) thru (49) for years 2003-2014, using the relative off-market time data from ATUS. Each of these routines requires separately estimating the model under the assumptions that $\rho \in (0, 1)$ (final goods are substitutes) and $\rho \in (-\infty, 0)$ (final goods are complements). Note that by looking at Figure 2 we can rule out very easily that $\rho = 0$ since relative expenditure is clearly not constant over time. We then compute the posterior odds of the complements model being true verses the substitutes model by using bridge sampling techniques to back out Bayes factors.¹⁷ For both the 1947-2014 regression on KLEMS data only and the 2003-2014 regression with time use data, Bayes factor estimates suggest the complements model fits the data better. As a result, we only present the structural parameters and productivity estimates for the complements estimations with $\rho \in (-\infty, 0)$.

The Hamiltonian Monte Carlo estimation routine involves using the No U-Turn sampling algorithm of Gelman et al. (2013b) with the HMC estimation software Stan (Stan Development Team 2016). We use Stan’s default tuning parameter values except for the adaptation acceptance rate, which we find from simulations is best picked to be 0.9, as opposed to the default value of 0.8. The HMC is run in parallel on four independent chains each with a post-warm-up sampling space of size 4000 for a total posterior sample size of $N = 16000$. Table 1 presents summary statistics for the posterior distribution estimates of reduced-form parameters for the 1947-2014 sample without time use. Table 2 presents summary statistics for the 2003-2014 sample estimation with time use. The last three columns of each table represent distribution percentiles, with a 95% confidence region buttressed on either side by the 2.5-percentile and 97.5-percentile. The mean and standard deviation of the posterior distributions are also presented, along with the $\hat{R}$ and effective sample size n_{eff}/N diagnostic statistics. Diagnostic analysis suggests that HMC chains converge on all four estimations. We make this assessment by examining $\hat{R}$ and the effective sample size n_{eff}/N for the log-posterior density, $U(\mathcal{P})$. $\hat{R} \rightarrow_+ 1$ as the numerical integration converges. Generally speaking, we want $\hat{R} < 1.1$ and $n_{eff}/N > 0.10$, especially for $U(\mathcal{P})$ (Gelman et al. 2013a). Our HMC integrations have effectively converged. This is especially true for the long series complements model and both estimates on the shorter series. For a more in depth discussion about HMC see Appendix B.1.

¹⁷The Bayes factor is the ratio of posterior expectations, which each require computation of the marginal likelihood of realizing the observed data given the model under question (see Rossi, Allenby, and McCulloch (2005) for a thorough explanation of model selection using Bayes factors). Bridge sampling is the computational procedure used to compute the marginal likelihood required for Bayes factor analysis (see Meng and Wong (1996) and Gronau et al. (2017) for information regarding the computational procedure).

Table 1: HMC Posterior Results, 1947-2014 Sample with No Time Use

$\rho \in (0, 1)$	$\hat{R}$	n_{eff}/N	Mean	S.D.	2.5%	50%	97.5%
β_0	1.000	0.222	-0.395	0.238	-0.881	-0.389	0.060
β_1	1.017	0.012	-1.388	0.309	-2.199	-1.311	-1.018
β_2	1.012	0.019	1.504	0.407	1.020	1.400	2.520
β_3	1.001	0.125	-0.417	0.227	-0.921	-0.398	-0.016
α_g	1.000	0.369	0.934	0.049	0.819	0.942	0.997
α_s	1.001	0.196	0.647	0.048	0.554	0.646	0.742
$U(\mathcal{P})$	1.001	0.076	1108.034	3.480	1100.385	1108.336	1113.886
$\rho \in (-\infty, 0)$	$\hat{R}$	n_{eff}/N	Mean	S.D.	2.5%	50%	97.5%
β_0	1.000	0.717	-0.233	0.192	-0.619	-0.229	0.132
β_1	1.001	0.258	-0.541	0.209	-0.939	-0.532	-0.152
β_2	1.000	0.356	0.335	0.246	0.012	0.285	0.887
β_3	1.000	0.417	-0.085	0.160	-0.413	-0.077	0.210
α_g	1.000	0.819	0.929	0.051	0.807	0.938	0.997
α_s	1.000	0.747	0.645	0.051	0.545	0.645	0.747
$U(\mathcal{P})$	1.000	0.361	1107.884	3.321	1100.432	1108.217	1113.301

Table 2: HMC Posterior Results, 2003-2014 Sample with Time Use

$\rho \in (0, 1)$	$\hat{R}$	n_{eff}/N	Mean	S.D.	2.5%	50%	97.5%
β_0	1.000	0.530	-0.031	0.152	-0.332	-0.031	0.273
β_1	1.000	0.443	-1.474	0.395	-2.463	-1.377	-1.015
β_2	1.001	0.412	2.182	1.046	1.039	1.911	4.843
β_3	1.000	0.375	-0.003	1.058	-2.354	0.091	1.836
β_4	1.000	0.000	1.693	0.000	1.693	1.693	1.693
α_g	1.000	0.775	0.772	0.199	0.246	0.829	0.993
α_s	1.000	0.353	0.719	0.223	0.180	0.773	0.991
$U(\mathcal{P})$	1.000	0.301	172.037	3.815	163.800	172.368	178.585
$\rho \in (-\infty, 0)$	$\hat{R}$	n_{eff}/N	Mean	S.D.	2.5%	50%	97.5%
β_0	1.000	0.657	-0.056	0.103	-0.254	-0.057	0.150
β_1	1.000	0.401	-0.498	0.260	-0.960	-0.496	-0.041
β_2	1.000	0.948	0.440	0.284	0.017	0.414	0.964
β_3	1.000	0.620	0.081	0.487	-0.854	0.091	0.922
β_4	1.000	0.000	1.693	0.000	1.693	1.693	1.693
α_g	1.000	0.774	0.759	0.208	0.205	0.816	0.993
α_s	1.000	0.485	0.735	0.215	0.211	0.789	0.992
$U(\mathcal{P})$	1.001	0.327	171.241	3.962	162.600	171.586	177.988

Recall the improper priors imposed on β_1 and β_2 under Assumption 6 when final goods are substitutes, $\rho \in (0, 1)$. This assumption does not appear to affect model performance, as both the posterior distributions of β_1 and β_2 appear tight in both sample estimates. Further, recall β_1 thru β_3 represent the elasticities of relative demand $q_{g,t}/q_{s,t}$ to prices and wages. In three of the four regressions, we find the sign of β_3 to be negative, consistent with the goods to services structural transformation story discussed in Herrendorf, Rogerson, and Valentinyi (2013) and Boppart (2014). Only for the 2003-2014 estimation with complements do we find that the mean of the posterior distribution of $\beta_3 > 0$, though the 95% confidence region includes negative values. We find as well that the output elasticity of capital for goods production $\alpha_g > \alpha_s$ for all four estimations, though the contrast is notably starker using the longer time series without off-market time use data.

Before diving into deeper analysis of the structural parameters, we first engage in model selection to determine whether the complements or substitutes story best fits the data. Let $\mathcal{M}_c^{1947}$ denote the complements model where $\rho \in (-\infty, 0)$ and let $\mathcal{M}_s^{1947}$ denote the substitutes model where $\rho \in (0, 1)$ each for the 1947-2014 sample without time use. For the 2003-2014 sample with time use, simply change the superscript to 2003. Posterior model odds computed using Bayes factors are

$$\text{Posterior Odds}_{\mathcal{M}_c^{1947} \text{ vs } \mathcal{M}_s^{1947}} = \frac{Pr(\mathcal{D} | \mathcal{M}_c^{1947})}{Pr(\mathcal{D} | \mathcal{M}_s^{1947})} = 8.609 \gg 1 \Rightarrow \text{Moderate Evidence for } \mathcal{M}_c^{1947} \quad (68)$$

$$\text{Posterior Odds}_{\mathcal{M}_c^{2003} \text{ vs } \mathcal{M}_s^{2003}} = \frac{Pr(\mathcal{D} | \mathcal{M}_c^{2003})}{Pr(\mathcal{D} | \mathcal{M}_s^{2003})} = 1.930 > 1 \Rightarrow \text{Weak Evidence for } \mathcal{M}_c^{2003} \quad (69)$$

The evidence classification rubric we follow comes from Harold Jeffreys' original scale of evidence as presented in Wasserman (2000), a summary of Bayesian model selection techniques. Selecting on the complements models confirms that our formulation can capture the general nature of the motivating empirical relationship we observe in Figure 1. Price elasticities for relative consumption and relative time use indeed have opposite sign, sending relative time use up while relative consumption falls. For these reasons, we select on the complements model for both time series, backing out structural parameters and productivity residuals.

Table 3 presents the posterior mean estimates for structural utility parameters ρ , ω_g , ω_s , and θ_s for both the long time series and the shorter estimation. Recall ω_i is the elasticity of home production output with respect to market inputs, so that $1 - \omega_i$ is the elasticity with respect to off-market time. Note that for both the long time series and short time series output of the final commodity associated with services is more sensitive to market inputs than that of goods. This fact makes sense in light of the average share of off-market time devoted to goods over the 2003-2014 interval — 0.631 as opposed to 0.369 for services. Time use factor intensity is thus higher for production of the final commodity associated with market goods. Comparing the differences in structural parameters between the long and short time series estimation, it is not surprising that posterior standard deviations for the short time series are larger given the smaller sample

size. The elasticity of substitution is $\frac{1}{1-\rho}$, which comes out to 0.535 for the long time series and 0.492 for the short one. These values are very close and actually contained within one standard deviation of each other's posterior distributions.

Table 3: Structural Parameters

<i>1947-2014 Sample with No Time Use</i>			
ρ	ω_g	ω_s	θ_s/θ_g
-0.869 (1.030)	0.623 (0.353)	0.856 (0.222)	1.821 (7.727)
<i>2003-2014 Sample with Time Use</i>			
ρ	ω_g	ω_s	θ_s/θ_g
-1.031 (2.753)	0.524 (0.353)	0.807 (0.240)	4.084 (20.643)

*Posterior standard deviation in parentheses.

3.3.1 Productivity Estimates

In Figures 3 and 4 we present posterior mean estimates of ξ_t after unravelling the first-difference errors using the procedure described in Assumption 7. Red lines are estimates for the full time series and black lines describe the shortened time series with off-market time use. In Figure 4, goods absolute productivities are solid while services productivities are dashed.

First, turn to Figure 3 which describes the log relative home productivity of services to goods.¹⁸ Within the household, goods productivity increased during the post-war era until 1970, when stagflation and oil crises made using services to achieve final utility relatively more productive. This particular trend makes macroeconomic sense: in the United States the era from 1950 to approximately 1970 was characterized by mass adoption of modern home appliances like dishwashers, laundry machines, televisions, and air conditioners (Gordon 2016). Following the post-war boom in 1970, import competition and major swings in oil-related commodity prices helped make services consumption relatively more efficient for the household. Relative productivity seems to level off in the 1990s before increasing again during the 2000s, as mass internet connectivity and new software for both personal computers and mobile technology made it easier for households to take advantage of certain services. Consider for example how the internet has made it possible to engage in banking and other financial activities without ever having to go to a physical bank branch or call a stock broker.

¹⁸Note that the ratio for the relative home productivity is inverted in comparison to the quantity ratio (see Equation (43)).

Since the Great Recession, other aspects of household life have been disrupted by the proliferation of new web-based services, almost instantaneously accessible on smart phones with the push of a button. Think of the efficiency gains experienced by consumers who can now use ride-share services like Uber and Lyft, whereas before one would have to hail a cab outside or call the cab company and hope the dispatcher answered the phone. Think further of the efficiency gains to consumers from simply having access to a trip itinerary in a single application on a smart phone: airlines now warn you when flights will be delayed and connections will be missed, offering you the option to select a new route of your choosing. Prior to the proliferation of this technology, an air traveler would have to make a phone call and be placed on hold for extensive lengths or stand in the customer service line at the airport with hundreds of other stranded passengers.

The Great Recession itself came about during a major decline in the relative home productivity of services. We speculate that this is related to the decline in income experienced by consumers due to the recession. First, the relative consumption of services to goods is increasing in income, thus if income falls we expect the relative consumption to fall. Now suddenly at a lower level of income, consumers behave as if services are less productive relative to goods than before, foregoing unnecessary services expenditures like eating out and flying for example, instead spending time using goods to achieve the same end — cooking meals at home and driving their car. They are thus behaving as if goods are relatively more productive than before. Similarly, during the oil crisis of the 1970s, public transportation services may have become more attractive as gas guzzling V8 automobiles became increasingly costly to fuel.

Turning now to firm log productivity estimates, notice that the long time series for goods $\xi_{g,t}^2$ seems to be an approximate horizontal reflection of ξ_t^1 . As firms get better at making goods, households also get relatively better at using them. The drop in goods productivity during the 1970s reflects both oil crises and stagflation. We obviously do not feature a monetary mechanism in our model. Nordhaus (2004) shows that energy-intensive industries experienced the greatest productivity slowdowns during the 1970s, suggesting that the oil crisis played a significant role in the overall goods productivity decline. Without oil and its related products, there is less energy to power the equipment already owned by firms, so the machines and the workers using them become less productive. Returning to our example of consumers using public transit more in response to fuel price increases, the co-movement of ξ_t^1 and $\xi_{g,t}^2$ in opposite directions during the 1970s confirms work by Lee and Ni (2002), showing that oil price shocks affect both firm input costs and household resource utilization.

Figure 3: ξ_t^1 is log relative home productivity of services to goods. Services have been relatively more productive since the 1970s.

Figure 4: $\xi_{i,t}^2$ is log absolute productivity. Overall productivity of services has been higher than goods since 2004.

3.4 Home Production and Structural Change

As an application of our estimation for the long time series, we seek to better understand how changes to relative home productivities are linked to long-run changes in relative market consumption. To accomplish this, we simulate relative demand under five counterfactual scenarios: 1) fix the level of $p_{g,t}$ at 1947 prices; 2) repeat for $p_{s,t}$; 3) repeat for w_t ; 4) no relative productivity changes; 5) fix relative prices at their 1947 value, but allow the absolute level of $p_{g,t}$ to inflate at the observed rate. In each scenario, variables that are not explicitly adjusted are taken at their observed data values for the entire time series.

Figure 5: $\ln \tilde{q}_t$ under different counterfactual changes to the right hand side variables of the relative consumption equation. (1947-2014 sample)

Figure 5 shows the actual log expenditure time series in black, listing the subsequent counterfactuals by line type and color. The absolute market price changes have the largest effect on log expenditure. Holding wages fixed at their 1947 value while allowing prices and productivity to change demonstrates the leverage income effects have over the long run structural transformation of the economy from goods to services. Actual relative consumption has declined 43.6% since 1947. With no changes to income, we would expect to see only a 25.8% decline. Letting the services price remain fixed, we would expect a 74.4% decline in relative consumption, while if goods prices were to remain fixed, we would expect an increase of 83%. The absolute difference between these latter two values is due to the entrance of $\omega_i, \forall i \in \{g, s\}$ into the relative demand price elasticities. Different production processes require higher levels of market inputs than others, and the final goods produced by all these processes are complementary, so price sensitivity varies as

a function of the production intensities. In our long time series estimation, consumers are more price sensitive for goods than services since $|\beta_1| > |\beta_2|$. Here again, a combination of income and substitution effects are at play: when goods prices rise consumers spend more time using goods relative to services. Perhaps instead of buying ready-made microwavable dinners from the grocery store, they spend less money and purchase raw commodities to make their own meal.

Finally, changes in the relative home productivity of services to goods appear to have had little impact on U.S. structural change. We hold $\xi_t^1 = 0$ fixed for all t , the normalized 1947 value, and then simulate relative market demand from 1947-2014, finding a 48.4% decline in relative consumption. The fact that relative services consumption increases in this counterfactual scenario, despite services being relatively more productive than goods within the household, is a direct result of the within-home final outputs being complements: consumers need to buy more lower-quality services to make up for the efficiency decline.

Figure 6: $\ln \tilde{q}_t$ under different counterfactual changes to the right hand side variables of the relative consumption equation. (2003-2014 sample)

We perform the same exercises as above on the shorter time series but also examine counterfactual time use ratios. The demand counterfactuals are presented in Figure 6 with time use counterfactuals in Figure 7. The complementarities between final home-production outputs induce negative co-movement between relative market consumption and relative time use. Looking first at the price counterfactuals, notice that reducing the goods price pushes relative consumption toward goods and relative time use toward services. Here, services price levels change very little from 2003 to 2014, so there is little counterfactual effect to fixing this value at 2003 levels on

Figure 7: $\ln \tilde{n}_t$ under different counterfactual changes to the right hand side variables of the relative consumption equation. (2003-2014 sample)

relative demand and relative time use. Meanwhile, ξ_t^1 moves according to a sine curve pattern over this period, which can be seen by looking at the black line in Figure 3. Fixing ξ_t^1 at its 2003 level, induces an initial shift toward services market purchases followed by a strong shift toward goods market purchases. This is because from 2008 to 2011, the relative productivity of services to goods actually falls before rising again in the recovery from the Great Recession. Here again, the complementarities between final goods play a major role: consumers purchase relatively more goods in a counterfactual environment when services to goods productivity is higher, but they spend comparatively less time using those same goods.

4 Conclusion

Whether you are talking about boats, beds, or restaurant meals, a market good is not valuable unless a household can allocate time to consume it. This idea is consistent with early home production models. In this paper, we formalize the concept by allowing households to explicitly choose time to spend consuming individual market goods. As in other macroeconomic models with home production, by embedding this idea in a general equilibrium framework, we can simultaneously think about efficiency gains from both home and market production. Finally, we use the resulting estimation of the structural model to understand long run productivity trends and consider the implications for observed changes in the relative time allocated to goods and

services within the home and how this has affected the structural change that has occurred in the United States economy.

Future work might consider a more detailed look at how households allocate their off-market or non-working time and how that relates to their consumption behavior. In our empirical work, because we are considering the general equilibrium implications of the required allocation of time to purchased commodities, we generalize by categorizing all market commodities as either goods or services. This is done as a way of thinking explicitly about the motivating fact that households are allocating increasingly more time to goods relative to services. However, ATUS contains more detailed information about how individuals allocate their off-market time, and so it might be possible to make a more detailed connection between the allocation of market time, off-market time, and market expenditures by estimating an extended version of the model over a more finely partitioned commodity space.

Last but certainly not least, the broader impacts of our work should resonate with researchers concerned with mis-measurement of aggregate output and its associated welfare gains. As touched on in Gordon (2016) and Syverson (2017), our measures of output fail to capture the final value-added from home production. By failing to consider how households use the market commodities they buy, conventional measures of GDP fail to quantify the value of activities undertaken by a very large sector of the economy. Measuring the value of these activities is of course difficult but should not be thought impossible. Under reasonable assumptions there exist powerful models in the stables of many economists which can be adapted to better quantify the aggregate impacts of these activities. We hope that our paper can be a useful springboard for future work that helps quantify this core issue at the frontier of our field: how valuable indeed *are* the off-market activities of households?

A Proofs

Lemma 1. Assume the household is a utility maximizer. Under Assumptions 1 and 2, and under Theorem 1 of Green (1964) attributed to Leontief (1947) we can restrict our analysis to

$$\tilde{u}_t(q_{1,t}, \dots, q_{i,t}, \dots, q_{I,t}, n_{1,t}, \dots, n_{I,t}) \quad (\text{A.1})$$

where $q_{i,t}$ is some index that describes the grouping of market goods $\{q_{i,1,t}, \dots, q_{i,j_i,t}, \dots, q_{i,j_i,t}\}$.

Proof. For notational simplicity, denote the marginal final utility for the consumption of market good $q_{i,j_i,t}$ by $MU_t(q_{i,j_i,t})$ which is

$$MU_t(q_{i,j_i,t}) = \frac{\partial u}{\partial c_{i,t}} \frac{\partial f_{i,t}}{\partial q_{i,j_i,t}} \quad (\text{A.2})$$

Let j_i and h_i index two distinct components of $q_{i,t}$. Theorem 1 of Green (1964) states that the indices comprising grouped market goods must be constructed so that

$$\frac{\partial}{\partial q_{i',m_{i'},t}} \left(\frac{MU_t(q_{i,j_i,t})}{MU_t(q_{i,h_i,t})} \right) = 0 \quad \forall i' \neq i \quad (\text{A.3})$$

Note that we can write

$$\left(\frac{MU_t(q_{i,j_i,t})}{MU_t(q_{i,h_i,t})} \right) = \frac{\frac{\partial u}{\partial c_{i,t}} \frac{\partial f_{i,t}}{\partial q_{i,j_i,t}}}{\frac{\partial u}{\partial c_{i,t}} \frac{\partial f_{i,t}}{\partial q_{i,h_i,t}}} \quad (\text{A.4})$$

$$= \frac{\frac{\partial f_{i,t}}{\partial q_{i,j_i,t}}}{\frac{\partial f_{i,t}}{\partial q_{i,h_i,t}}} \quad (\text{A.5})$$

Note that (A.5) only depends on goods in $q_{i,t}$. By Assumption 2, (A.3) thus holds. Together under Assumption 1, Theorem 1 of Green (1964) is satisfied. $\square$

Proposition 1. Under Lemma 1, CES utility for final consumption, and Cobb-Douglas aggregated home production, the relative demand for market good j to market good i can be written

$$\left(\frac{q_{j,t}}{q_{i,t}} \right) = \left[\frac{\theta_i \omega_i [(1 - \omega_i)/\omega_i]^{(1-\omega_i)\rho}}{\theta_j \omega_j [(1 - \omega_j)/\omega_j]^{(1-\omega_j)\rho}} \right]^{\frac{1}{\rho-1}} p_{j,t}^{\frac{1-\rho+\rho\omega_j}{\rho-1}} p_{i,t}^{\frac{1-\rho+\rho\omega_i}{1-\rho}} w_t^{\frac{\rho(\omega_i-\omega_j)}{\rho-1}} \left[\frac{z_{i,t}}{z_{j,t}} \right]^{\frac{\rho}{\rho-1}} \quad (\text{A.6})$$

Proof. Start with the marginal rate of substitution for goods i and j :

$$\frac{\frac{\partial \tilde{u}}{\partial q_{i,t}}}{\frac{\partial \tilde{u}}{\partial q_{j,t}}} = \frac{p_{i,t}}{p_{j,t}} \quad (\text{A.7})$$

$$\Rightarrow \frac{\tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)^{1-\rho} \omega_i \theta_i z_{i,t}^\rho q_{i,t}^{\rho \omega_i} n_{i,t}^{\rho-\rho \omega_i} \left(\frac{1}{q_{i,t}}\right)}{\tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)^{1-\rho} \omega_j \theta_j z_{j,t}^\rho q_{j,t}^{\rho \omega_j} n_{j,t}^{\rho-\rho \omega_j} \left(\frac{1}{q_{j,t}}\right)} = \frac{p_{i,t}}{p_{j,t}} \quad (\text{A.8})$$

Substitute the implicit function $n_{i,t}(q_{i,t})$ expressed in (18) for each process then cancel like terms to get

$$\frac{\omega_i \theta_i z_{i,t}^\rho q_{i,t}^{\rho \omega_i} (p_{i,t}(1-\omega_i)/(w_t \omega_i) q_{i,t})^{\rho-\rho \omega_i} \left(\frac{1}{q_{i,t}}\right)}{\omega_j \theta_j z_{j,t}^\rho q_{j,t}^{\rho \omega_j} (p_{j,t}(1-\omega_j)/(w_t \omega_j) q_{j,t})^{\rho-\rho \omega_j} \left(\frac{1}{q_{j,t}}\right)} = \frac{p_{i,t}}{p_{j,t}} \quad (\text{A.9})$$

$$\Rightarrow \frac{\omega_i \theta_i z_{i,t}^\rho (p_{i,t}(1-\omega_i)/(w_t \omega_i))^{\rho-\rho \omega_i} q_{i,t}^{\rho-1}}{\omega_j \theta_j z_{j,t}^\rho (p_{j,t}(1-\omega_j)/(w_t \omega_j))^{\rho-\rho \omega_j} q_{j,t}^{\rho-1}} = \frac{p_{i,t}}{p_{j,t}} \quad (\text{A.10})$$

Collect like terms and move everything but quantities to the right side:

$$\left(\frac{q_{i,t}}{q_{j,t}}\right)^{\rho-1} = \left[\frac{\theta_j \omega_j [(1-\omega_j)/\omega_j]^{(1-\omega_j)\rho}}{\theta_i \omega_i [(1-\omega_i)/\omega_i]^{(1-\omega_i)\rho}} \right] p_{j,t}^{\rho-\rho \omega_j-1} p_{i,t}^{1-\rho+\rho \omega_i} w_t^{\rho(\omega_i-\omega_j)} \left[\frac{z_{j,t}}{z_{i,t}} \right]^\rho \quad (\text{A.11})$$

Rewrite the left side so that $q_{j,t}$ is on top, then raise both sides to $\frac{1}{1-\rho}$ power to get

$$\left(\frac{q_{j,t}}{q_{i,t}}\right) = \left[\frac{\theta_i \omega_i [(1-\omega_i)/\omega_i]^{(1-\omega_i)\rho}}{\theta_j \omega_j [(1-\omega_j)/\omega_j]^{(1-\omega_j)\rho}} \right]^{\frac{1}{\rho-1}} p_{j,t}^{\frac{1-\rho+\rho \omega_j}{\rho-1}} p_{i,t}^{\frac{1-\rho+\rho \omega_i}{1-\rho}} w_t^{\frac{\rho(\omega_i-\omega_j)}{\rho-1}} \left[\frac{z_{i,t}}{z_{j,t}} \right]^{\frac{\rho}{\rho-1}} \quad (\text{A.12})$$

□

Corollary 1. The same procedure can be applied as in Proposition 1 to express the time devoted toward production process j relative to process i as follows:

$$\left(\frac{n_{j,t}}{n_{i,t}}\right) = \left[\frac{\theta_i (1-\omega_i) [\omega_i/(1-\omega_i)]^{\rho \omega_i}}{\theta_j (1-\omega_j) [\omega_j/(1-\omega_j)]^{\rho \omega_j}} \right]^{\frac{1}{\rho-1}} p_{j,t}^{\frac{\rho \omega_j}{\rho-1}} p_{i,t}^{\frac{\rho \omega_i}{1-\rho}} w_t^{\frac{\rho(\omega_i-\omega_j)}{\rho-1}} \left[\frac{z_{i,t}}{z_{j,t}} \right]^{\frac{\rho}{\rho-1}} \quad (\text{A.13})$$

Proof. Start with the marginal rate of substitution for time use between processes i and j :

$$\frac{\frac{\partial \tilde{u}}{\partial n_{i,t}}}{\frac{\partial \tilde{u}}{\partial n_{j,t}}} = 1 \quad (\text{A.14})$$

$$\Rightarrow \tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)^{1-\rho} (1-\omega_i) \theta_i z_{i,t}^\rho q_{i,t}^{\rho \omega_i} n_{i,t}^{\rho-\rho \omega_i} \left(\frac{1}{n_{i,t}}\right) = \tilde{u}_t(\mathbf{q}_t, \mathbf{n}_t)^{1-\rho} \omega_j \theta_j z_{j,t}^\rho q_{j,t}^{\rho \omega_j} n_{j,t}^{\rho-\rho \omega_j} \left(\frac{1}{n_{j,t}}\right) \quad (\text{A.15})$$

Substitute (18) for each good and cancel like terms to get

$$(1 - \omega_i) \theta_i z_{i,t}^\rho (\lambda_{q_{i,t}} w_t \omega_i / [p_{i,t}(1 - \omega_i)] n_{i,t})^{\rho \omega_i} (\lambda_{n_{i,t}} n_{i,t})^{\rho - \rho \omega_i} \left(\frac{1}{n_{i,t}} \right) =$$

$$(1 - \omega_j) \theta_j z_{j,t}^\rho (\lambda_{q_{j,t}} w_t \omega_j / [p_{j,t}(1 - \omega_j)] n_{j,t})^{\rho \omega_j} (\lambda_{n_{j,t}} n_{j,t})^{\rho - \rho \omega_j} \left(\frac{1}{n_{j,t}} \right) \quad (\text{A.16})$$

Collect like terms and isolate relative time use on the left hand side:

$$\left(\frac{n_{i,t}}{n_{j,t}} \right)^{\rho-1} = \left[\frac{\theta_j (1 - \omega_j) [\omega_j / (1 - \omega_j)]^{\rho \omega_j}}{\theta_i (1 - \omega_i) [\omega_i / (1 - \omega_i)]^{\rho \omega_i}} \right] p_{j,t}^{-\rho \omega_j} p_{i,t}^{\rho \omega_i} w_t^{\rho(\omega_j - \omega_i)} \left[\frac{z_{j,t}}{z_{i,t}} \right]^\rho \quad (\text{A.17})$$

Rewrite the left side so that $n_{j,t}$ is on top, then raise both sides to $\frac{1}{1-\rho}$ power to get

$$\left(\frac{n_{j,t}}{n_{i,t}} \right) = \left[\frac{\theta_i (1 - \omega_i) [\omega_i / (1 - \omega_i)]^{\rho \omega_i}}{\theta_j (1 - \omega_j) [\omega_j / (1 - \omega_j)]^{\rho \omega_j}} \right]^{\frac{1}{\rho-1}} p_{j,t}^{\frac{\rho \omega_j}{\rho-1}} p_{i,t}^{\frac{\rho \omega_i}{1-\rho}} w_t^{\frac{\rho(\omega_i - \omega_j)}{\rho-1}} \left[\frac{z_{i,t}}{z_{j,t}} \right]^{\frac{\rho}{\rho-1}} \quad (\text{A.18})$$

□

Lemma 2. Suppose $\rho = 0$ so that $u(c_t)$ is Cobb-Douglas. Then

$$\frac{p_{g,t} q_{g,t}}{p_{s,t} q_{s,t}} = \frac{\theta_g \omega_g}{\theta_s \omega_s} \in \mathbb{R}_+ \quad (\text{A.19})$$

That is, expenditure ratios are constant over time.

Proof. If $u(c_t)$ is Cobb-Douglas, then $\tilde{u}_t(q_t, n_t)$ is

$$\tilde{u}_t(q_t, n_t) = \left[z_{g,t} q_{g,t}^{\omega_g} n_{g,t}^{1-\omega_g} \right]^{\theta_g} \left[z_{s,t} q_{s,t}^{\omega_s} n_{s,t}^{1-\omega_s} \right]^{\theta_s} \quad (\text{A.20})$$

and marginal utilities are

$$\frac{\partial \tilde{u}_t}{\partial q_{i,t}} = \frac{\theta_i \omega_i \tilde{u}_t(q_t, n_t)}{q_{i,t}} \quad \forall i \in \{g, s\} \quad (\text{A.21})$$

Thus the marginal rate of substitution for market commodities is

$$\frac{\frac{\partial \tilde{u}_t}{\partial q_{s,t}}}{\frac{\partial \tilde{u}_t}{\partial q_{g,t}}} = \frac{q_{g,t} \theta_s \omega_s}{q_{s,t} \theta_g \omega_g} \quad (\text{A.22})$$

$$= \frac{p_{s,t}}{p_{g,t}} \quad (\text{A.23})$$

$$\Rightarrow \frac{p_{g,t} q_{g,t}}{p_{s,t} q_{s,t}} = \frac{\theta_g \omega_g}{\theta_s \omega_s} \quad (\text{A.24})$$

□

Proposition 2. Assume $\omega_i \in (0, 1)$ for all $i \in \{g, s\}$ and suppose $\rho \neq 0$. Then one of the following reduced-form restrictions must hold:

i. Final goods are substitutes so that $\rho \in (0, 1)$ and:

$$\begin{aligned}\beta_1 &\in (-\infty, -1) \\ \beta_2 &\in (1, \infty) \\ \beta_3 &\in (-\infty, \infty)\end{aligned}\tag{A.25}$$

ii. Final goods are complements so that $\rho \in (-\infty, 0)$ and:

$$\begin{aligned}\beta_1 &\in (-1, 0) \\ \beta_2 &\in (0, 1) \\ \beta_3 &\in (-1, 1)\end{aligned}\tag{A.26}$$

For completeness, $\beta_0 \in (-\infty, \infty)$ and $\beta_4 \in (0, \infty)$ always.

Proof. i. Final goods are substitutes so that $\rho \in (0, 1)$. Start with $\beta_1 = \frac{1-\rho+\rho\omega_g}{\rho-1}$. Assume ρ is interior.

$$\lim_{\omega_g \rightarrow +0} \frac{1-\rho+\rho\omega_g}{\rho-1} = -1 \quad \text{and} \quad \lim_{\omega_g \rightarrow -1} \frac{1-\rho+\rho\omega_g}{\rho-1} = \frac{1}{\rho-1}\tag{A.27}$$

Now take ρ to 0 and 1, holding ω_g interior and fixed. As $\rho \rightarrow -1$, $\frac{1}{\rho-1} \rightarrow -\infty$. So

$$\begin{aligned}\lim_{\rho \rightarrow +0} \frac{1-\rho+\rho\omega_g}{\rho-1} &= -1 \\ \text{and} \quad \lim_{\rho \rightarrow -1} \frac{1-\rho+\rho\omega_g}{\rho-1} &= -\infty\end{aligned}\tag{A.28}$$

For $\rho \in (0, 1)$, these cases are exhaustive. Thus $\beta_1 \in (-\infty, -1)$.

Consider $\beta_2 = \frac{1-\rho+\rho\omega_s}{1-\rho}$. Again, assume ρ is interior. By applying the same logic as in (A.27) and (A.28) and reversing the sign, $\beta_2 \rightarrow 1$ as $\omega_s \rightarrow +0$, $\beta_2 \rightarrow \frac{1}{1-\rho}$ as $\omega_s \rightarrow -1$, $\beta_2 \rightarrow 1$ as $\rho \rightarrow +0$, and $\beta_2 \rightarrow \infty$ as $\rho \rightarrow -1$ with $\frac{1}{1-\rho} \rightarrow \infty$ as $\rho \rightarrow -1$. Thus $\beta_2 \in (1, \infty)$.

Now consider $\beta_3 = \frac{\rho(\omega_s - \omega_g)}{\rho-1}$. First, assume ω_s and ω_g are interior and send ρ to its endpoints:

$$\lim_{\rho \rightarrow +0} \frac{\rho(\omega_s - \omega_g)}{\rho-1} = 0 \quad \text{and} \quad \lim_{\rho \rightarrow -1} \frac{\rho(\omega_s - \omega_g)}{\rho-1} = -\infty\tag{A.29}$$

Now assume ω_g is interior. Note that

$$\lim_{\rho \rightarrow -1} \lim_{\omega_s \rightarrow +0} \frac{\rho(\omega_s - \omega_g)}{\rho-1} = -\infty \quad \text{and} \quad \lim_{\rho \rightarrow -1} \lim_{\omega_s \rightarrow +0} \frac{\rho(\omega_s - \omega_g)}{\rho-1} = \infty\tag{A.30}$$

The same procedure can be applied assuming ω_s is interior and sending ω_g to its endpoints, flipping the sign. It follows that $\beta_3 \in (-\infty, \infty)$ if $\rho \in (0, 1)$.

- ii. Final goods are complements so that $\rho \in (-\infty, 0)$. Again, start with $\beta_1 = \frac{1-\rho+\rho\omega_g}{\rho-1}$. Consider the double limits:

$$\lim_{\rho \rightarrow -0} \lim_{\omega_g \rightarrow +0} \frac{1-\rho+\rho\omega_g}{\rho-1} = \lim_{\rho \rightarrow -0} \lim_{\omega_g \rightarrow -1} \frac{1-\rho+\rho\omega_g}{\rho-1} = -1 \quad (\text{A.31})$$

Taking ρ to $-\infty$, note that

$$\lim_{\omega_g \rightarrow +0} \lim_{\rho \rightarrow -\infty} \frac{1-\rho+\rho\omega_g}{\rho-1} = \frac{\infty}{-\infty} \quad (\text{A.32})$$

$$\Rightarrow \text{By L'Hopital's Rule: } \lim_{\omega_g \rightarrow +0} \lim_{\rho \rightarrow -\infty} (\omega_g - 1) = -1$$

$$\text{and } \lim_{\rho \rightarrow -\infty} \lim_{\omega_g \rightarrow -1} \frac{1-\rho+\rho\omega_g}{\rho-1} = 0 \quad (\text{A.33})$$

Thus, $\beta_1 \in (-1, 0)$.

Turning to β_2 , the logic is the same except the signs are flipped, so $\beta_2 \in (0, 1)$.

Consider $\beta_3 = \frac{\rho(\omega_s - \omega_g)}{\rho-1}$. First, assume ω_s and ω_g are interior. Note that as $\rho \rightarrow -0$, $\beta_3 \rightarrow 0$. Consider the following

$$\lim_{\rho \rightarrow -\infty} \frac{\rho(\omega_s - \omega_g)}{\rho-1} = \frac{\infty}{\infty} \quad (\text{A.34})$$

$$\Rightarrow \text{By L'Hopital's Rule: } \lim_{\rho \rightarrow -\infty} (\omega_s - \omega_g) = (\omega_s - \omega_g)$$

Now send ω_s and ω_g to their endpoints, using the result of applying L'Hopital's Rule to the limit as $\rho \rightarrow -\infty$. Clearly since $\omega_i \in (0, 1)$, $\forall i \in \{g, s\}$, then $(\omega_s - \omega_g) \in (-1, 1)$. Thus $\beta_3 \in (-1, 1)$ when $\rho \in (-\infty, 0)$.

For completeness, let us now show that β_0 can take any real value, regardless of ρ . We require $\theta_s > 0$. Recall

$$\beta_0 = \left(\frac{1}{\rho-1} \right) \ln \left[\frac{\theta_s \omega_s [(1-\omega_s)/\omega_s]^{(1-\omega_s)\rho}}{\omega_g [(1-\omega_g)/\omega_g]^{(1-\omega_g)\rho}} \right] \quad (\text{A.35})$$

Using the additive property of logarithms:

$$\begin{aligned} \beta_0 = \left(\frac{1}{\rho-1} \right) & \left[\ln \theta_s + \ln \omega_s - \ln \omega_g + (1-\omega_s)\rho (\ln(1-\omega_s) - \ln \omega_s) \right. \\ & \left. - (1-\omega_g)\rho (\ln(1-\omega_g) - \ln \omega_g) \right] \end{aligned} \quad (\text{A.36})$$

Suppose θ_s is finite and ω_s and ω_g are interior. As $\rho \rightarrow 0$ from both sides, β_0 can be any positive or negative real number depending on the values of θ_s , ω_s , and ω_g . As $\rho \rightarrow -1$, $\beta_0 \rightarrow -\infty$. Now consider the case where θ_s , ω_s , and ω_g are finite and $\rho \rightarrow -\infty$:

$$\lim_{\rho \rightarrow -\infty} \left(\frac{1}{\rho - 1} \right) \left[\ln \theta_s + \ln \omega_s - \ln \omega_g + (1 - \omega_s) \rho (\ln(1 - \omega_s) - \ln \omega_s) - (1 - \omega_g) \rho (\ln(1 - \omega_g) - \ln \omega_g) \right] = \pm \frac{\infty}{\infty} \quad (\text{A.37})$$

where the sign of the limit depends on whether ω_s and ω_g are less than or greater than $\frac{1}{2}$. Applying L'Hôpital's Rule:

$$\begin{aligned} & \lim_{\rho \rightarrow -\infty} \left[(1 - \omega_s) (\ln(1 - \omega_s) - \ln \omega_s) - (1 - \omega_g) (\ln(1 - \omega_g) - \ln \omega_g) \right] \\ &= \lim_{\rho \rightarrow -\infty} (1 - \omega_s) (\ln(1 - \omega_s) - \ln \omega_s) - \lim_{\rho \rightarrow -\infty} (1 - \omega_g) (\ln(1 - \omega_g) - \ln \omega_g) \end{aligned} \quad (\text{A.38})$$

Now consider the double limits:

$$\lim_{\omega_i \rightarrow +0} \lim_{\rho \rightarrow -\infty} (1 - \omega_i) (\ln(1 - \omega_i) - \ln \omega_i) = -\infty \quad \forall i \in \{g, s\} \quad (\text{A.39})$$

$$\begin{aligned} & \lim_{\omega_i \rightarrow -1} \lim_{\rho \rightarrow -\infty} (1 - \omega_i) (\ln(1 - \omega_i) - \ln \omega_i) \\ &= \lim_{\omega_i \rightarrow -1} \lim_{\rho \rightarrow -\infty} \frac{\ln(1 - \omega_i) - \ln \omega_i}{\frac{1}{1 - \omega_i}} = \frac{-\infty}{\infty} \\ \Rightarrow \text{By L'Hôpital's Rule} & \lim_{\omega_i \rightarrow -1} \lim_{\rho \rightarrow -\infty} \frac{-\frac{1}{1 - \omega_i} - \frac{1}{\omega_i}}{\frac{1}{(1 - \omega_i)^2}} \\ &= \lim_{\omega_i \rightarrow -1} \lim_{\rho \rightarrow -\infty} \frac{\omega_i - 1}{\omega_i} = 0 \quad \forall i \in \{g, s\} \end{aligned} \quad (\text{A.40})$$

Depending on whether ω_g or ω_s goes to 0, (A.39) implies that $\beta_0 \rightarrow \pm\infty$. Sending $\omega_i \rightarrow 1$ sends the corresponding term to 0, so this limiting condition does not affect the bounds for β_0 . The cases here are enough to show $\beta_0 \in (-\infty, \infty)$ for any feasible ρ .

Finally, $\beta_4 \in (0, \infty)$ since $0 < (1 - \omega_i)/\omega_i < \infty$ always, and

$$\lim_{\omega_i \rightarrow +0} (1 - \omega_i)/\omega_i = \infty \quad (\text{A.41})$$

$$\lim_{\omega_i \rightarrow -1} (1 - \omega_i)/\omega_i = 0 \quad (\text{A.42})$$

Thus both the numerator and denominator of $\beta_4 = \frac{(1 - \omega_s)/\omega_s}{(1 - \omega_g)/\omega_g}$ independently range over the positive real line, so $\beta_4 \in (0, \infty)$.

□

Proposition 3. The elasticities of relative consumption with respect to prices have opposite sign from the elasticities of relative time use with respect to prices if and only if final goods are complements, so that $\rho \in (-\infty, 0)$.

Proof. ($\Rightarrow$) If $\rho \in (-\infty, 0)$ then clearly from Proposition 2, $\beta_1 \in (-1, 0)$ and $\beta_2 \in (0, 1)$ thus implying $\beta_1 + 1 > 0$ and $\beta_2 - 1 < 0$.

($\Leftarrow$) Suppose elasticities have opposite sign, so that $\beta_1 = \frac{1-\rho+\rho\omega_g}{\rho-1} < 0$ and $\frac{1-\rho+\rho\omega_g}{\rho-1} + 1 > 0$. Assume $\omega_g \in (0, 1)$ and recall that $\rho < 1$. Then referring solely to the elasticity of relative time use with respect to $p_{g,t}$

$$\frac{1 - \rho + \rho\omega_g}{\rho - 1} + 1 > 0 \quad (\text{A.43})$$

$$\Leftrightarrow 1 - \rho + \rho\omega_g < 1 - \rho \quad (\text{A.44})$$

$$\Leftrightarrow \rho\omega_g < 0 \quad (\text{A.45})$$

Thus final goods are complements since $\rho \in (-\infty, 0)$. $\square$

B Estimation Details

B.1 Description of Hamiltonian Monte Carlo

For a thorough and readable, detailed treatment of Hamiltonian Monte Carlo (HMC), we recommend reading Neal (2011). For an equally thorough, more technical treatment, read Betancourt and Stein (2011). For a brief overview of HMC see Gelman et al. (2013b). Here, we provide the “Reader’s Digest” overview of the sampler’s properties and benefits since there are very few examples of this particular estimation technique being used in the econometrics literature.¹⁹ This Appendix is not intended to be a full treatment, only an outline.

Recall, in traditional Bayesian MCMC posterior sampling (i.e. Gibbs sampling or Metropolis-Hastings sampling), the state of the sampler at iteration m is the m^{th} draw of the parameter set $\mathcal{P}^m$ from the posterior distribution with density $\pi(\mathcal{P} \mid \mathcal{D}, \mathcal{H})$. Thus a traditional sampler is a Markov Chain operating in a discrete, countable sampling space, with discrete dynamics. For example, the m^{th} draw of a Metropolis-Hastings sampler depends on the $m - 1$ draw, and the econometrician knows this distribution has converged when autocorrelation between draws has been sufficiently minimized, depending on the nature of the estimation problem. HMC extends this concept to a sample space with continuous dynamics.²⁰ In Hamiltonian dynamics commonly employed in physical mechanics, the state of the system depends on both the forward “momentum” of the

¹⁹The only published example in the economics literature we were able to find is Burda (2015). Martin Burda also has a recent working paper providing an application of HMC to a discrete choice model (Burda and Daviet 2018).

²⁰Recall, “dynamics” in this context refer to the dynamics of the parameter sampler, *not* actual time, as in the modeled structural dynamics of the underlying agent’s decision process. One dynamic operates over the sampler dimension m while the other over the data dimension t . This can be confusing when estimating models wrought from a dynamic decision process or which have dynamic time series components.

system $\mathcal{Q}$ and the “position” of the system $\mathcal{P}$, each of the same dimension. The Hamiltonian equation associated with this system is the sum of “potential” $U(\cdot)$ and “kinetic” $K(\cdot)$ energy:

$$H(\mathcal{Q}, \mathcal{P}) = U(\mathcal{P}) + K(\mathcal{Q}) \quad (\text{B.1})$$

The partial derivatives of this equation will determine how the parameter space $\mathcal{P}$ evolves as the sampler proceeds, as well as the rate of this evolution $\mathcal{Q}$.

Since the “dynamics” of our particular task involve traversing a parameter space over which some posterior density function $\pi(\mathcal{P} | \mathcal{D})$ is defined, the potential energy function for HMC is just the negative log of the right hand side of (62):

$$U(\mathcal{P}) = -\ln \left[\pi(\mathcal{D} | \mathcal{P}) \pi(\mathcal{P}) \right] \quad (\text{B.2})$$

Note that we can sample from this using HMC only if $U(\mathcal{P})$ is continuously differentiable over the entire sample space since HMC operates on Hamilton’s equations which require computation of the gradient vectors for both $U(\cdot)$ and $K(\cdot)$. Let $\#$ denote the cardinality of a countable set. In practice, the kinetic energy function is defined conditional on $\mathcal{P}$ and taken to be a quadratic of the form:²¹

$$K(\mathcal{Q} | \mathcal{P}) = \frac{1}{2} \sum_{k=1}^{\#\mathcal{Q}} \sum_{r=1}^{\#\mathcal{Q}} \mathcal{Q}_k \mathcal{Q}_r \Lambda_{k,r}(\mathcal{P}) - \frac{1}{2} \ln \left[\det(\Lambda(\mathcal{P})) \right] \quad (\text{B.3})$$

where $\Lambda(\mathcal{P})$ is what is called the mass matrix and may be constant. It could also be restricted to the identity matrix or simply be diagonal. At most it is a dense symmetric positive definite matrix that represents the variance/covariance of an underlying conditional Gaussian distribution function for the momentum vector:

$$\mathcal{Q} | \mathcal{P} \sim \mathcal{N}(\mathbf{0}, \Lambda(\mathcal{P})) \quad (\text{B.4})$$

A dense $\Lambda(\mathcal{P})$ can help account for high local non-linearities in $U(\mathcal{P})$, though can be difficult to compute (Betancourt and Stein 2011). In our estimation, we allow $\Lambda(\mathcal{P})$ to be diagonal and tune $\Lambda(\mathcal{P})$ during a warm-up period using Stan’s HMC implementation (Stan Development Team 2016).²²

Having defined the objects on which we operate, we can summarize the algorithm described in detail in Neal (2011). Here is where the kinetic energy component of the Hamiltonian helps greatly speed up convergence and reduce autocorrelation in our sampling routine. After a suffi-

²¹This is a more flexible form of the kinetic energy function than that in Neal (2011). See Betancourt and Stein (2011) for more details.

²²Stan is free software for high-performance HMC which utilizes automatic differentiation, LAPACK, and BLAS libraries for computational efficiency and which can be implemented and executed in a number of top-end scientific computing programs like R, Python, Matlab, or executed simply from the shell. See <http://mc-stan.org>. For examples using the Stan language to execute HMC models, see Gelman et al. (2013a).

cient warm-up period,^{23,24} a sampling step proceeds as follows:

- i. Given the position from the previous iteration $\mathcal{P}$, draw a new $\mathcal{Q}$ from (B.4). In a sense (B.4) is thus the HMC analog of what is commonly called a “proposal distribution” for an MCMC Metropolis-Hastings algorithm.
- ii. Given $(\mathcal{Q}, \mathcal{P})$, iterate on Hamilton’s equations using the leapfrog method for L steps to get a pair $(\mathcal{Q}', \mathcal{P}')$.²⁵ This is your proposed new state.
- iii. Similar to the Metropolis-Hastings algorithm, draw a uniform random deviate $u \sim \mathcal{U}[0, 1]$ and accept $(\mathcal{Q}', \mathcal{P}')$ as the next state if

$$u < \exp \{ -H(\mathcal{Q}', \mathcal{P}') + H(\mathcal{Q}, \mathcal{P}) \} \quad (\text{B.5})$$

HMC is most useful when sampling from $\pi(\mathcal{P} \mid \mathcal{D})$ is computationally burdensome and may require an incredibly long chain of discrete draws in order to achieve convergence. This happens in cases where posterior draws are likely to be highly autocorrelated in a traditional MCMC, especially when using Metropolis-Hastings algorithms where high autocorrelation may also require low acceptance probabilities, thus further giving reason for a long sampling chain.

²³This is akin to the “burnin” period for an MCMC operating under discrete dynamics.

²⁴This is handled automatically by Stan and discussed thoroughly in the software manual, (Stan Development Team 2016).

²⁵See Neal (2011) for a detailed description of this iterative method and how it operates on Hamilton’s equations.

References

- Aguiar, Mark, and Erik Hurst. 2007. "Life-Cycle Prices and Production". *The American Economic Review* 97 (5). (Cit. on pp. 1, 3).
- Aguiar, Mark, Erik Hurst, and Loukas Karabarbounis. 2013. "Time Use During the Great Recession". *The American Economic Review* 103 (5). (Cit. on p. 3).
- Barnett, William. 1977. "Pollak and Wachter on the Household Production Function Approach". *Journal of Political Economy* 85 (5). (Cit. on p. 5).
- Becker, Gary. 1965. "A Theory of the Allocation of Time". *The Economic Journal* 75 (299). (Cit. on pp. 2, 3).
- Benhabib, Jess, Richard Rogerson, and Randall Wright. 1991. "Homework in Macroeconomics: Household Production and Aggregate Fluctuations". *Journal of Political Economy* 99 (6). (Cit. on p. 3).
- Betancourt, Michael, and Leo Stein. 2011. "The Geometry of Hamiltonian Monte Carlo". *arXiv:1112.4118*. (Cit. on pp. 14, 36, 37).
- Boppart, Timo. 2014. "Structural Change and the Kaldor Facts in a Growth Model with Relative Price Effects and Non-Gorman Preferences". *Econometrica* 82 (6). (Cit. on pp. 12, 15, 22).
- Buera, Francisco, and Joseph Kaboski. 2012. "The Rise of the Service Economy". *The American Economic Review* 102 (6). (Cit. on p. 12).
- Burda, Martin. 2015. "Constrained Hamiltonian Monte Carlo in BEKK GARCH with Targeting". *Journal of Time Series Econometrics* 7 (1). (Cit. on p. 36).
- Burda, Martin, and Remi Daviet. 2018. "Hamiltonian Sequential Monte Carlo with Application to Consumer Choice Behavior". *University of Toronto, Department of Economics Working Paper #618*. (Cit. on p. 36).
- Bureau of Economic Analysis. 2017. "National Income and Product Accounts (NIPA): Tables 1.1.4, 1.1.5, 1.1.6, 1.5.4, 1.5.5, 2.4.3". Visited on 06/27/2017. <https://www.bea.gov/national/>. (Cit. on p. 18).
- Bureau of Labor Statistics. 2017. "American Time Use Survey (ATUS)". Visited on 11/20/2017. <https://www.bls.gov/tus/home.htm>. (Cit. on pp. 2, 18).
- Caselli, Francesco, and Wilbur Coleman. 2001. "The U.S. Structural Transformation and Regional Convergence: A Reinterpretation". *Journal of Political Economy* 109 (3). (Cit. on p. 15).
- Echevarria, Cristina. 1997. "Changes in Sectoral Composition Associated with Economic Growth". *International Economic Review* 38 (2). (Cit. on p. 15).
- Gelman, Andrew, et al. 2013a. *Computation in R and Stan*. (Cit. on pp. 14, 20, 37).
- . 2013b. *Computationally efficient Markov chain simulation*. (Cit. on pp. 14, 20, 36).

- Gomme, Paul, Finn Kydland, and Peter Rupert. 2001. "Home Production Meets Time to Build". *Journal of Political Economy* 109 (5). (Cit. on p. 3).
- Goolsbee, Austan, and Peter Klenow. 2006. "Valuing consumer products by the time spent using them: An application to the Internet". *American Economic Review* 96 (2): 108–113. (Cit. on p. 1).
- Gordon, Robert. 2016. *The Rise and Fall of American Growth*. Princeton University Press. (Cit. on pp. 23, 29).
- Gorman, WM. 1961. "On a class of preference fields". *Metroeconomica* 13 (2): 53–56. (Cit. on p. 6).
- Graham, John, and Carole Green. 1984. "Estimating the Parameters of a Household Production Function with Joint Products". *The Review of Economics and Statistics* 66 (2). (Cit. on p. 3).
- Green, H.A. John. 1964. *The Concept of Functional Separability*. (Cit. on pp. 7, 30).
- Gronau, Quentin, et al. 2017. "A Tutorial on Bridge Sampling". *arXiv:1703.05984v2*. (Cit. on p. 20).
- Gronau, Reuben. 1977. "Leisure, Home Production, and Work – the Theory of the Allocation of Time Revisited". *Journal of Political Economy* 85 (6). (Cit. on p. 3).
- Herrendorf, Berthold, Richard Rogerson, and Akos Valentinyi. 2013. "Two Perspectives on Preferences and Structural Transformation". *American Economic Review* 103 (7). (Cit. on pp. 12, 15, 22).
- Ingram, Beth, Narayana Kocherlakota, and N.E. Savin. 1997. "Using theory for measurement: An analysis of the cyclical behavior of home production". *Journal of Monetary Economics* 40:435–456. (Cit. on p. 3).
- Jorgenson, Dale, Mun Ho, and Jon Samuels. 2016. "Education, Participation, and the Revival of U.S. Economic Growth". *NBER Working Paper No. 22453*. (Cit. on pp. 2, 18, 19).
- Kongsamut, Piyabha, Sergio Rebelo, and Danyang Xie. 2001. "Beyond Balanced Growth". *Review of Economic Studies* 68:869–882. (Cit. on p. 15).
- Kuznets, Simon. 1966. *Economic growth and structure*. Heinemann Educational. (Cit. on p. 16).
- Lee, Kiseok, and Shawn Ni. 2002. "On the dynamic effects of oil price shocks: a study using industry level data". *Journal of Monetary Economics* 29. (Cit. on p. 24).
- Leontief, Wassily. 1947. "Introduction to a Theory of the Internal Structure of Functional Relationships". *Econometrica* 15 (4). (Cit. on pp. 7, 30).
- Lewandowski, Daniel, Dorota Kurowicka, and Harry Joe. 2009. "Generating random correlation matrices based on vines and extended onion method". *Journal of Multivariate Analysis* 100 (9). (Cit. on p. 14).
- McGrattan, Ellen, Richard Rogerson, and Randall Wright. 1993. "Household Production and Taxation in the Stochastic Growth Model". *Federal Reserve Bank of Minneapolis: Staff Report #166*. (Cit. on p. 3).

- Meng, Xiao-Li, and Wing Hung Wong. 1996. "Simulating Ratios of Normalizing Constants Via a Simple Identity: A Theoretical Exploration". *Statistica Sinica* 6. (Cit. on p. 20).
- Neal, Radford. 2011. *MCMC using Hamiltonian dynamics*. Ed. by Steve Brooks et al. (Cit. on pp. 14, 36–38).
- Ngai, Rachel, and Christopher Pissarides. 2007. "Structural Change in a Multisector Model of Growth". *American Economic Review* 97 (1). (Cit. on p. 12).
- Nordhaus, William. 2004. "Retrospective on the Postwar Productivity Slowdown". *Cowles Foundation Discussion Paper No. 1494*. (Cit. on p. 24).
- Pollak, Robert, and Michael Wachter. 1975. "The Relevance of the Household Production Function and Its Implications for the Allocation of Time". *Journal of Political Economy* 83 (2). (Cit. on p. 5).
- Rossi, Peter, Greg Allenby, and Robert McCulloch. 2005. *Model Choice and Decision Theory*. (Cit. on p. 20).
- Rupert, Peter, Richard Rogerson, and Randall Wright. 1995. "Estimating Substitution Elasticities in Household Production Models". *Economic Theory* 6 (1). (Cit. on p. 3).
- Stan Development Team. 2016. *Stan Modeling Language: User's Guide and Reference Manual*. (Cit. on pp. 14, 20, 37, 38).
- Syverson, Chad. 2017. "Challenges to Mismeasurement Explanations for the US Productivity Slowdown". *Journal of Economic Perspectives* 31 (2). (Cit. on p. 29).
- Wasserman, Larry. 2000. "Bayesian Model Selection and Model Averaging". *Journal of Mathematical Psychology* 44. (Cit. on p. 22).