Money Mining and Price Dynamics*

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This version: December 2018

Abstract

We develop a random-matching model where the private production of money results from an occupation choice. While there exists a unique perfect-foresight equilibrium where the value of money reaches a steady state, there is a continuum of equilibria where the value of money inflates and bursts. In the early stage of the transition, money is not used as means of payment and its value grow at a rate equal or larger than the rate of time preference. We study divisible, indivisible, interest-bearing, and competing monies, the role of mining to confirm trades, and the implementation of the constrained-efficient allocation.

Keywords: Money, Search, Private Monies, Mining

JEL codes: E40, E50

*We thank for their comments seminar participants at UC Riverside, UC Irvine, and the 2018 West Coast Search-and-Matching Workshop at UC Irvine. Usual disclaimers apply.

1 Introduction

Since the creation of Bitcoins a decade ago, hundreds of new crypto-currencies – digital objects with no intrinsic value whose ownership can be transferred through encryption techniques – have been introduced. The total market capitalisation of crypto-assets peaked at \$830 billion at the beginning of 2018, with an estimated \$290 billion for Bitcoins alone (Financial Stability Board, 2018), see Figure 1. Over the year, Bitcoin lost 70% of its value while other cryptocurrencies lost on average 95% of their value since the peak. Hence, some view cryptocurrencies as "*the mother of all bubbles*" (Roubini, 2018) and believe they do not contribute to the common good while others view them as useful financial innovations.¹ Irrespective of one's views, this new phenomenon has brought to the forefront some of the foundational questions of monetary theory. Can intrinsically useless objects serve as media of exchange and have a positive value? Are all privately-produced monies bubbles condemned to burst? How is the initial value of a new money determined? Is the private production of money socially efficient?

Figure 1: (Left) Market capitlization of various cryptocurrencies. (Right) Market capitalization in different equilibria in our model.

The object of this paper is to revisit these questions by studying the dynamics of an economy where money is privately produced through mining — a costly and time-consuming occupation. Our theory applies to the mining of commodity monies, e.g., gold and silver, as well as the production of fiat currencies, e.g., Bitcoins. Because the determination of currency prices is better understood in the context of models where there is

¹For instance, Jean Tirole (Financial Times, 11/29/2017) asks and answers the following questions: “Bitcoin raises two distinct questions. Is it sustainable? And assuming it is, does it contribute to the common good? My answers are: probably not (the jury is still out) and definitely not.”

an essential role for a medium of exchange, we adopt the search-theoretic model of monetary exchange of Shi (1995) and Trejos and Wright (1995). In this environment, trades take place within pairwise meetings that are formed randomly. Heterogeneity in preferences and specialization in production generate a lack-of-double-coincidence-of-wants problem and rule out barter trades. In addition, agents, who are anonymous, cannot finance random consumption opportunities by issuing private debts, hence a role for money. Money is indivisible and there is an upper bound on money holdings. While this assumption was originally made for tractability, it captures the notion that the quantity of liquid assets is scarce and affects the number of transactions in fully decentralized markets.² We will also study a version of the model with perfectly divisible interest-bearing money where agents can readjust their asset holdings in competitive exchanges in continuous time, in a similar spirit as in Lagos and Wright (2005). In this version liquidity can be abundant and it can be redistributed across agents periodically, thereby mitigating any shortage. We will show that our results are robust to this extension.

We add two components to the New Monetarist model. First, we introduce a mining technology allowing agents to unearth units of money. This technology is time consuming and such that the speed at which units of money are dug decreases with the quantity of money already discovered. Second, we add an occupation choice allowing us to endogenize the opportunity cost of mining. Agents can either be producers looking for opportunities to sell goods in pairwise meetings or they can be miners searching for opportunities to dig units of money. The formalization of mining as an occupation choice is realistic in the case of gold and silver. It is also relevant in the case of cryptocurrencies as the computers use to mine coins can have alternative uses to produce goods and services, the value of which depends on the overall liquidity in the economy. One can also interpret mining as an occupation in the finance industry to produce new assets that satisfy investors' liquidity needs. For completeness, we also investigate alternative assumptions where agents can look for trading partners while mining.

Our answers to our initial questions are as follows. We show that fiat money can be valued in the long run if agents are sufficiently patient, trading frictions are not too severe, and the market power of producers is not too large. In addition, the maximum amount of money that can be mined and money discoveries cannot happen too frequently. This finding captures the common wisdom that objects that serve as means of

²In theory, crypto-currencies are perfectly divisible. In practice, though, divisibility is limited. For instance, a bitcoin is divisible by eight decimal points. The smallest unit is called the Satoshi, so that each Bitcoin can be broken down into 10^8 Satoshis.

payment must be scarce to maintain their value. Increases in the potential money supply or improvements in the mining technology lower the value of money but raise its supply in the long run, a negative correlation that is consistent with the quantity theory. In contrast, a reduction in producers' bargaining power raises both the value of money and its supply in the long run.

We characterize all dynamic equilibria starting from a zero money supply. The set of initial values for the new currency that are consistent with an equilibrium is a nonempty, half-closed interval. The largest value in this interval is the one corresponding to the unique equilibrium leading to a steady state. The infimum of the interval is zero. For all initial value in the interior of the interval, the equilibrium path for the value of money is first increasing and then decreasing, and it reaches zero asymptotically. So unless agents can coordinate on the highest equilibrium – one equilibrium among a continuum – the life cycle of a privately-produced currency is composed of a boom where agents mine money and a bust where agents stop mining but trade a depreciating money. For all these equilibria, the peak for the value of money is positively correlated with its initial value and the new currency disappears in the long run. So initial beliefs regarding the value of a new currency serve as a bootstrap to raise its future value, possibly toward a steady state. The existence of booms and busts in our models are not sustained by irrational behavior or beliefs: bubbles inflate and burst rationally as agents have perfect foresight. Hence, agents are willing to produce a new currency knowing with certainty it will ultimately fail. Even when the initial value is at its maximum, it might stop growing and start declining before reaching a steady state. Hence, it is impossible for an outside observer to infer from the price trajectory whether the currency will be successful in the long run or not.

A new money can be successful and have a transactional role in the long run even if does not serve as means of payment in the short run. In all equilibria, the new money does not circulate initially. We also show that this outcome is part of a constrained-efficient allocation. In the initial phase of the introduction of the new currency, its price grows at a rate equal or larger than the rate of time preference. For an outside observer the new currency looks like a speculative bubble since it is only held for its capital gains. It is only when money is sufficiently abundant that agents start trading with it and that the transaction velocity of money increases, which reduces its rate of appreciation as lower capital gains are compensated for by liquidity services.

Crypto-currencies have been introduced to mimic the role of gold as a store of value. In the context of our model we show that a new currency can be designed so as to keep its value over time. First, inflationary

equilibria where the value of money vanishes asymptotically can be eliminated by requiring that money pays a real (possibly small) dividend. For such interest-bearing monies the equilibrium converging to a steady state is unique and money can be priced above its fundamental value permanently. Second, the new currency can be designed so as to keep its value constant along the transition path. For this to be the case, the money growth rate must decrease gradually over time and it must approach zero when the money supply is close to its steady-state value. In addition, we provide a simple policy rule setting the money growth rate as a function of the value of money (on and off the equilibrium path) in order to guarantee that the equilibrium with price stability is unique. This rule raises the money growth rate when the value of money is below its target and reduces it when the value of money is above the target.

We also ask whether the equilibrium generates overproduction of money relative to what a social planner would like. We show that the planner’s allocation can be implemented with an incentive-feasible trading mechanism, with time-varying bargaining powers, provided that agents are sufficiently patient and the speed of mining is not too large.

We conclude our analysis with a divisible-money version of our model and show that the set of equilibria and dynamics of prices are qualitatively equivalent to those in the indivisible money model. We extend the analysis to the case of a constant money growth rate where the seigniorage revenue is the reward of successful miners. An increase in inflation reduces trade sizes but increases the measure of miners. We assume that miners play a role to confirm trades and enter transactions into an open ledger. As a result, the measure of successful transaction increases with the measure of miners. We show that the optimal inflation rate is positive and if the mining intensity is sufficiently large, it implements the maximum number of trades.

1.1 Empirical evidence on the mining of money

In this section we present three facts concerning money mining.

Fact 1A: Gold mining is an occupation choice. In our model, money mining is an occupation choices that diverts resources from the production of consumption goods and services. There is plenty of evidence to support this assumption in the case of commodity monies such as gold and silver. The California Gold Rush (1848–1855) is a case in point. Before the Gold Rush, the population in California was approximately 160,000 and the majority were Native Americans. The gold rush brought approximately 300,000 people to California from the rest of the world (see Britannica). As an aside, the large increase in population allowed

California to move to statehood in the Compromise of 1850 (see Santos (1998) for references). South Africa offers another example where gold mining has had a large impact on workers' occupation choices (Gilbert, 1933). After the discoveries of gold mines around 1886, the mining sector in South Africa attracted labor from other industries by offering high wages. It has been argued that the discovery of gold mines hurt South Africa's manufacturing sector significantly. According to Gilbert (1933):

“... in the case of such products as wool, skins, hides and diamonds, whose market was almost entirely foreign and whose exports all declined to 1899. It can hardly be doubted that the reason lay in the development of the gold industry, the rise of wages and prices the course of which has been traced, and the loss of a comparative advantage in these fields.”

Fact 2A: The resource put into gold and silver mining is largely driven by the price of the commodity being mined. Rockoff (1984) claims that the long-run world supply of gold is endogenous — major gold discoveries are lagged responses to changes in the real price of gold.³ To test whether the long-run world supply of money gold is endogenous, one can compare deviations from trend in the purchasing power of gold to that in the production of gold. If the production is endogenous, then one expects the deviations from trend in the purchasing power of gold would lead the corresponding changes in the production of gold.⁴ The top panel of Figure 2 confirms this conjecture in the period 1880-1970. We test this causality in a two-variable vector autoregression model and find that the purchasing power of gold Granger-causes gold production in the same time period at the 5-percent significance level.⁵ Similarly the bottom panel of Figure 2 shows that the deviations from trend in the purchasing power of silver leads the corresponding changes in the production of silver in 1985-2017. The purchasing power of silver Granger-causes production at a 5-percent significance level.⁶

Fact 3: The supply of gold and the price of gold can be positively or negatively correlated over time. Figure 3 shows the relationship between the monetary gold stock and the real gold price from 1879 to 1955. Most of the time a sharp increase in the gold stock is accompanied by a drop in the value of gold, which is consistent with the quantity theory. But from 1880 to 1890 the gold stock and the gold price increase

³Rockoff: “... many discoveries, particularly those that influenced world gold production between 1890 and 1905, should be regarded as movements along a long-run supply curve rather than as shifts in the curve.”

⁴Bordo (1981) uses a similar idea to show a rising purchasing power of gold would induce an increased in the monetary gold stock.

⁵A variable x_t Granger-cause another variable y_t if the lags of x_t are able to increase the accuracy of the prediction of y_t with respect to a forecast that considers only past values of y_t .

⁶The data on purchasing power and production are from Jastram (2009) and the World Silver Survey (2018).

Figure 2: Mine production and price of gold and silver

together. This fact is consistent with a key prediction of our model (Proposition 1 and 2).

Figure 3: Gold Stock and Price

Fact 1B: Bitcoin mining is an occupation choice. Bitcoin was created in 2008 and its software was released to public in 2009. It is often regarded as the first *cryptocurrency*, namely a currency that is unbacked and only exists electronically. Bitcoin owners can transfer coins over the internet and the transactions are recorded in a ledger that is open to the public. Bitcoin's programming decides the supply of Bitcoin and a key feature

Figure 4: Difficulty level of Bitcoin mining and price

is that the supply is predictable — the number of new coins issued in each time period is pre-determined.⁷ Any individual can “mine” new Bitcoins by using computers to solve a complex mathematical puzzle. The miner that solves the puzzle first is rewarded by certain amount of new coins. Since mining requires time and computational resources, one can view it as an occupation choice — individuals specializing in crypto assets can allocate the time and resources to other economic activities.

Fact 2B: The resource put into Bitcoin mining is largely driven by the price of Bitcoin. Bitcoin controls the aggregate money supply by varying the difficulty level of the mathematical puzzles. If there is a sudden increase in the number of miners, then the difficulty level increases accordingly so that the number of new coins remains unchanged. It follows that one can infer the intensity of the mining activities (the number of miners, how much CPU time they put into mining etc) by looking at the difficulty level of the puzzles. Figure 4 shows that the growth rate of the price of Bitcoin and that of the difficulty level of mining in 2010-2018. The growth rate of prices Granger-causes the growth rate of difficulty level at a 1-percent significance level. This suggests that the intensity of Bitcoin mining is driven by the purchasing power of Bitcoins. The data is from bitcoinity.org.

⁷Bitcoin was created by the person or people under the name “Satoshi Nakamoto”. Satoshi Nakamoto wrote: “The steady addition of a constant of amount of new coins is analogous to gold miners expending resources to add gold to circulation. In our case, it is CPU time and electricity that is expended.”

1.2 Literature review

Our benchmark model builds on the search-theoretic models of monetary exchange of Shi (1995) and Trejos and Wright (1995). Relative to these models, we add an occupation choice and a time-consuming mining activity, thereby allowing the money supply to be endogenous. We study both fiat and commodity monies and consider both equilibrium analysis and an implementation approach. The closest papers are Burdett, Trejos and Wright (2001) where the quantity of commodity money (cigarettes) is endogenous and Cavalcanti and Wallace (1999) where a subset of agents, called banks, can issue inside money (circulating IOUs). A survey of this class of model is provided by Lagos et al. (2017).

We extend the model to have divisible assets in a continuous-time version of the Lagos and Wright (2005) model, as in Rocheteau et al. (2018). The description of the asset as a Lucas tree is analogous to Geromichalos et al. (2007) and Lagos (2010). The supply of assets is endogenous as in Lagos and Rocheteau (2008), Rocheteau and Rodriguez (2014), and Geromichalos and Herrenbrueck (2018). Relative to these papers, we emphasize the creation of assets through an occupation choice and formalize an explicit, time-consuming mining technology. The study of dynamic equilibria in this class of models includes Lagos and Wright (2003) and Rocheteau and Wright (2013). The continuous-time assumption allows us to eliminate some exotic dynamics (such as cycles or chaotic dynamics) for a similar reason as in Oberfield and Trachter (2012). Our extension with two competing assets is related to Zhang (2014) and Gomis-Porqueras et al. (2017).

A closely related paper is Fernandez-Villaverde and Sanches (2018) who study currency competition in the context of the Lagos-Wright model with divisible money. Fernandez-Villaverde and Sanches study currency competition in an environment where entrepreneurs can issue distinguishable tokens. Instead, we focus on characterizing price dynamics in a single currency economy. We adopted a model where the money supply matters for the extensive margin of trade, thereby generating non-trivial dynamics in continuous time to a steady state. Our description of money mining also differs. We described the decision to mine as an occupation choice. Therefore, the cost of mining in our model is the opportunity cost of not being a producer in a decentralized goods market. Mining is a time-consuming activity that generates units of money at random times. We also consider variants of the model where agents can trade while mining and can choose their intensity of money. We showed that these different assumptions matter for price dynamics. Finally, even closer to Fernandez-Villaverde and Sanches's model, our last section describes a version of the

model with divisible money. Key differences are that mining takes place in continuous time and the measure of miners determines the measure of transactions that can be confirmed and entered in an open ledger.

Chiu and Koepl (2017) study the optimal design of cryptocurrencies to overcome double-spending and show that the current Bitcoin scheme generates a large welfare loss. Chiu and Koepl (2018) determine necessary conditions for blockchain-based settlement to be feasible. Biais et al. (2018) formalize the proof-of-work blockchain protocol as a stochastic game and show it has multiple equilibria, including equilibria with forks and orphaned blocks. They also identify negative externalities implying that equilibrium investment in computing capacity is excessive.

There is also a related literature on rational bubbles in the context of OLG models, e.g., Wallace (1980) and Tirole (1985), among many others. Search-theoretic models like the ones we use emphasize the transactional role of money in economies where agents meet bilaterally. We think this description is more palatable to describe cryptocurrencies and other assets that aim to serve as media of exchange.

2 Environment

The environment is based on Shi (1995) and Trejos and Wright (1995). Time is continuous and indexed by $t \in \mathbb{R}_+$. The economy is composed of a unit measure of ex ante identical, infinitely-lived agents and a perishable good, q , that comes in different varieties. Each agent can produce a single variety and consumes a subset of all varieties that does not include his own production good, thereby creating a need for trade. The utility of consumption is $u(q)$ with $u(0) = 0$, $u' > 0$, $u'(0) = +\infty$, and $u'' < 0$. There exists a $\bar{q} < +\infty$ such that $u(\bar{q}) = \bar{q}$. The disutility of production is q . Agents discount future utility at rate $r > 0$.

Agents meet bilaterally and at random times according to a Poisson process with arrival rate $\alpha > 0$. All agents participate in this meeting process irrespective of their occupation or asset holdings. We assume that preferences and specialization rule out double-coincidence-of-wants matches. Conditional on meeting a potential producer, the probability of a single-coincidence match is $\sigma \in [0, 1]$. Agents are anonymous (i.e., there is no public record of their trading histories), they lack commitment, and there is no technology to enforce private debt contracts. These frictions create a need for a medium of exchange.

There is an intrinsically useless object, called money, that is perfectly storable and durable. For now it is indivisible and holdings of this object are restricted to $\{0, 1\}$. We relax this assumption in Section 5 by making money perfectly divisible. The overall fixed quantity of this object is $\bar{A} \in (0, 1)$. The amount held

by agents at time t is $A_t \leq \bar{A}$, where A_0 is given. The remaining $\bar{A} - A_t$ is buried in the ground and requires time to extract.

At any point in time agents without money must select one among two occupations. They can choose to operate their technology to produce a variety of good q or they can temporarily give up on this technology and search for undug money—an activity called *mining*. The mining technology is such that each miner finds a unit of money according to a Poisson process with arrival rate $\lambda(\bar{A} - A_t)$ where $\lambda > 0$. This assumption captures the idea that the time that it takes to dig a unit of money is proportional to the stock that remains in the ground. When miners meet other individuals in the economy, they cannot produce for them – an assumption we relax later in the paper. Agents can transition back and forth between occupations at no cost.

3 Equilibria

We characterize monetary equilibria where money has a positive value at all dates. First, we define an equilibrium as a list of Bellman equations, bargaining outcomes, occupation choices, and a law of motion for the money supply. Second, we examine steady-state equilibria where the money supply is constant over time. Third, we describe all dynamic monetary equilibria starting from arbitrary initial conditions.

3.1 Definition

Bellman equations

We denote $V_{a,t}$ the value function of an agent holding $a \in \{0, 1\}$ units of money at time t . Throughout most of the paper we keep the time argument as implicit. The lifetime expected discounted utility of an agent with $a = 1$ solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rV_1 = \alpha\sigma(1 - A - m)[u(q) + V_0 - V_1] + \dot{V}_1, \quad (1)$$

with m denotes the measure of agents engaging in mining and a dot over a variable represents a time derivative. According to the right side of (1) the agent meets another agent with Poisson rate α ; this potential trading partner drawn at random from the whole population is a producer with probability $1 - A - m$ (he does not hold money and is not a miner); and he produces a good that the agent likes with probability σ . In that case, a trade takes place where one unit of money buys q units of output. The last term on the right side is the change in the value function over time in nonstationary equilibria.

The value function of agent with $a = 0$ solves the following HJB equation:

$$rV_0 = \max \left\{ \alpha\sigma A(-q + V_1 - V_0), \lambda(\bar{A} - A)(V_1 - V_0) \right\} + \dot{V}_0. \quad (2)$$

At any point in time, unmatched agents without money can choose among two occupations: mining or producing a consumption good. According to the first term in the maximization problem, an agent who chooses to become a producer meets a money holder who likes his good with Poisson arrival rate $\alpha\sigma A$ in which case he produces q at a linear cost in exchange for one unit of money. According to the second term, an agent who chooses to be a miner becomes a money holder at Poisson arrival rate $\lambda(\bar{A} - A)$ when he successfully digs a unit of money out of the ground. A miner can meet other agents in the economy but cannot produce for them (his occupation is to mine) or buy from them (he has no money), hence those encounters lead to no trade.

Bargaining over the value of money

The quantity q produced in a bilateral match is determined according to the Kalai (1977) bargaining solution that specifies that the buyer receives a constant fraction of the match surplus.⁸ Formally,

$$u(q) + V_0 - V_1 = \theta[u(q) - q], \quad (3)$$

where $\theta \in [0, 1]$ is the buyer's share. Solving for the value of money, $V_1 - V_0$, we obtain:

$$V_1 - V_0 = \omega(q) \equiv (1 - \theta)u(q) + \theta q. \quad (4)$$

Substituting $V_1 - V_0$ by its expression given by (4) into (1) and (2) leads to:

$$rV_1 = \alpha\sigma(1 - A - m)\theta[u(q) - q] + \dot{V}_1 \quad (5)$$

$$rV_0 = \max \left\{ \alpha\sigma A(1 - \theta)[u(q) - q], \lambda(\bar{A} - A)\omega(q) \right\} + \dot{V}_0. \quad (6)$$

Subtracting (6) from (5) and making use of (4) we obtain that q is the solution to the following ODE:

$$r\omega(q) = \alpha\sigma(1 - A - m)\theta[u(q) - q] - \max \left\{ \alpha\sigma A(1 - \theta)[u(q) - q], \lambda(\bar{A} - A)\omega(q) \right\} + \omega'(q)\dot{q}. \quad (7)$$

Occupation choice

The net instantaneous gain from being a miner rather than a producer is:

$$\Delta(q, A) \equiv \lambda(\bar{A} - A)\omega(q) - \alpha\sigma A(1 - \theta)[u(q) - q]. \quad (8)$$

⁸We use Kalai bargaining instead of Nash because of its simplicity. For the merits of this solution, see Aruoba et al. (2007).

From (6) or (7) the measure of miners is given by:

$$m = \begin{cases} 1 - A & \text{if } \Delta(q, A) > 0 \\ \in [0, 1 - A] & \text{if } \Delta(q, A) = 0 \\ 0 & \text{if } \Delta(q, A) < 0 \end{cases} \quad (9)$$

The solution for m is interior when $\Delta = 0$, which from (8) can be expressed as:

$$A = \mu(q) \equiv \frac{\lambda \bar{A} \omega(q)}{\alpha \sigma (1 - \theta) [u(q) - q] + \lambda \omega(q)}. \quad (10)$$

Since $\omega(q)/[u(q) - q]$ increases in q by the concavity of $u(q)$, so does $\mu(q)$. Therefore, as A increases, so must q for agents to be indifferent across occupations.

Money growth

The law of motion for the supply of money in circulation in the economy is:

$$\dot{A} = m \lambda (\bar{A} - A). \quad (11)$$

There is a measure m of agents who specialize in mining and each agent extracts one unit of money from the ground at Poisson rate $\lambda (\bar{A} - A)$. We are now in position to define an equilibrium.

Definition 1 *An equilibrium is a pair of value functions, $V_{0,t}$ and $V_{1,t}$, the quantity traded in each match, q_t , the measure of miners, m_t , and the quantity of money in circulation, A_t , that solve: (5)-(6), (7), (9), (11), and the initial condition A_0 .*

3.2 Steady states

We first describe steady-state equilibria where q and A are constant over time and $m = 0$. We focus on the steady state with the lowest A as it is the one that will be reached from the initial condition $A_0 = 0$. From (7):

$$r \omega(q) = \alpha \sigma (\theta - A) [u(q) - q]. \quad (12)$$

Substituting $\omega(q)$ by its expression given by (4) and rearranging,

$$r q = \{\alpha \sigma (\theta - A) - r(1 - \theta)\} [u(q) - q]. \quad (13)$$

There is a unique $q > 0$ solution to (13) provided that $r < \alpha \sigma (\theta - A)/(1 - \theta)$. Hence, a necessary (but not sufficient) condition for a monetary equilibrium to exist is $\theta > A$. Moreover, $\partial q / \partial A < 0$, i.e., an increase in the money supply reduces the purchasing power of money.

The condition for $m = 0$, $\Delta(q, A) \leq 0$, holds if $A \geq \mu(q)$, which from (10) and (12) can be reexpressed as

$$rA(1 - \theta) \geq \lambda(\bar{A} - A)(\theta - A). \quad (14)$$

We represent inequality (14) in Figure 5. The left side is linear in A while the right side is quadratic with two roots, $A = \bar{A}$ and $A = \theta$. They intersect for two values, $A_1 < \min\{\bar{A}, \theta\}$ and $A_2 > \max\{\bar{A}, \theta\}$. The left side is located above the right side for all $A \in (A_1, A_2)$. Since A cannot be greater than θ for a monetary equilibria to exist, we must have $A < \min\{\bar{A}, \theta\}$. So a steady-state monetary equilibrium exists for all $A \in [A_1, \min\{\bar{A}, \theta\})$. In the following we focus on the steady state $A^s = A_1$.

Figure 5: Steady states

The steady-state equilibrium is determined recursively. First, A^s is obtained as the smallest solution to (14). Given $A = A^s$, q^s exists if and only if $r < \alpha\sigma(\theta - A^s)/(1 - \theta)$ by (13) or, equivalently,

$$A^s < \theta - \frac{r(1 - \theta)}{\alpha\sigma}. \quad (15)$$

Figure 5 provides a graphical representation of the determination of the equilibrium.

Proposition 1 (Steady-state monetary equilibria) *There exists a unique steady-state monetary equilibrium (where $\Delta(q, A) = 0$) if and only if*

$$r < \frac{\alpha\sigma}{1 - \theta} \left[\frac{\theta\alpha\sigma + \lambda(\theta - \bar{A})}{\alpha\sigma + \lambda} \right] \quad (16)$$

where the steady-state money supply is

$$A^s = \frac{\lambda\theta + \lambda\bar{A} + r(1-\theta)}{2\lambda} - \sqrt{\left(\frac{\lambda\theta + \lambda\bar{A} + r(1-\theta)}{2\lambda}\right)^2 - \bar{A}\theta}. \quad (17)$$

Comparative statics are summarized in the following table:

	$\partial\lambda$	$\partial\bar{A}$	$\partial\theta$	∂r	$\partial(\alpha\sigma)$
$\partial A^s /$	+	+	+	-	0
$\partial q^s /$	-	-	$\pm$	-	+
$corr(q^s, A^s)$	-	-	$\pm$	+	0

Table 1: Comparative statics

Under which conditions can fiat money be privately produced and maintain a positive value in the long run? According to Condition (16), the existence of a monetary equilibrium requires agents to be sufficiently patient — a standard “folk theorem” in monetary theory. The threshold for r below which money is valued decreases in the two parameters that affect the private production of money, i.e., $\bar{A}$ and λ . Indeed, higher $\bar{A}$ or λ are associated with higher incentives to mine, and hence a higher money supply at the steady state. But a higher A^s reduces buyers’ trading opportunities, thereby making it harder to sustain a positive value for money.

Comparative statics in Table 1 have implications for the correlation between the endogenous money supply and price level. The sign of this correlation (the bottom row in Table 1) depends on which fundamentals drive the movements of A^s and q^s . If λ or $\bar{A}$ increase, then A^s increases and q^s decreases. In this case there is a positive correlation between money supply and price level ($1/q^s$), which is consistent with the quantity theory. However, if r increases, then both A^s and q^s decrease. There is now a negative correlation between the money supply and the price level. A change in θ generates a non-monotone relation between A^s and q^s : numerical examples show that for low θ , there is a positive correlation between A^s and q^s while for high θ there is a negative correlation.⁹ Finally, an increase in α or σ does not affect incentives to mine and the money supply but it raises q^s .

3.3 Dynamics

We now turn to transitional dynamics from an arbitrary initial condition, A_0 . Without loss of generality we will focus on $A_0 = 0$ because the equilibrium is time consistent, i.e., if the equilibrium reaches A_t at time t from $A_0 = 0$, then the path onward is the same as the one obtained from the initial condition A_t . One can

⁹The parameter values and functional forms are: $\alpha\sigma = 1/2$, $\lambda = 0.4$, $\bar{A} = 1$, $\theta = 0.9$, $r = 0.05$, $u(q) = 2\sqrt{q}$.

give several interpretations to A_0 , including some initial stock of fiat money provided by the government.¹⁰ In that case, government-supplied money has the same physical characteristics as privately-produced money.

From (7) and (11) (q, A, m) solve the following system of differential equations:

$$\omega'(q)\dot{q} = [r + \lambda(\bar{A} - A)]\omega(q) - \alpha\sigma(1 - A - m)\theta[u(q) - q] \quad (18)$$

$$\dot{A} = m\lambda(\bar{A} - A) \quad (19)$$

$$m \leq 1 - A \quad \text{"="} \quad \text{if } \Delta(q, A) > 0. \quad (20)$$

Equation (18) is an asset pricing equation for the value of money. The first term on the right side is the appreciation of the value of money over time if it does not provide transactional services: ω grows at rate $r + \lambda(\bar{A} - A)$, which compensates the buyer for his rate of time preference and the foregone opportunities of mining. The second term on the right side corresponds to the liquidity services that money provides to a buyer as measured by the expected surplus from a trade match. These liquidity services constitute a nonpecuniary return that reduces the appreciation rate of money. Equation (19) is the law of motion of the stock of money. Equation (20) is the optimality condition for the occupation choice between being a producer or a miner.

We distinguish two regimes. In the first regime all agents without money (the potential producers) engage in mining, namely $m = 1 - A$. Then by (18) and (19):

$$\dot{q} = [r + \lambda(\bar{A} - A)] \frac{\omega(q)}{\omega'(q)} \quad (21)$$

$$\dot{A} = (1 - A)\lambda(\bar{A} - A). \quad (22)$$

Note from (21) that the value of money, ω , grows at a rate larger than r because it provides no liquidity services yet. Along the equilibrium path, by virtue of (21) and (22), the relation between q and A is given by

$$\left. \frac{\partial q}{\partial A} \right|_{m=1-A} = \frac{\omega(q)}{\omega'(q)} \frac{r + \lambda(\bar{A} - A)}{(1 - A)\lambda(\bar{A} - A)}. \quad (23)$$

The path is upward sloping in the (A, q) space.

Consider next the regime where miners and producers coexist, $m \in (0, 1 - A)$. In that case $A = \mu(q)$

¹⁰Aiyagari, Wallace, and Wright (1996) established that in a Shi-Trejos-Wright model two fiat monies can have different values if two buyers can trade money for some output. In our interpretation of A_0 as government-produced money, we focus on equilibria where private and government monies are traded at the same price.

and, from (19), the measure of miners is

$$m = \frac{\mu'(q)}{\lambda [\bar{A} - \mu(q)]} \dot{q}. \quad (24)$$

The measure of miners increases with the capital gain $\dot{q}$. This result is consistent with our evidence for gold, silver, and bitcoin, where the appreciation of the value of a money causes increases in its production. The next proposition characterizes the unique dynamic equilibrium that converges to (q^s, A^s) .

Proposition 2 (*Transitional Dynamics to SME*) *Suppose (16) holds and $A_0 = 0$. There exists a unique monetary equilibrium such that (q_t, A_t) converges to $(q^s, A^s) > 0$. The equilibrium path is such that q_t and A_t increase over time. Moreover:*

1. *There exists $t_0 > 0$, such that for all $t < t_0$, $m_t = 1 - A_t$, and*

$$A_t = \frac{\bar{A} [1 - e^{-\lambda(1-\bar{A})t}]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \quad (25)$$

$$\omega(q_t) = e^{rt} \omega_0 \left[\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}} \right]. \quad (26)$$

2. *If*

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1 - \theta}{\theta}, \quad (27)$$

then $m_t < 1 - A_t$ in the neighborhood of the steady state and convergence to (q^s, A^s) is asymptotic.

Otherwise, $m_t = 1 - A_t$ until the steady state is reached in finite time.

Proposition 2 proves the existence and uniqueness of a dynamic equilibrium leading to (q^s, A^s) starting from an initial condition $A_0 = 0$. It allows us to answer the following question: how do the supply of privately-produced money and its price covary over time? The equilibrium features monotone trajectories for q and A over time. As the money supply increases, the price level falls, and quantities traded in pairwise meetings increase.¹¹ This result seems in contradiction with the quantity theory according to which the price level increases with the money supply and the long-run comparative statics in Table 1 where an increase in $\bar{A}$ reduces q . Intuitively, the value of money must appreciate over time for agents to be willing to mine it because it becomes harder to mine as A increases and the production sector becomes more attractive as the number of potential buyers increases.

¹¹As shown in the Appendix, we can obtain less dramatic results with alternative matching functions, i.e., agents trade at all dates, but the insight that market tightness measured by the ratio of producers to buyers increases over time is robust.

Figure 6: Dynamic equilibria with mining

Proposition 2 also answers the question: Can money be valued if it does not serve as a medium of exchange? Early on, when A is close to 0, all agents without money choose to be miners and all agents with money hoard it because they have no opportunities to use it as a medium of exchange. From the viewpoint of an outside observer, money resembles a pure speculative bubble: it does not play any role in exchange, and hence it should not have any liquidity premium, but its value grows at a rate larger than r . This path for the value of money is sustainable because in finite time money starts being used as a medium of exchange.

Can the government prevent the emergence of a private money? The government can remove the incentives to produce money by supplying $A_0 > A^s$. If the initial quantity of money is sufficiently abundant, then the benefits from being in the production sector outweigh the gain from mining. As shown by Table 1, however, the larger λ and $\bar{A}$, the larger A_0 will have to be to prevent money mining. In 5.1 and 5.2 we will revisit this question in a version of the model where agents can hold a portfolio of currencies.

Finally, we showcase the tractability of the model by solving the equilibrium path in closed form in (25)-(26). This result follows from the observation that the law of motion for A , (19), when $m = 1 - A$, is a Riccati equation that admits an analytical solution.

In the neighborhood of the steady state we distinguish two types of equilibria illustrated in Figure 6. There are equilibria where occupations coexist, i.e., the economy features both miners and producers. In this case the steady state is only reached asymptotically. There is another type of equilibrium where

all agents without money choose to mine until the steady-state money supply has been reached, which happens in finite time. These regimes have implications for the transaction velocity of money measured by $\mathcal{V}_t = \alpha\sigma(1 - A_t - m_t)$. In the initial phase, $\mathcal{V}_t = 0$ since all potential producers prefer to mine. If (27) holds, then $\mathcal{V}_t$ becomes positive for some $t > 0$ and it increases as m_t decreases toward its steady-state value. If (27) does not hold, then $\mathcal{V}_t = 0$ until the steady state is reached, in which case $\mathcal{V} = \alpha\sigma(1 - A^s)$. Along the equilibrium path there is a positive correlation between the velocity of money, its price, and its supply.¹² The next lemma provides conditions for the existence of these two regimes.

Lemma 1 (*Coexistence of trades and mining*)

1. If $\epsilon(q) \equiv u'(q)q/u(q)$ is non-increasing in q , then there exists $\lambda^* < +\infty$ and $\kappa^* \in (0, +\infty)$ such that $m_t < 1 - A_t$ in the neighborhood of (A^s, q^s) if and only if $\lambda < \lambda^*$ or $\sigma\alpha \geq \kappa^*$. Moreover, if $\alpha\sigma > \lambda\theta/(1 - \theta)$, then the equilibrium features at most one regime switch.
2. If $\theta \leq 1/2$ and λ is sufficiently large, then $m_t = 1 - A_t$ for all t such that $A_t < A^s$.

The condition on the elasticity of $u(q)$ in Lemma 1 is satisfied by $u(q) = q^{1-a}$ or $u(q) = 1 - e^{-aq}$. Part 1 of Lemma 1 establishes that if the efficiency of mining is low and the matching rate is high, then mining and trades coexist close to the steady state. Part 2 of Lemma 1 provides a global characterization of the occupation choice. If the mining technology is sufficiently efficient and if producers have more bargaining power than buyers, then no trade takes place until the supply of money has reached its steady-state level.

Proposition 2 establishes the existence of a unique equilibrium, denoted $(\hat{A}_t, \hat{q}_t)$, that converges to (A^s, q^s) . The following proposition aims to address the question of the determination of the initial value of money by characterizing the set of all initial values of the new currency that are consistent with a monetary equilibrium. It shows that money can be valued and privately produced even it is anticipated that it will disappear in the long run.

Proposition 3 (*Boom/Bust Equilibria*)

1. For all $q_0 \in (0, \hat{q}_0)$, there exist $0 < T_0 \leq T_1 < +\infty$ such that a monetary equilibrium exists with the following properties:

¹²In the appendix we describe equilibria assuming that only buyers (money holders) and producers participate in the matching process according to a constant returns to scale matching function. The matching probability of a buyer is $\alpha(\tau)$, where $\tau = (1 - A - m)/A$ is market tightness expressed as the ratio of sellers to buyers. We assume that $\alpha' > 0$, $\alpha'' < 0$, $\alpha'(0) = +\infty$, $\alpha'(+\infty) = 0$. If $\lim_{\tau \rightarrow 0} \alpha(\tau)/\tau = +\infty$, then $\tau > 0$ for all t along the equilibrium path. The velocity of money is $\mathcal{V}_t = \alpha(\tau_t)\sigma$ and it increases along the transitional path leading to the steady state.

Figure 7: (Left) Phase diagram of different equilibria (Right) The value of money under different equilibria.

(a) Boom phase: For all $t \leq T_0$, $m_t = 1 - A_t$ and $\dot{\omega}/\omega = r + \lambda(\bar{A} - A) > 0$.

(b) Bust phase: For $t > T_1$, $m_t = 0$, $\dot{\omega}_t = r\omega_t - \alpha\sigma(\theta - A_{T_1})[u(q_t) - q_t] < 0$, and $\lim_{t \rightarrow +\infty} \omega_t = 0$.

2. If (27) holds, then there is a continuum of monetary equilibria indexed by $T \in \{t \in \mathbb{R}_+ : \hat{A}_t = \mu(\hat{q}_t)\}$ such that $q_0 = \hat{q}_0$ and:

(a) Boom phase: For all $t \leq T$, $(A_t, q_t) = (\hat{A}_t, \hat{\omega}_t)$.

(b) Bust phase: For $t > T$, $m_t = 0$, $A_t = \hat{A}_T$, $\dot{\omega}_t = r\omega_t - \alpha\sigma(\theta - \hat{A}_T)[u(q_t) - q_t] < 0$, and $\lim_{t \rightarrow +\infty} \omega_t = 0$.

There is a continuum of monetary equilibria featuring a boom and a bust of the currency price indexed by the initial value of money in the interval $(0, \hat{q}_0)$. If the initial beliefs are not optimistic enough to bootstrap the value of money to $\hat{q}_0$, then a boom/bust equilibrium exists.¹³ Along the equilibrium path the value of money first increases at a rate larger than r . It reaches a maximum at which point agents stop mining. Even though the money supply remains constant afterwards, the value of money declines and converges to 0 asymptotically. In the phase diagram of Figure 7, the equilibrium path is upward sloping until it reaches the locus $A = \mu(q)$. At that point it becomes vertical since the money supply remains constant with arrows of motion oriented toward the horizontal axis since money loses its value over time.¹⁴

¹³Such equilibria capture the idea that new currencies might be likely to fail in the absence of coordination mechanisms. See Selgin (1994) for historical examples.

¹⁴The result according to which the initial value of a new currency is indeterminate was acknowledged by earlier adopters of

There can also be boom/bust equilibria where $q_0 = \hat{q}_0$. Such equilibria occur when agents are indifferent between mining or producing in the neighborhood of the steady state, i.e., (27) holds, and they are indexed by the time T at which the value of money starts falling. Such an equilibrium path is represented in the left panel of Figure 7 by a trajectory starting at $q_0 = \hat{q}_0$. The trajectory is upward sloping and follows the $A = \mu(q)$ locus for a while until it becomes vertical and oriented toward the horizontal axis. From the viewpoint of an outside observer it would be impossible to tell whether the currency will be successful until the time T at which the currency starts declining has been reached.

3.4 Stabilizing the value of money

The high volatility of crypto-currency prices has made many observers wonder whether such currencies could be designed so as to become stable stores of value. We think of the design of the currency as the choice of the mining rate, λ_t , and the choice of a payoff structure for the currency. If the currency pays a real dividend flow $d > 0$, then the next proposition shows that there is a unique equilibrium and it is the one leading to the steady state.

Proposition 4 (*Interest-Bearing Money*) *Suppose money pays an arbitrarily small interest $d > 0$. Then, there exists a unique equilibrium and it is such that $(A_t, q_t) \rightarrow (A^s, q^s)$ as $t \rightarrow \infty$.*

If money generates a flow dividend $d > 0$ (e.g., a utility flow from a commodity money or a real interest payment), then any equilibrium path must be such that the value of holding money is bounded below by the discounted sum of dividends, d/r .¹⁵ This observation rules out the continuum of boom-and-bust equilibria where the value of money vanishes asymptotically. The important insight from Proposition 4 is that d does not need to be large to prevent the collapse of the currency. As long as $d > 0$, cryptocurrencies can be priced above their fundamental value, d/r , without risking to lose their value in the long run.

Focusing on the equilibrium selected by this refinement, we ask whether it is possible to choose the time path for the rate of growth of the money supply so as to achieve price stability. Denote the rate of money creation at time t by π_t , i.e., $\dot{A}_t = \pi_t A_t$. Given this aggregate rate, each individual miner digs units of Bitcoins. Luther (2018) reports the following post on bitcoin-list in January 2009, the month when Bitcoin was first introduced:

“One immediate problem with any new currency is how to value it. Even ignoring the practical problem that virtually no one will accept it at first, there is still a difficulty in coming up with a reasonable argument in favor of a particular non-zero value for the coins.”

¹⁵This commodity-money refinement is often used to reduce the equilibrium set of pure currency economies. See, e.g., Wallace and Zhu (2004) or Garret and Wallace (2018).

money at the rate $\lambda_t = \pi_t A_t / m_t$. The next proposition examines first the case where π_t is kept constant at $\bar{\pi}$ until some quantity $\bar{A}$ has been reached. The second part allows π_t to vary over time.

Proposition 5 (*Stabilizing the value of money*) 1. Suppose the money growth rate is constant, $\dot{A} = \bar{\pi} A$. For A close to $\bar{A}$,

$$\frac{\dot{\omega}}{\omega} \approx \min \left\{ \frac{\bar{\pi} \theta}{1 - \theta}, r + \frac{\bar{\pi} \bar{A}}{1 - \bar{A}} \right\}. \quad (28)$$

2. In any equilibrium where the value of money is constant, $\omega = \omega^s(\bar{A})$, the rate of money creation must evolve according to:

$$\pi_t^* = \pi^*(A_t) \equiv \frac{1 - \theta}{\theta} \left[\alpha \sigma (\theta - A_t) \frac{u(q^s) - q^s}{\omega^s} - r \right]. \quad (29)$$

3. There exists a unique equilibrium and it features price stability if money pays an arbitrarily small dividend and the money growth rate is determined by the following rule:

$$\pi(\omega, A) = \begin{cases} \pi^*(A) & \omega = \omega^s \\ 0 & \text{if } \omega < \omega^s \\ \frac{\alpha \sigma (1 - \theta) [u(q) - q] (1 - A)}{\omega} & \omega > \omega^s. \end{cases} \quad (30)$$

A constant money growth rate does not implement price stability. As shown by (28), close to the steady state, the rate of appreciation of the value of money increases with π .¹⁶ From (29) the money growth path that achieves price stability is such that π decreases over time and it approaches $\pi = 0$ as A approaches $\bar{A}$. It implies that the measure of miners decreases over time and converges to 0. The last part of Proposition 5 provides a policy rule — a Taylor rule for crypto-currency economies — to implement price stability as a unique equilibrium. The rule, (30), is such that the money growth rate is lowered to 0 if ω falls below its steady-state value, ω^s , and it is raised to a level such that $m = 1 - A$ if ω rises above ω^s . As a result, the only path consistent with an equilibrium is the one where $\omega_t = \omega^s$ for all t .

3.5 Mine discoveries and price waves

Historically, the world supplies of silver and gold have increased through sequential discoveries of new mining sites, e.g., South America during the 16-17th centuries, South Africa, and Australia during the 19-20th centuries. In order to capture such discoveries and their impact on price dynamics, we describe a sequence of unanticipated shocks on $\bar{A}$ starting from a steady state. Initially, $\bar{A} = \bar{A}_0$ and the economy is at a stationary equilibrium (q_0^s, A_0^s) . At time 0, the maximum amount of money agents can dug, $\bar{A}$, increases from $\bar{A}_0$ to

¹⁶When the money growth rate is fixed the value of money ω_t can be U-shaped — it falls over time when A is close to 0 and rises when A is large. See Figure 20 for a numerical example.

$\bar{A}_1$. This could correspond to a new estimate of the gold resources of the planet or a coordinated change in the supply of Bitcoins. After the economy reaches a new steady state, (q_1^s, A_1^s) , another discovery happens that raises the potential money supply from $\bar{A}_1$ to $\bar{A}_2$. And so on.

In the phase diagram of Figure 8 the locus $A = \mu(q)$ shifts to the right. The new steady state is such that the money supply increases, $A_1^s > A_0^s$, and money loses some value, $q_1^s < q_0^s$. At time 0^+ , q falls below q_1^s so that the value of money overshoots its steady state. Along the transition to the new steady state the value of money increases. The sequence of unanticipated increases in $\bar{A}$ generate fluctuations in the value of money around a downward trend. We summarize these results in the following proposition.

Proposition 6 (*Unanticipated mine discoveries*) *Consider a sequence of unanticipated increases in $\bar{A}$ (i.e., mine discoveries). Every mine discovery triggers a downward jump in q followed by a gradual recovery. Over time q exhibits a downward trend.*

Figure 8: Mine discoveries: Unanticipated increases of $\bar{A}$

The response to an unanticipated increase in the mining intensity, λ , are similar to those of an increase in $\bar{A}$: the value of money falls on impact and increases afterwards to reach a new steady state with a lower q and higher A . One can also think of a situation where there is some initial stock of fiat money, A_0 , before a new currency is introduced. In the initial steady state, the value of money is $q^s(A_0)$. When the new currency is introduced, the value of all units of money drop and, over the transitional path leading to the new steady state, the value of money recovers partially.

4 Efficient mining

We now ask whether the private production of money can generate a socially efficient outcome. We describe the problem of a social planner who is subject to the mining technology (as it would be the case for gold and silver) and the matching technology between asset holders and producers. Implicit in the latter constraint is the requirement that all trades take the form of one unit of money for some q , i.e., trades are quid pro quo. The planner chooses agents' occupation and output in pairwise meetings in order to maximize the discounted sum of all agents' utilities. (The planner's problem is written explicitly in the Appendix.) We then provide an incentive-feasible mechanism to implement such constrained-efficient allocations.

Proposition 7 (Constrained efficient allocation) *Assume $A_0 = 0$.*

1 Efficient allocations are such that $q_t = q^*$ for all t and $m_t = 1 - A_t$ for all $t < T^*$ where $T^* > 0$ is the time it takes to mine A^* where

$$A^* = \frac{1}{4} \left[\left(2\bar{A} + 1 + \frac{r}{\lambda} \right) - \sqrt{\left(2\bar{A} + 1 + \frac{r}{\lambda} \right)^2 - 8\bar{A}} \right]. \quad (31)$$

For all $t \geq T^*$, $m_t = 0$ and $A_t = A^*$.

2 Implementatation. If

$$r \leq \frac{\alpha\sigma(1 - A^*)[u(q^*) - q^*]}{q^*} \quad (32)$$

$$\lambda(\bar{A} - A^*) \leq \frac{\alpha\sigma A^*(1 - A^*)[u(q^*) - q^*] - rq^*}{(1 - A^*)u(q^*) + A^*q^*}, \quad (33)$$

then the constrained-efficient allocation is implementable with

$$\begin{aligned} \theta_t &= 1 \text{ if } t < T^* \\ &= \theta^* \equiv \frac{rq^* + (\alpha\sigma A^* + r)[u(q^*) - q^*]}{[u(q^*) - q^*](\alpha\sigma + r)} \text{ if } t \geq T^*. \end{aligned} \quad (34)$$

The planner chooses q^* in all trade matches and it assigns all non-asset holders to mining until the efficient quantity of money has been produced. We show in the Appendix that along the optimal path the shadow value of money ξ (i.e. the co-state variable associated with A_t) satisfies

$$\frac{\dot{\xi}}{\xi} = r + \lambda(1 + \bar{A} - 2A).$$

If we compare with the equilibrium ODE, (18), when $m = 1 - A$,

$$\frac{\dot{\omega}}{\omega} = r + \lambda(\bar{A} - A),$$

we see that the rate of growth of ξ is larger than the rate of growth of ω by a term equal to $\lambda(1 - A)$. This additional term reflects the fact that the planner internalizes the scarcity effect of digging new coins on future miners: as more money is taken out of the ground, it becomes harder for future miners to find new units of money. As a result, the growth in the shadow value of money must compensate for that increased scarcity. The optimal quantity of money, A^* , is less than $1/2$, which is the quantity that would maximize the measure of trades. As agents become infinitely patient, $\lim_{r \rightarrow 0} A^* = \min\{1/2, \bar{A}\}$. By comparing (31) and (17) one can check that $A^s > A^*$ if $\theta > 1/2$ and $A^s < A^*$ if $\theta < 1/2$. There is over-production of money in the decentralized equilibrium if buyers receive a larger share of the match surplus than producers. Even if $\theta = 1/2$ so that $A^s = A^*$, the output produced in the decentralized equilibrium is different from the first best.

In the first part of Proposition 7 we restricted trades to take place between money holders and producers but we did not impose that the actual trades in pairwise meetings satisfy individual rationality constraints. In the second part we propose an incentive-feasible trading mechanism that implements the constrained-efficient allocation. The mechanism is incentive feasible if it satisfies the individual rationality constraints of the buyer, $u(q) + V_0 - V_1 \geq 0$, and the producer, $-q + V_1 - V_0 \geq 0$, in a pairwise meeting. Any incentive-feasible trading mechanisms can be described by a sequence of time-varying bargaining shares, θ_t . From (34) the incentive-feasible trading mechanism that implements the constrained-efficient allocation is such that buyers have all the bargaining power until the efficient quantity of money, A^* , has been dug at time T^* . Giving no bargaining power to producers initially guarantees that agents without money choose to be miners rather than producers. Following T^* the buyer's bargaining power is $\theta^* > 0$, which is the value that implements q^* in all pairwise meetings.

Condition (32) is the standard implementation condition of the first best in monetary search models (see, e.g., Wright 1999). It requires that the opportunity cost of holding money as measured by rq^* is smaller than the expected surplus from holding money assuming the buyer has all the bargaining power, $\alpha\sigma(1 - A^*)[u(q^*) - q^*]$. A key difference from the existing literature is that the money supply here is endogenous and depends on fundamentals. As r approaches 0, A^* tends to $\min\{\bar{A}, 1/2\}$. Hence the implementation condition is satisfied for r sufficiently small.

Condition (33) is new and guarantees that agents have no incentive to over-produce money. Assuming that the buyer's bargaining share is θ^* , it requires that the expected gain from mining, $\lambda(\bar{A} - A^*)\omega^*$ where

$\omega^* = (1 - \theta^*)u(q^*) + \theta^*q^*$, is no greater than the expected gain from being a producer $\alpha\sigma(1 - \theta^*)[u(q^*) - q^*]$. If $\bar{A} < 1/2$, then this condition holds for r sufficiently close to 0.

5 Mining divisible assets

So far, we have described an economy with indivisible assets and a shortage of liquidity – not all agents can hold one unit of money. We now make money/assets perfectly divisible and we remove the upper bound on asset holdings.¹⁷ We add a centralized market (CM) that opens concurrently with the decentralized market (DM) composed of random, pairwise meetings, in the same spirit as the Lagos-Wright (2005) model, except our model is in continuous time. The CM is a competitive exchange where agents can readjust their holdings of liquid assets in-between pairwise meetings. When in the CM agents can produce or consume a numeraire good at a linear cost or utility. Pairwise meetings occur outside of the CM where agents cannot produce the numeraire. The liquid asset takes the form of a Lucas tree that pays a dividend flow $d \geq 0$. The case $d = 0$ corresponds to fiat money. The CM price of the asset is denoted ϕ_t .

Figure 9: Timing of model with divisible assets

Let $V(a)$ denote the value function of an agent with a units of assets expressed in terms of the numeraire. After a pairwise meeting or a mining discovery, an agent can readjust its asset holdings by consuming or producing discrete amounts of the numeraire. Formally,

$$V(a) = \max_h \{-h + V(a + h)\} = a + \max_{a^* \geq 0} \{-a^* + V(a^*)\},$$

where a^* is the agent's desired asset holdings. It solves $V'(a^*) = 1$. The lifetime expected utility of the

¹⁷The model is a continuous-time version of the Lagos-Wright model, as in Rocheteau and Rodriquez (2014) or Rocheteau, Weill and Wong (2018).

agent holding his targeted real balances solves the following HJB equation:

$$\begin{aligned} rV(a^*) = & \alpha(1-m)\sigma\theta \{u[q(a^*)] - q(a^*)\} + \max_h [-h + V'(a^*)(h + \rho a^*)] \\ & + \max \{ \alpha\sigma(1-\theta) \{u[q(\bar{a})] - q(\bar{a})\}, \lambda(\bar{A} - A)\phi \}, \end{aligned} \quad (35)$$

where the rate of return of assets is

$$\rho = \frac{d + \dot{\phi}}{\phi}. \quad (36)$$

The agent receives an opportunity to consume at Poisson arrival rate $\alpha\sigma$. The partner can produce if he is not a miner, with probability $1-m$. The buyer's surplus is $u(q) + V(a^* - p) - V(a^*)$ where p is the payment to the producer in the form of assets. By the linearity of the value function, this surplus is equal to $u(q) - p$. The terms of trade, (q, p) , are set in accordance with the proportional solution, i.e., $u(q) - p = \theta[u(q) - q]$. According to the second term, the agent can readjust his asset holdings continuously through time. The change in asset holdings is $\dot{a} = h + \rho a$, where ρ is the rate of return of assets composed of the dividend and the capital gain. Using that $V'(a^*) = 1$, this second term is equal to ρa^* . The third term on the right side of (35) corresponds to the occupational choice according to which agents can choose to be producers and enjoy the flow payoff $\alpha\sigma(1-\theta) \{u[q(\bar{a})] - q(\bar{a})\}$ or miners and enjoy $\lambda(\bar{A} - A)\phi$. The term $\bar{a}$ represents asset holdings' of other agents in the economy. The expected gain from mining describes the assumption that at Poisson arrival rate $\lambda(\bar{A} - A)$ the miner digs a unit of money which is worth ϕ units of numeraire.

The envelope condition associated with (35) gives

$$r - \rho = \alpha(1-m)\sigma\theta \left\{ \frac{u'[q(a^*)] - 1}{(1-\theta)u'[q(a^*)] + \theta} \right\}. \quad (37)$$

The opportunity cost of holding the asset on the left side of (37) is the difference between the rate of time preference and the real rate of return of the asset. The right side is the marginal value of an asset if a consumption opportunity arises.

The measure of miners solves

$$m \begin{cases} \in (0, 1) \\ = 1 \\ = 0 \end{cases} \quad \text{if } \lambda(\bar{A} - A)\phi \begin{cases} = \\ \geq \\ \leq \end{cases} \alpha\sigma(1-\theta)[u(q) - q], \quad (38)$$

where q solves $\omega(q) = \min\{\omega(q^*), \phi a^*\}$. By market clearing:

$$a^* = \phi A. \quad (39)$$

The supply of assets on the right side is endogenous and determined by the measure of agents who choose to mine as follows:

$$\dot{A} = \lambda m(\bar{A} - A). \quad (40)$$

An equilibrium is a list, $\langle a_t^*, m_t, \phi_t, A_t \rangle$, that solves (37), (38), (39), and (40).

Figure 10: Phase diagram with divisible assets

From (38) and (39) the locus of pairs (A, ϕ) such that agents are indifferent between mining and producing is given by:

$$\lambda(\bar{A} - A)\phi = \alpha\sigma(1 - \theta)S(A\phi), \quad (41)$$

where $S(A\phi) = u(q) - q$ is the total surplus from a pairwise trade where quantities solve $\omega(q) = \min\{\omega(q^*), \phi A\}$. For all $A > \lambda\bar{A}/(\alpha\sigma + \lambda)$, there is a unique $\phi > 0$ solution to (41). In the (A, ϕ) space, this solution is upward sloping and it has a vertical asymptote, $A = \bar{A}$. To the left of this locus all agents choose to mine, $m = 1$.

The locus of stationary equilibria where $m = 0$ and $\dot{\phi} = 0$ is given by

$$r = \frac{d}{\phi} + \alpha\sigma\theta \left\{ \frac{u'(q) - 1}{(1 - \theta)u'(q) + \theta} \right\}, \quad (42)$$

where q is an implicit function of ϕA . It is a standard asset pricing equation where the effective rate of return of the asset is composed of the pecuniary return, d/ϕ , and the liquidity return. It gives a negative relationship between ϕ and A with $\phi = d/r$ for all A such that $A \geq r\omega(q^*)/d$. The two loci, (41) and (42), are represented in the phase diagram of Figure 10 that allows us to characterize stationary and non-stationary equilibria.

Proposition 8 (Mining divisible assets) Suppose assets are perfectly divisible and asset holdings are unrestricted.

1. **(Abundant liquidity)** If

$$\frac{\bar{A}d}{r} \geq \frac{\alpha\sigma(1-\theta)[u(q^*) - q^*]}{\lambda} + \omega(q^*), \quad (43)$$

then there exists a unique equilibrium and it is such that $\phi_t = d/r$, $q_t = q^*$ for all t , and $m_t = \mathbf{1}_{\{t < T\}}$ where

$$T = \ln \left\{ \frac{\lambda d \bar{A}}{r \alpha \sigma (1 - \theta) [u(q^*) - q^*]} \right\}^{\frac{1}{\lambda}} < +\infty.$$

Moreover,

$$\begin{aligned} A_t &= \bar{A} (1 - e^{-\lambda t}) \quad \text{for all } t < T \\ A_t &= A^s = \bar{A} - \frac{r \alpha \sigma (1 - \theta) [u(q^*) - q^*]}{\lambda d} \quad \text{for all } t \geq T. \end{aligned}$$

2. **(Scarce liquidity)** If $d > 0$ and (43) does not hold, then there exists a unique equilibrium and it is such that $\phi_t > d/r$ and $q_t < q^*$ for all t . If $A_0 = 0$ then $m_t = 1$ for all t sufficiently small and the economy transitions to $m_t < 1$ before reaching the steady state if

$$\frac{\theta}{1-\theta} > \frac{\nu(A\phi) + A/(\bar{A} - A)}{\nu(A\phi) [1 - \nu(A\phi)]} \quad (44)$$

where $\nu(x) \equiv S'(x)x/S(x)$. If $\epsilon(q)$ falls in q , then there is at most one regime switch from $m_t = 1$ to $m_t < 1$ and m_t is non-increasing in t .

3. **(Fiat money)** If $d = 0$, then there exists a unique equilibrium leading to a monetary steady state if $r < \alpha\sigma\theta/(1-\theta)$. The equilibrium is such that $q_t < q^*$ and $\dot{\phi}_t \geq 0$ for all t and the steady state solves

$$r = \alpha\sigma\theta \left\{ \frac{u'(q^s) - 1}{(1-\theta)u'(q^s) + \theta} \right\} \quad (45)$$

$$\frac{A^s}{\bar{A}} = \frac{\lambda\omega(q^s)}{\alpha\sigma(1-\theta)[u(q^s) - q^s] + \lambda\omega(q^s)} \quad (46)$$

Proposition 8 provides two main insights relative to the model with indivisible money. First, there is a regime where the asset supply at the steady state is abundant enough to satiate agents' liquidity needs and allow agents to trade q^* in all matches. In such equilibria, the asset is priced at its fundamental value at all dates. A necessary (but not sufficient) condition is that the potential asset supply when valued at its

fundamental price, $\bar{A}d/r$, is larger than agents' liquidity needs, $\omega(q^*)$. It is the standard condition in the literature for abundant liquidity since Geromichalos et al. (2007), except that it applies to the potential asset supply, $\bar{A}$, and not the actual asset supply, A , which is endogenous. Condition (43) has an extra term that captures agents' incentives to stop mining. This term decreases with λ so that liquidity is more likely to be abundant when the mining speed is large. Maybe surprisingly, $q_t = q^*$ even before market capitalization has reached $\omega(q^*)$. The reason for this result is that as long as the steady-state asset supply has not been reached, all agents mine and hence there is no demand for liquidity. When the steady state is reached, in finite time, then agents start trading.

The second insight is that there is a regime with scarce liquidity that is qualitatively similar to the equilibria of the model with indivisible money. Hence, the results of our first model are robust to an environment with divisible money and competitive exchanges for liquidity. The price of the asset is above its fundamental value at all dates and it keeps increasing over time until it reaches a steady state.¹⁸ Under a condition on the elasticity of $u(q)$ that there is at most one transition from the regime with $m = 1$ to the regime with $m < 1$.

In the case where the asset is a pure fiat money, $d = 0$, then q^s is independent of $\bar{A}$ and λ . The endogenous money supply is proportional to the potential supply with a coefficient that increases with λ . Note that the condition for a regime switch, (44), in the case of fiat money ($d = 0$) can be rewritten as:

$$\frac{r(1 - \theta) + \theta\lambda}{r\theta} < \frac{\alpha\sigma\theta [u(q^s) - q^s] - r\omega(q^s)}{\alpha\sigma\theta [u(q^s) - q^s]}.$$

This condition does not hold when r is close to 0, i.e., agents specialize in mining until the steady state is reached. If $\alpha\sigma$ goes to $+\infty$ this condition becomes $\theta\lambda < r(2\theta - 1)$, which requires $\theta > 1/2$ and λ to be small. The fact that this condition depends on λ shows that even though λ does not affect the steady-state equilibrium, it does matter for the transition leading to it.

Corollary 1 (*Money neutrality*) *Assume $d = 0$. An increase in $\bar{A}$ leads to a proportional increase in A^s in the long run and no real effects. In the short run, aggregate real balances and output fall. During the transition to the new steady state, the inflation rate is negative.*

¹⁸If agents can search for trading partners while mining, $\eta = 1$, and $r < \alpha\sigma\theta/(1 - \theta)$, then the unique equilibrium leading to the steady state is such that ϕ decreases over time. The phase diagram is analogous to the middle panel in Figure 14 in the Appendix. The logic for the price dynamics is now consistent with the quantity theory. The value of money is high initially because the supply is low. As the supply increases, the value of money decreases. A key difference with respect to the model with indivisible studied earlier is that all agents irrespective of their money holdings receive the revenue from mining.

5.1 Competiting private monies

Suppose now that there are two assets that can serve as means of payment, silver (A^g) and gold (A^u).¹⁹ We normalize asset supplies so that both assets yield the same dividend, $d > 0$. Potential asset supplies are $\bar{A}^g$ and $\bar{A}^u$ and $\bar{A} = \bar{A}^g + \bar{A}^u$. Mining rates are given by λ^g and λ^u . With no loss in generality, we assume that $\lambda^g \bar{A}^g \geq \lambda^u \bar{A}^u$. Because these assets are perfect substitutes as means of payment, their common price is ϕ . The occupation choice of an agent is now given by

$$\max \left\{ \alpha \sigma (1 - \theta) [u(q) - q], \lambda^g (\bar{A}^g - A^g) \phi, \lambda^u (\bar{A}^u - A^u) \phi \right\},$$

where A^g and A^u are the amounts of silver and gold in circulation with $A = A^g + A^u$. The characterization of the equilibrium is analogous to the one in previous sections and is detailed in the Appendix. In the following proposition m^g is the measure of silver miners, m^u is the measure of gold miners, and $m = m^g + m^u$.

Proposition 9 (Dual asset equilibrium) *Assume $A_0^u = A_0^g = 0$. For all $\bar{A}^u > 0$, there is a $\kappa_0 > 0$ independent of $\{\lambda^u, \lambda^g, \bar{A}^u, \bar{A}^g\}$ such that if*

$$\frac{\lambda^g \bar{A}^g - \lambda^u \bar{A}^u}{\lambda^g \lambda^u \bar{A}^u} < \kappa_0, \quad (47)$$

then the long-run equilibrium features two competing assets as media of exchange. For all t such that $A_t < \hat{A} \equiv (\lambda^g \bar{A}^g - \lambda^u \bar{A}^u) / \lambda^g$, the economy has a single means of payment (monometallism), $m_t^u = A_t^u = 0$; For all t such that $A_t > \hat{A}$, the economy has two means of payment (bimetallism) with

$$\begin{aligned} A_t^u &= \frac{\lambda^g}{\lambda^u} (A_t^g - \hat{A}) > 0 \\ m_t^u &= \frac{\lambda^g}{\lambda^g + \lambda^u} m_t > 0. \end{aligned}$$

According to (47), if the initial mining rates of the two assets are not too far apart, then the economy transitions from using a single asset as medium of exchange to using two assets when the mining rates have been equalized.

5.2 Competing private and government monies

Suppose now that there are two fiat monies: the privately produced money (labelled b) and the government-produced money (labelled g). Following Lester et al. (2012), the two monies differ by their acceptabilities in

¹⁹The periodic symbol of silver and gold are Ag and Au respectively.

different meetings. There is fraction χ_b of meetings where only privately-produced money is acceptable. For instance, the private money is accepted as means of payment for illegal transactions whereas government money is not. By symmetry, there is a fraction χ_g of meetings where only government-produced money is acceptable, e.g., in transactions that involve the government. In the remaining fraction, χ_2 , both types of monies are acceptable. We denote $\alpha_j = \alpha\chi_j$ for $j \in \{b, g, 2\}$. By the same reasoning as above the demand for the two monies satisfies the following Euler equations for $j = b, g$:

$$r - \rho_j = \alpha_j(1 - m)\sigma\theta \left\{ \frac{u'(q_j) - 1}{(1 - \theta)u'(q_j) + \theta} \right\} + \alpha_2(1 - m)\sigma\theta \left\{ \frac{u'(q_2) - 1}{(1 - \theta)u'(q_2) + \theta} \right\} \quad (48)$$

where q_j indicates output in a match of type $j \in \{b, g, 2\}$. The term on the left side, $r - \rho_j$, is the cost of holding money j . The terms on the right side represent the liquidity services that money j provides at the margin in different types of matches. The second term is common to both monies while the first term is specific to each money.

We assume that monetary policy aims at keeping q_g constant by varying ρ_j . As a result, aggregate real balances supplied by the government, $\omega(q_g)$, are constant. It follows that the quantity traded in type- b and type-2 matches solve:

$$\omega(q_2) = \min \{ \omega(q_g) + \phi_b A_b, \omega(q^*) \} \quad (49)$$

$$\omega(q_b) = \phi_b A_b. \quad (50)$$

If the real value of the two types of monies is larger than $\omega(q^*)$, then agents trade q^* in type-2 matches and spend only a fraction of their money holdings. Otherwise, they spend all their money, both private and public. The indifference condition between occupations becomes

$$\lambda(\bar{A}_b - A_b)\phi_b = \sum_{j \in \{b, g, 2\}} \alpha_j \sigma(1 - \theta) [u(q_j) - q_j]. \quad (51)$$

As before, this condition gives a positive relationship between ϕ_b and A_b but, differently from before, as A_b goes to 0, $\phi_b = (\alpha_g + \alpha_2)\sigma(1 - \theta) [u(q_g) - q_g] / \lambda\bar{A}_b > 0$. This implies that $m = 1$ is no longer guaranteed for low values of A_b . As before, the law of motion for A_b is

$$\dot{A}_b = \lambda(\bar{A}_b - A_b)m. \quad (52)$$

An equilibrium is a list $(\rho_b, \rho_g, \phi_b, q_2, q_b, m, A_b)$ that solves (48), (49), (50), (51), (52) and $\rho_b = \dot{\phi}_b / \phi_b$.

Figure 11: Two numerical examples of equilibrium. (Left) Agents are indifferent between producing and mining even when $A_b = 0$. (Right) Agents specialize in mining early on.

The phase diagram is characterized by the indifference locus, (51), and the isocline for ϕ_b , from (48),

$$r = \sigma\theta \left\{ \alpha_b \left[\frac{u'(q_b) - 1}{(1 - \theta)u'(q_b) + \theta} \right] + \alpha_2 \left[\frac{u'(q_2) - 1}{(1 - \theta)u'(q_2) + \theta} \right] \right\}, \quad (53)$$

where q_b and q_2 are given by (49)-(50). Equation (53) gives a negative relationship between ϕ_b and A_b . The phase diagram is qualitatively similar to the one studied earlier. In particular, ϕ_b and A_b increase over time until they reach a steady state.

At the steady state $m = 0$ and $\rho_b = 0$, which pins down q_b and q_2 . The indifference condition between occupations gives ϕ_b . We can now determine a condition under which the government can prevent the emergence of the private money.

Proposition 10 (*Preventing the emergence of a private money.*) *There is no equilibrium with production of private money if*

$$\frac{\alpha_b \sigma \theta}{1 - \theta} + \alpha_2 \sigma \theta \left[\frac{u'(q_g) - 1}{(1 - \theta)u'(q_g) + \theta} \right] < r. \quad (54)$$

The proof is as follows. When (54) holds, the real balances $\phi_b^s A_b^s$ in (53) cannot be positive, and thus there is no mining along the equilibrium path. By raising q_g the government reduces the liquidity shortage, and hence incentives to produce the private money. However, if $\alpha_b \sigma \theta > r(1 - \theta)$, it is not enough to eliminate the private money. The government must reduce the fraction of matches where the private money is the only means of payment.

5.3 The role of miners to validate transactions

In the case of crypto-currencies, miners play a role to confirm transactions and enter them in an open ledger. They are remunerated for this role with the seigniorage revenue arising from the creation of cryptocurrencies. Because trades cannot go through without being confirmed, the mining activity must keep going on at the steady state. Hence, we will consider equilibria with fiat money and constant money growth. Successful miners receive n_t units of money where n_t is allowed to vary over time. The mining speed is constant and equal to λ . So money growth obeys

$$\dot{A} = \lambda mn. \quad (55)$$

There is an interior measure of agents choosing to be miners if

$$\lambda \phi n = \alpha \sigma (1 - \theta) [u(q) - q]. \quad (56)$$

The reward to the miners, $\{n_t\}$, is chosen to guarantee a constant money growth rate, $\dot{A}/A = \pi$. From (40),

$$n = \frac{\pi A}{\lambda m}. \quad (57)$$

The seigniorage revenue, πA , is divided evenly among all succesful miners, λm . The indifference between mining and trade, (38), and the equilibrium condition $a = a^*$ gives:

$$m = \frac{\pi \omega(q)}{\alpha \sigma (1 - \theta) [u(q) - q]}. \quad (58)$$

The measure of miners is equal to the ratio of the seigniorage revenue to the net expected gains from trading.

Proposition 11 (*Constant money growth*) *If $r + \pi < \alpha \sigma \theta / (1 - \theta)$ and $\pi > 0$, then there exists a unique steady-state monetary equilibrium where mining and trading coexist. For π sufficiently small, $\partial m / \partial \pi > 0$ and $\partial q / \partial \pi < 0$.*

If π is not too large but positive and θ not too low, a steady-state monetary equilibrium exists where both traders and miners coexist. In the steady state the money stock $\dot{A} > 0$ but all real variables are stationary.

In order to capture the role of mining to confirm trades, suppose that a trade can be completed only if it is entered by a miner into the ledger. We describe the mining technology as follows. When two traders agree on a transaction, they send it to the community of miners. The aggregate measure of transactions that are processed is given by

$$\min \{ \alpha \sigma (1 - m), m \lambda \}.$$

The first term is the total measure of potential transaction while the second term is the total mining activity. So we use the total mining activity as an upper bound for the number of transactions that can be entered into the ledger. Provided that $\alpha\sigma(1 - m) \leq m\lambda$ all transactions are entered into the ledger.²⁰ We denote β the probability that a trade is entered into the ledger. It is given by:

$$\beta = \min \left\{ \frac{m\lambda}{\alpha\sigma(1 - m)}, 1 \right\}.$$

If $m\lambda \geq \alpha\sigma(1 - m)$, then there are enough miners to enter all transactions into the ledger and hence $\beta = 1$. Otherwise, $\beta < 1$ and the effective measure of transactions is $m\lambda$. The measure of miners that is required to enter a fraction β of all transactions into the ledger is:

$$m = m_\beta^* \equiv \frac{\beta\alpha\sigma}{\lambda + \beta\alpha\sigma}. \quad (59)$$

The required measure of miners m_β^* increases with β but decreases with λ . The measure of miners will be implemented with some money growth π provided that m is not too large.

Proposition 12 (*Optimal inflation when miners confirm transactions*) *Assume $\theta = 1$ and the elasticity of $u(q)$ is decreasing in q . Then, the optimal inflation rate is strictly positive. Moreover, there exists $\lambda^* \in (0, \infty)$ such that if $\lambda \geq \lambda^*$, then the optimal inflation rate implements m_1^* .*

The optimal inflation rate is positive because miners play an essential role to confirm trades and they are remunerated with the revenue from money creation. If the mining speed is fast so that a small measure of miners is necessary to confirm all trades, then the optimal inflation rate is such that $\beta = 1$, i.e., all potential trades are entered into the ledger.

6 Conclusion

²⁰Note that we assume that transactions are entered into the ledger immediately even though mining takes time. It is consistent with the fact that verifying transactions is very fast while mining is essentially a mechanism to allocate the seigniorage revenue in a probabilistic fashion.

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Appendix A: Proofs

Proof of Proposition 1. The steady-state money supply is the lowest root of (14) at equality, i.e.,

$$A^2 - \left[\frac{(\theta + \bar{A}) \lambda + r(1 - \theta)}{\lambda} \right] A + \bar{A}\theta = 0. \quad (60)$$

The lowest root of (60) is (17). If $\theta > 0$, then $A^s > 0$. From (13), $q^s > 0$ is a solution to

$$\Gamma(q) \equiv \{\alpha\sigma(\theta - A^s) - r(1 - \theta)\} [u(q) - q] - rq = 0. \quad (61)$$

If $\alpha\sigma(\theta - A^s) - r(1 - \theta) > 0$ then $\Gamma(q)$ is strictly concave with $\Gamma'(0) = +\infty$ and $\lim_{q \rightarrow \infty} \Gamma(q) = -\infty$. Hence, there exists a unique $q^s > 0$ solution to $\Gamma(q^s) = 0$. If $\alpha\sigma(\theta - A^s) - r(1 - \theta) \leq 0$ then $\Gamma(q)$ is decreasing with $\Gamma(0) = 0$. Hence, there is no $q^s > 0$ solution to $\Gamma(q^s) = 0$. Comparative statics are straightforward. As an example, consider the effects of changes in λ . Differentiating (60) with respect to A and λ we obtain:

$$\frac{\partial A^s}{\partial \lambda} = \frac{(\bar{A} - A^s)(A^s - \theta)}{2\lambda A^s - [(\theta + \bar{A})\lambda + r(1 - \theta)]}.$$

Using that A^s is the lowest root of (60), it follows that the denominator is negative (graphically, the slope of the parabola is negative when it intersects the horizontal axis at A^s). Using that $A^s < \min\{\bar{A}, \theta\}$, the numerator is negative and $\partial A^s / \partial \lambda > 0$. From (61) $\Gamma(q)$ decreases with A^s for all q such that $u(q) - q > 0$. Hence, $\partial q^s / \partial \lambda > 0$. As another example, consider a change in r . From (60),

$$\frac{\partial A^s}{\partial r} = \frac{(1 - \theta)A^s}{2\lambda A^s - [(\theta + \bar{A})\lambda + r(1 - \theta)]} < 0.$$

Since agents are indifferent between mining or producing at A^s , namely $\Delta(q^s, A^s) = 0$, by (8) we have

$$\frac{u(q^s) - q^s}{\omega(q^s)} = \left\{ 1 - \theta + \frac{q^s}{u(q^s) - q^s} \right\}^{-1} = \frac{\lambda(\bar{A} - A^s)}{\alpha\sigma A^s(1 - \theta)}.$$

The left side falls in q^s and the right side falls in A^s . Consequently, q^s and A^s co-move when r changes and $\partial q^s / \partial r < 0$. ■

Proof of Proposition 2. Parts 1 and 2. We first provide a condition that determines which regime, $m = 1 - A$ or $m \in (0, 1 - A)$, is relevant along the equilibrium trajectory. We use the condition to create two ODEs that fully characterize the dynamics of q_t and A_t . Then we will use a standard result for systems of ODEs to prove the existence and uniqueness of q_t and A_t .

Suppose the economy is at (A', q') where $A' < A^s$ and consider the trajectory as we move backward in time. If $\mu(q') > A'$, then the equilibrium path cannot follow (10) because the solution path is continuous. To see this, note that the path of A_t is continuous by (11) since A_t cannot jump and $|\dot{A}| \leq \lambda \bar{A}$. The value of money, $\omega(q_t) \equiv V_{1,t} - V_{0,t}$, is continuous over time because the continuation values $V_{1,t}$ and $V_{0,t}$ are integrals of payoffs that arrive randomly according to Poisson processes. As a result $m = 1 - A'$ when $\mu(q') > A'$. In this case it is optimal for agents to mine because $\Delta(q', A')$ in (8) is strictly positive when $\mu(q') > A'$.

Next suppose $\mu(q') = A'$. The equilibrium is in the regime with $m = 1 - A$ if and only if

$$\left. \frac{\partial q}{\partial A} \right|_{m=1-A} \leq \left. \frac{\partial q}{\partial A} \right|_{m \in (0, 1-A)} \quad (62)$$

where the first derivative is defined by (23) and the second is obtained by differentiating (10) with respect to A , namely $\partial q / \partial A = 1 / \mu'(q)$. If (62) is binding, then both regimes implies $m = 1 - A$ and thus they imply the same trajectory. If (62) holds strictly, then the trajectory defined by (21) and (22) converges to (A', q') from the left of the line $\mu(q) = A$, and thus it is optimal for all agents without money to mine, $m = 1 - A$. The trajectory $\mu(q) = A$ is not an equilibrium near (A', q') because by (18) and (19)

$$\frac{\partial q}{\partial A} = \frac{m^{-1} \{ [r + \lambda (\bar{A} - A)] \omega(q) - \alpha \sigma (1 - A) \theta [u(q) - q] \} + \alpha \sigma \theta [u(q) - q]}{\lambda (\bar{A} - A) \omega'(q)}. \quad (63)$$

For any (A, q) such $A = \mu(q)$, the right side rises in m because the expression in the braces is negative by (14) and (10). Using that the left side of (62) coincides with (63) when $m = 1 - A$, it follows that if (62) holds strictly at (q', A') , then the measure of miners m implied by the trajectory $A = \mu(q)$ must strictly exceed $1 - A'$, and thus it cannot be an equilibrium path.

If (62) does not hold, then the trajectory defined by (21) and (22) converges to (q', A') from below $A = \mu(q)$ and hence no agent has an incentive to mine. In this case the equilibrium path is in the regime where $m \in (0, 1 - A)$ and the measure of miners m implied by $A = \mu(q)$ satisfies $m < 1 - A$ by (63).

We are now ready to characterize the equilibrium by a system of ODEs in backward time. Let $y \equiv A^s - A$. Then we can define q as a function of y along the equilibrium path. From (23) and (10), the two sides of the inequality (62) can be expressed as:

$$\left. \frac{\partial q}{\partial y} \right|_{m=1-A} = g_1(y, q) \equiv -\frac{\omega(q)}{\omega'(q)} \frac{[r + \lambda (\bar{A} - A^s + y)]}{(1 - A^s + y) \lambda (\bar{A} - A^s + y)} \quad (64)$$

$$\left. \frac{\partial q}{\partial y} \right|_{m \in (0, 1-A)} = g_2(y, q) \equiv -\frac{1}{\mu'(q)}. \quad (65)$$

By the discussion above, the equilibrium path $q(y)$ solves the following ODE:

$$\frac{\partial q}{\partial y} = f(y, q) \equiv \max\{g_1(y, q), g_2(y, q)\} \mathbf{1}_{\{\mu(q) \leq A^s - y\}} + g_1(y, q) \mathbf{1}_{\{\mu(q) > A^s - y\}} \quad (66)$$

with the initial condition $q(0) = q^s$, where $\mathbf{1}_{\{\cdot\}}$ is an indicator function. It is easy to check that $f(y, q)$ is bounded and continuous for $y \in [0, A^s]$ and $q \in (0, \bar{q}]$ where $\bar{q} > 0$ is the solution to $u(\bar{q}) - \bar{q} = 0$.

Next we show that the equilibrium eventually enters the regime with $m = 1 - A$ as y increases. As q tends to 0, then $\omega(q)\mu'(q)/\omega'(q) \rightarrow 0$ and thus there exists $\underline{q} > 0$ such that (64) exceeds (65) for all $q < \underline{q}$. Therefore the equilibrium stays in the regime $m = 1 - A$ for all $q < \underline{q}$ as shown in Figure 12. Let $\underline{y} = A^s - \mu(\underline{q})$ so that the equilibrium has $m = 1 - A$ for all $y \geq \underline{y}$.

Figure 12: The equilibrium path is contained in the shaded area.

Consider the existence and uniqueness of equilibrium for $q \in [\underline{q}, q^s]$ and $y \in [0, \underline{y}]$. By Theorem 58.5 in Tennenbaum and Pollard (1985), there is a unique solution for $q(y)$ in (66) provided that $f(y, q)$ is Lipschitz continuous, namely there is a real constant $K > 0$ such that

$$|f(y, q') - f(y, q'')| \leq K|q' - q''|$$

for every $y \in [0, \underline{y}]$ and $q', q'' \in [\underline{q}, q^s]$. For any $y \in [0, \underline{y}]$, one can check that the slope of f with respect to q is bounded provided that u'' is bounded. Therefore, f is Lipschitz continuous.

Next we express m as a function of y . By using the solution of $q(y)$ and (63),

$$m(y) = \frac{\alpha \sigma (1 - A^s + y) \theta \{u[q(y)] - q(y)\} - [r + \lambda (\bar{A} - A^s + y)] \omega[q(y)]}{\lambda (\bar{A} - A^s + y) \omega'(q) \partial q / \partial y + \alpha \sigma \theta \{u[q(y)] - q(y)\}}. \quad (67)$$

By (19) the ODE that determines y_t is

$$\dot{y}_t = -\lambda m(y_t) (\bar{A} - A^s + y_t).$$

One can check that the slope of the right side with respect to y is bounded for all $y \leq \underline{y}$. Therefore the right side is Lipschitz continuous and the solution for y_t is unique for any given initial condition. Since $q(y)$ and y_t exist and are unique, q_t and A_t exist and also are unique.

For $A < A^s - \underline{y}$, the equilibrium is in the regime with $m = 1 - A$. The ODE for A , (22), is a Riccati equation, which has a closed form solution:

$$A_t = \frac{\bar{A} - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}, \quad (68)$$

where $A_0 = 0$ is the initial condition. By (68) and (21) we can solve for the value of money in closed form.

$$\omega(q_t) = \omega_0 e^{rt} \left\{ 1 + \frac{\bar{A} [1 - e^{-\lambda(1-\bar{A})t}]}{(1 - \bar{A})} \right\}. \quad (69)$$

where ω_0 is the initial value of $\omega(q_t)$ at $t = 0$. We can solve for ω_0 by first solving t_0 in $A_{t_0} = A^s - \underline{y}$ where A_t is given by (68). Then we derive ω_0 by solving $\omega(q_{t_0}) = \omega(\underline{q})$ where $\omega(q_{t_0})$ is derived by evaluating (69) at $t = t_0$. Since $\underline{q} > 0$ as discussed above, $\omega_0 > 0$ by (69).

Part 3. By (23) the inequality (62) is equivalent to

$$\frac{\omega(q)}{\omega'(q)} \frac{r + \lambda(\bar{A} - A)}{(1 - A)\lambda(\bar{A} - A)} \leq \frac{1}{\mu'(q)}. \quad (70)$$

Suppose $A = A^s$ and $q = q^s$. Since $A^s = \mu(q^s)$ and A^s solves the equality (14), the inequality above is equivalent to

$$\frac{\mu'(q^s)}{\mu(q^s)} \frac{\omega(q^s)}{\omega'(q^s)} \leq \frac{1 - \theta}{\theta}.$$

This proves the claim concerning (27) in the proposition. Finally, if $m = 1 - A$ near the neighborhood of the steady state, then $\dot{A}$ in (11) is strictly positive because $\bar{A} - A_t > 0$ and $1 - A_t > 0$ near the steady state. Therefore A_t converges to A^s in finite time near the steady state. When $m < 1 - A$, $m \rightarrow 0$ as $(q, A) \rightarrow (q^s, A^s)$ because the denominator in (67) is positive by (10) and (14), and it vanishes as $(q, A) \rightarrow (q^s, A^s)$. Moreover, since $\partial q / \partial y = -1 / \mu'[q(y)]$ when $m < 1 - A$, the numerator in (67) can be written as

$$-\lambda(\bar{A} - A^s + y) \frac{\omega'[q(y)]}{\mu'[q(y)]} + \alpha \sigma \theta \{u[q(y)] - q(y)\} = (1 - \theta) \alpha \sigma \{u[q(y)] - q(y)\} \left[\frac{\theta}{1 - \theta} - \frac{\omega'[q(y)] / \omega[q(y)]}{\mu'[q(y)] / \mu[q(y)]} \right]. \quad (71)$$

The equation is true because when $m < 1 - A$ we have $A = \mu(q)$, and thus by (10) and $A = A^s - y$

$$\lambda(\bar{A} - A^s + y) = \frac{(1 - \theta) \alpha \sigma \mu[q(y)] \{u[q(y)] - q(y)\}}{\omega(q)}. \quad (72)$$

As $q(y) \rightarrow q^s$, the right side of (71) converges to a strictly positive value by (27) and therefore m in (67) vanishes as $(q, A) \rightarrow (q^s, A^s)$. It follows that $\dot{A}$ vanishes by (19) and thus $A_t \rightarrow A^s$ only asymptotically. ■

Proof of Lemma 1. Parts 1 and 2. By Proposition 2 mining and trades coexist near the steady state if

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}.$$

By differentiating (10),

$$\frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} = \frac{(\bar{A} - A^s) [1 - \epsilon(q^s)]}{\bar{A}\omega'(q^s)[1 - q^s/u(q^s)]} \quad (73)$$

Hence, (27) can be reexpressed as:

$$\frac{(\bar{A} - A^s) [1 - \epsilon(q^s)]}{\bar{A}\omega'(q^s)[1 - q^s/u(q^s)]} > \frac{1-\theta}{\theta}. \quad (74)$$

The left side rises in q^s by $\epsilon'(q) \leq 0$ and the concavity of $u(q)$. As λ rises, A^s rises and q^s falls by Proposition 1, and thus the left side of (74) falls. As $\lambda \rightarrow \infty$, $A \rightarrow \bar{A}$ and the right side of (74) vanishes, so λ^* is finite.

Next, by Proposition 1, as $\sigma\alpha$ increases, A^s remains constant but q^s rises. Hence, the left side of (74) rises. When $\alpha\sigma$ is sufficiently small, q is arbitrarily close to 0. As $q \rightarrow 0$, the left side of (74) vanishes by L'Hospital's Rule and thus the inequality fails. From (13), as $\alpha\sigma \rightarrow \infty$, $q^s/u(q^s) \rightarrow 1$ and the left side of (74) goes to infinity. This proves that (27) holds if $\alpha\sigma$ is sufficiently large and thus $\kappa^* \in (0, \infty)$.

Part 3. We show $m \in (0, 1 - A)$ is impossible when $\theta < 1/2$. As discussed before (71), the denominator in the right side of (67) is strictly negative for all $q < q^s$ and $A < A^s$. Suppose $m \in (0, 1 - A)$, then the numerator can be written as (71) and it is negative if and only if

$$\frac{\omega(q)}{\omega'(q)} \frac{\mu'(q)}{\mu(q)} > \frac{1-\theta}{\theta} \iff 1 - \frac{\omega(q)\{\alpha\sigma(1-\theta)[u'(q)-1] + \lambda\omega'(q)\}}{\omega'(q)\{\alpha\sigma(1-\theta)[u(q)-q] + \lambda\omega(q)\}} > \frac{1-\theta}{\theta}.$$

The second inequality uses the definition of μ in (10). If λ is sufficiently large, then $q^s < q^*$ by Proposition 1. In this case $u'(q) - 1 > 0$ for all $q \leq q^s$ and thus the left side is less than 1. If $\theta < 1/2$, then $(1-\theta)/\theta > 1$ and therefore this condition always fails. As a result $m < 0$ and thus it is impossible for the equilibrium path to be in the regime with $m \in (0, 1 - A)$.

Part 4. The transition from the regime with $m = 1 - A$ to one with $m \in (0, 1 - A)$ must occur on the line $A = \mu(q)$ and the slope of the two trajectories must be the same, namely

$$\left. \frac{\partial q}{\partial A} \right|_{m=1-A} = \left. \frac{\partial q}{\partial A} \right|_{m \in (0, 1-A)}.$$

The equality holds because if the right side is strictly larger, then the measure of miner implied by the trajectory $A = \mu(q)$ strictly exceeds $1 - A$ and thus it cannot be an equilibrium. If the left side is strictly larger, then the trajectory in the $m = 1 - A$ regime will cut the line $A = \mu(q)$ from below, but in this case no agents has incentive to mine and thus it is impossible to have $m = 1 - A$. Therefore the slope of the trajectories must be the same at the transition point. From (23) and (10), the displayed equation is the same as

$$\frac{\omega(q)}{\omega'(q)} \frac{r + \lambda(\bar{A} - A)}{(1 - A)\lambda(\bar{A} - A)} = \frac{1}{\mu'(q)}.$$

By the definition of μ in (10) and $A = \mu(q)$, the equation can be rewritten as

$$\begin{aligned} & \left[1 - \frac{\omega(q)\{\alpha\sigma(1 - \theta)[u'(q) - 1] + \lambda\omega'(q)\}}{\omega'(q)\{\alpha\sigma(1 - \theta)[u(q) - q] + \lambda\omega(q)\}} \right] \frac{A[r + \lambda(\bar{A} - A)]}{(1 - A)\lambda(\bar{A} - A)} = 1 \\ \iff & \frac{\alpha\sigma(1 - \theta)[1 - \epsilon(q)]}{\omega'(q)} \left[\frac{u(q)}{\alpha\sigma(1 - \theta)[u(q) - q] + \lambda\omega(q)} \right] \frac{A[r + \lambda(\bar{A} - A)]}{(1 - A)\lambda(\bar{A} - A)} = 1. \end{aligned}$$

Now we argue that the left side rises monotonically as we move along the line $A = \mu(q)$ in the (A, q) space. As we move along $A = \mu(q)$, both q and A increase. The fraction $1 - \epsilon(q)$ rises in q provided that u has decreasing elasticity. The fraction $1/\omega'(q)$ rises in q by the concavity of u . The fraction in the large bracket rises in q when $\alpha\sigma(1 - \theta) > \lambda\theta$. The last fraction in the left side rises in A . Altogether the left side increases as the trajectory move along $A = \mu(q)$. It follows that there can be at most one transition from the $m = 1 - A$ regime to the one with $m \in (0, 1 - A)$.

Finally, once the equilibrium enters the regime with $m \in (0, 1 - A)$, it cannot switch regime again. For suppose it switches regime, then the equilibrium must stay in the regime with $m = 1 - A$ as explained above. But this implies the equilibrium trajectory cannot converge to (A^s, q^s) because it cannot intersect the line $A = \mu(q)$ anymore. ■

Proof of Proposition 3. Part 1 From the proof of Proposition 2, for any (A', q') where $A' < A^s$ and $A' \leq \mu(q')$, there is a unique q_0 such that $(A_t, q_t) = (A', q')$ for some $t > 0$ only if it starts at $(A_0, q_0) = (0, q_0)$. As a result if two equilibrium trajectories have different initial values for q , then they will not intersect in the (A, q) space. It follows that any equilibrium trajectory with $q_0 < q_0^*$ is located below the trajectory with $q_0 = q_0^*$ that converges to (A^s, q^s) as illustrated in the left panel of Figure 13. Since $A_0 = 0 \leq \mu(q_0)$, the

trajectory is in the regime with $m = 1 - A_t$ in the beginning. From (25) in this regime the trajectory solves:

$$\begin{aligned} A_t &= \frac{\bar{A} \left[1 - e^{-\lambda(1-\bar{A})t} \right]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \\ \omega(q_t) &= e^{rt} \omega(q_0) \left[\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}} \right]. \end{aligned}$$

Since the trajectory cannot intersect with the one that converges to steady state, it must cross the locus $A = \mu(q)$ in the (A, q) space at some $A < A^s$. Let T_0 be the first time the trajectory satisfies $A_t = \mu(q_t)$. Since $A_{T_0} < A^s$ and $m = 1 - A_t$ for all $t < T_0$, the value of A_t reaches A_{T_0} in finite time, thus $T_0 < +\infty$. This proves Part 1(a).

Next, we argue that the trajectory only crosses the locus $A = \mu(q)$ once. Suppose $A_t > \mu(q_t)$, then $m_t = 0$ because no agent wants to mine. Therefore A_t remains constant. The value of money $\omega_t = \omega(q_t)$ solves (7) with $m = 0$, i.e.,

$$\dot{\omega} = r\omega - \alpha\sigma(\theta - A_{T_1})[u(q) - q]. \quad (75)$$

Using that $\dot{\omega} = 0$ when $(A_t, q_t) = (A^s, q^s)$ and $\dot{\omega}/\omega$ increases in q , it follows that $\dot{\omega} < 0$. See right panel of Figure 13. It follows that the trajectory (A_t, q_t) falls vertically whenever (A_t, q_t) lies below the locus $A = \mu(q)$ and $A_t < A^s$. Since the trajectory must and can only cross $A = \mu(q)$ once, there is $T_1(q_0) \in (T_0(q_0), +\infty)$ such that $A_{T_1} = \mu(q_{T_1})$ and $A_t > \mu(q_t)$ for all $t > T_1$. For all $t < T_1$, $m_t > 0$ and $\dot{q}_t > 0$ since otherwise the equilibrium trajectory would fall permanently below the locus $A = \mu(q)$. These properties are also illustrated in the left panel of the Figure 13.

Figure 13: Failing currency equilibria

For all $t > T_1$, $m_t = 0$, $A_t = A_{T_1}$, and $\dot{\omega} < 0$ as discussed above. See right panel of Figure 13. By (75)

and the L'Hospital's Rule

$$\lim_{q \searrow 0} \frac{\dot{\omega}}{\omega} = r - \frac{\alpha \sigma (\theta - A_{T_1})}{u'(0)/[u'(0) - 1] - \theta}.$$

Since the right side is constant for $t \geq T_1$, ω_t falls at a constant percentage rate when $q \approx 0$. Therefore ω_t converges to 0 asymptotically. This proves Part 1(b).

Part 2 If (27) holds, then there is a $\underline{T} < +\infty$ such that for all $t \geq \underline{T}$, $A_t = \mu(q_t)$ along the unique equilibrium, (A_t^*, q_t^*) , leading to (A^s, q^s) . For all $T \geq \underline{T}$, we can construct an equilibrium such that $(A_t, q_t) = (A_t^*, q_t^*)$ for all $t \leq T$ and $m_t = 0$ for all $t \geq T$. The trajectory up to T is the solution to the system of ODEs in backward time characterized in the proof of Proposition 2. Since $A_T = \mu(q_T)$, at time T agents are indifferent between mining or not. We select $m_T = 0$. As a result, $\dot{A}_T = 0$ and

$$\dot{q}_T = \frac{r\omega(q_T) - \alpha\sigma(\theta - A_T)[u(q_T) - q_T]}{\omega'(q_T)} < 0.$$

Since $\dot{q}_T < 0$, the trajectory falls below the locus $A = \mu(q)$. As a result for all $t > T$, $m_t = \dot{A}_t = 0$, and $\dot{q}_t < 0$. The rest of the argument is similar to the proof of Part 1 of Proposition 3. ■

Proof of Proposition 4. For $d > 0$, the value functions and equilibrium conditions are detailed in Appendix 6. Suppose there is a $T > 0$ such that $A_t > \mu(q_t)$ for all $t > T$. Then, $m_t = 0$ and $A_t = A_T$ for all $t > T$. The law of motion of $\omega_t = \omega(q_t)$ is given by

$$\dot{\omega}_t = r\omega_t - d - \alpha\sigma(\theta - A_T)S(\omega_t).$$

where $S(\omega_t) = u(q_t) - q_t$, $S'(\omega_t) = [u'(q_t) - 1] / [(1 - \theta)u'(q_t) + \theta]$, and $S''(\omega_t) < 0$. Therefore, the right side is convex in ω_t , it approaches $-d$ as ω_t goes to 0 and $+\infty$ as ω_t grows large. Consider a trajectory where ω_t approaches 0. Then, $\dot{\omega}_t \rightarrow -d$, which implies that ω_t becomes negative for some finite t , which is inconsistent with $\omega(q_t) \geq 0$. This rules out equilibria where $A_t > \mu(q_t)$. The rest of the proof is similar to the one of Proposition 2. ■

Proof of Proposition 5.

From (18), an equilibrium under time-varying money growth rates, $\{\pi_t\}$, is a bounded solution, $\{\omega_t, A_t, m_t\}$,

to the following system:

$$\dot{\omega} = \left(r + \frac{\pi A}{m}\right) \omega - \alpha \sigma (1 - A - m) \theta [u(q) - q], \quad (76)$$

$$\dot{A} = \pi_t A \quad (77)$$

$$m = \min \left\{ \frac{\pi \omega}{\alpha \sigma (1 - \theta) [u(q) - q]}, 1 - A \right\}. \quad (78)$$

Part 1. Suppose first that $\pi_t = \bar{\pi}$. If $m < 1 - A$, then (76) and (78) can be rewritten as:

$$\frac{\dot{\omega}}{\omega} = r - \alpha \sigma (\theta - A) \frac{u(q) - q}{\omega} + \frac{\bar{\pi} \theta}{1 - \theta}. \quad (79)$$

At the steady state, $A^s = \bar{A}$ and $r\omega(q^s) = \alpha \sigma (\theta - \bar{A}) [u(q^s) - q^s]$. Hence, close to the steady state, $\dot{\omega}/\omega = \bar{\pi} \theta / (1 - \theta)$. The solution for m is interior if $\bar{\pi} (\theta - \bar{A}) / r(1 - \theta) < 1 - \bar{A}$. Otherwise, $m = 1 - A$ and from (76)

$$\frac{\dot{\omega}}{\omega} = r + \frac{\bar{\pi} \bar{A}}{1 - \bar{A}}.$$

Combining these two equations gives (28). One can check that $\dot{\omega}/\omega = \bar{\pi} \theta / (1 - \theta)$ in (28) if and only if the implied m is interior.

Part 2. A necessary condition for $\dot{\omega} = 0$ is $m < 1 - A$. Substituting m by its expression from (78) into (76) and setting $\dot{\omega} = 0$ we obtain (29).

Part 3. Consider an outcome (ω, A) such that $\omega < \omega^s$. From (30), $\pi = 0$ and from (78) $m = 0$. From (76),

$$\dot{\omega}/\omega \leq r - \alpha \sigma (\theta - A) \frac{[u(q) - q]}{\omega},$$

where we used $\pi/m \leq \alpha \sigma (1 - \theta) [u(q) - q] / \omega$, this inequality is true because agents prefer not to mine. Using $r = \alpha \sigma (\theta - A) [u(q^s) - q^s] / \omega^s$ and the fact that the right side is decreasing in ω^s , it follows that $\dot{\omega}/\omega < 0$. Moreover, $\dot{\omega}/\omega$ decreases with ω so that ω converges to 0. By Proposition 4, if money pays an arbitrarily small interest, such trajectories are not part of an equilibrium. Consider next an outcome (ω, A) such that $\omega > \omega^s$. From (30) and (78) $m = 1 - A$. From (76)

$$\frac{\dot{\omega}}{\omega} = r + \frac{\alpha \sigma (1 - \theta) [u(q) - q] A}{\omega} > 0.$$

The value of money grows at a rate higher than r which violates the transversality condition, $\lim_{t \rightarrow +\infty} e^{-rt} \omega_t = 0$. Therefore, the only trajectory consistent with an equilibrium is $\omega_t = \omega^s$. ■

Lemma 2 *There exists a pair of (m_t, A_t) that solves the planner's problem (80)-(82), provided that $q_t = q^*$.*

Proof. The planner's problem is given by:

$$\max_{q_t, m_t, A_t} \int_0^{+\infty} e^{-rt} \alpha \sigma A_t (1 - A_t - m_t) [u(q_t) - q_t] dt \quad (80)$$

$$\text{s.t. } \dot{A} = m_t \lambda(\bar{A} - A_t), \quad (81)$$

$$m_t \leq 1 - A_t \text{ and } A(0) = A_0. \quad (82)$$

The objective is the discounted sum of all trade match surpluses where the aggregate measure of trade matches between a money holder and a producer is $\alpha \sigma A(1 - A - m)$. The state variable is the money supply which increases with the measure of miners who successfully did money from the ground. There are two control variables, the measure of miners and output in a match. The measure of miners has an upper bound given the measure of agents without money. If the planner can dictate the output traded in each match, he will choose $q_t = q^*$ for all t . One can rewrite (81) as

$$\dot{A} = \min\{m_t, 1 - A_t\} \lambda(\bar{A} - A_t). \quad (83)$$

This reformulation is useful because it ensures the planner never chooses $m_t > 1 - A_t$ even when it is feasible. As a result we could drop the constraint $m_t \leq 1 - A_t$ and only impose $m_t \in [0, 1]$. Next we apply a standard result to show the existence of solution for an infinite horizon optimization problem. By Theorem 15 in Seierstad and Sydsaeter (1986) there exists a (A_t, m_t) that solves the new planner's problem if

1. The right side of (83) and the integrand in (80) are continuous in m_t and A_t .
2. There exists a function $\phi(t)$ such that $\phi(t) \geq |e^{-rt} \alpha \sigma A_t (1 - A_t - m_t) [u(q^*) - q^*]|$ for all admissible (m_t, A_t) and $\int_0^\infty \phi(t) dt < \infty$.
3. There exists non-negative function $a(t)$ and $b(t)$ such that

$$\min\{m_t, 1 - A_t\} \lambda(\bar{A} - A_t) \leq a(t) A_t + b(t)$$

for all $A_t \in [0, \bar{A}]$ and $m_t \in [0, 1]$.

4. The set

$$N(A, t) = \{(e^{-rt} \alpha \sigma A_t (1 - A_t - m_t) [u(q^*) - q^*] + \gamma, \min\{m_t, 1 - A_t\} \lambda(\bar{A} - A_t)) | m_t \in [0, 1], \gamma \leq 0\}$$

is convex for all A_t and t .

It is easy to see condition (1) is satisfied. Condition (2) is satisfied by assuming $\phi(t) = e^{-rt} \alpha \sigma \bar{A} [u(q^*) - q^*]$. Condition (3) is satisfied because the right side of (83) is bounded above for all A_t and m_t by $\lambda \bar{A}$. The last condition is satisfied because the first component of $N(A, t)$ is linear in m_t and γ and the second component is concave in m_t and constant in γ . It follows that there is a pair of (A_t, m_t) that solves the planner's problem provided that $q_t = q^*$. ■

Proof of Proposition 7. Part 1 It is obvious that the optimal output is $q_t = q^*$ provided that trade happens. By Lemma 2 there is a solution to the planner's problem provided that $q_t = q^*$. Now we characterize this solution and argue it is unique. The current value Hamiltonian corresponding to (80)-(82) is:

$$\mathcal{H}(A, m, \xi, \nu) = \alpha \sigma A (1 - A - m) [u(q^*) - q^*] + \xi m \lambda (\bar{A} - A) + \nu (1 - A - m),$$

where ξ is the co-state variable associated with A , and ν is the Lagrange multiplier associated with $m \leq 1 - A$. The FOC with respect to m is:

$$m \begin{cases} = 0 \\ \in [0, 1 - A] \end{cases} \quad \text{if} \quad -\alpha \sigma A [u(q^*) - q^*] + \xi \lambda (\bar{A} - A) - \nu \begin{cases} < 0 \\ = 0 \end{cases}, \quad (84)$$

together with the complementary slackness condition, $\nu(1 - A - m) = 0$. The co-state variable satisfies the following ODE:

$$r\xi = \alpha \sigma (1 - 2A - m) [u(q^*) - q^*] - \xi m \lambda - \nu + \dot{\xi}. \quad (85)$$

The stationary solutions to (81) and (85), $\dot{A} = \dot{\xi} = 0$, are such that $m = 0$ and

$$r\xi = \alpha \sigma (1 - 2A) [u(q^*) - q^*] \quad (86)$$

$$\xi \lambda (\bar{A} - A) \leq \alpha \sigma A [u(q^*) - q^*]. \quad (87)$$

We denote A^* the lowest value of A that satisfies (87). It is the lowest root of the following quadratic equation,

$$2A^2 - \left(1 + 2\bar{A} + \frac{r}{\lambda}\right) A + \bar{A} = 0.$$

In closed form:

$$A^* = \frac{(1 + 2\bar{A} + r/\lambda) - \sqrt{(1 + 2\bar{A} + r/\lambda)^2 - 8\bar{A}}}{4}.$$

It is easy to check that $A^* < \min\{1/2, \bar{A}\}$. We denote

$$\xi^* = \alpha \sigma (1 - 2A^*) [u(q^*) - q^*] / r.$$

Now we argue that A_t converges to A^* . Since A_t is continuous, non-decreasing and bounded above by $\bar{A}$, eventually it converges and m vanishes. The process A_t cannot converge to any $A' < A^*$. Suppose it does. Since A^* is the smallest solution to (86) and (87), for all $A' < A^*$ we have

$$-\alpha\sigma A' [u(q^*) - q^*] + \xi\lambda (\bar{A} - A') > 0.$$

This implies $\nu > 0$ by (84) and thus $m = 1 - A$ by the complementary slackness condition. Since $A' < 1$, $m = 1 - A' > 0$ and thus A_t cannot converge to A' . The process A_t also cannot converge to any $A > A^*$ because when A_t goes above A^* the inequality (87) holds strictly and thus $m = 0$ by (84). It follows that the optimal solution can only converge to A^* .

We conjecture and then verify that the solution to the planner's problem is such that for all $A < A^*$, $m = 1 - A$. Then, the ODEs (81) and (85) can be rewritten as

$$\dot{A} = \lambda(1 - A)(\bar{A} - A) \quad (88)$$

$$\dot{\xi} = [r + (1 + \bar{A} - 2A)\lambda] \xi. \quad (89)$$

The ODE for A , (88), is a Riccati equation, which can be solved in closed form. This leads to

$$A_t = \frac{\bar{A} [1 - e^{-\lambda(1-\bar{A})t}]}{1 - \bar{A}e^{-\lambda(1-\bar{A})t}} \quad \text{and} \quad \xi_t = \xi_0 e^{[r + \lambda(1-\bar{A})]t} \left(\frac{1 - \bar{A}e^{-\lambda(1-\bar{A})t}}{1 - \bar{A}^2} \right)^2,$$

where we used that $A_0 = 0$. Hence, there is a unique solution to (88)-(89). By the formula for A^* and A_t , one can solve for the time T^* . We denote the path defined by (88)-(89) by $\xi = \xi^p(A)$. From (88)-(89) the slope of $\xi = \xi^p(A)$ is

$$\xi^{p'}(A) = \frac{\dot{\xi}}{\dot{A}} = \frac{[r + (1 + \bar{A} - 2A)\lambda] \xi}{\lambda(1 - A)(\bar{A} - A)}.$$

From (84) $m = 1 - A$ is optimal only if

$$\xi^p(A) \geq \Omega(A) \equiv \frac{\alpha\sigma A [u(q^*) - q^*]}{\lambda(\bar{A} - A)} \quad \text{for all } A < A^*.$$

We now show that whenever $\xi^p(A) = \Omega(A)$ then $0 < \xi^{p'}(A) < \Omega'(A)$. To see this, we evaluate $\xi^{p'}(A)$ at $\xi = \Omega(A)$:

$$\begin{aligned} \xi^{p'}(A) \Big|_{\xi=\Omega(A)} &= \frac{A [r + (1 + \bar{A} - 2A)\lambda]}{\lambda(1 - A)\bar{A}} \frac{\alpha\sigma\bar{A} [u(q^*) - q^*]}{\lambda(\bar{A} - A)^2} \\ &< \Omega'(A) = \frac{\alpha\sigma\bar{A} [u(q^*) - q^*]}{\lambda(\bar{A} - A)^2} \quad \text{for all } A < A^*. \end{aligned}$$

Given that $\xi^p(A^*) = \Omega(A^*)$, there is no other solution $A < A^*$ to $\xi^p(A) = \Omega(A)$, and thus $\xi^p(A) \geq \Omega(A)$ for all $A < A^*$.

Finally we argue $m \in (0, 1 - A)$ cannot be optimal. Suppose $m \in (0, 1 - \tilde{A})$ at certain $(\tilde{A}, \tilde{\xi})$ where $\tilde{A} < A^*$. Then $\tilde{\xi} = \Omega(\tilde{A})$ by (84). By the ODE (81) and (85),

$$\xi^{p'}(\tilde{A}) = \frac{\dot{\xi}}{\tilde{A}} = \frac{\tilde{\xi}}{\lambda(\bar{A} - \tilde{A})} \left[\frac{1}{m} \left(r + \frac{\lambda(\bar{A} - \tilde{A})(2\tilde{A} - 1)}{\tilde{A}} \right) + \lambda + \frac{\alpha\sigma[u(q^*) - q^*]}{\tilde{\xi}} \right]. \quad (90)$$

Since $rA + \lambda(\bar{A} - A)(2A - 1) < 0$ for all $A < A^*$ by (86) and (87), the right side of (90) strictly increases in m . As discussed above, if $m = 1 - \tilde{A}$, then $\xi^{p'}(\tilde{A}) < \Omega'(\tilde{A})$ and thus $\xi^p(A)$ cuts $\Omega(A)$ from above at $(\tilde{A}, \tilde{\xi})$. By a similar argument, $\xi^p(A)$ must be lower than $\Omega(A)$ for all $A \in (\tilde{A}, A^*)$. But then it is impossible for $\xi(A)$ to reach ξ^* as $A \rightarrow A^*$ because $m = 0$ when $\xi^p(A) < \Omega(A)$ by (84). Therefore $m \in (0, 1 - A)$ is sub optimal.

Part 2 In order to guarantee that $m_t = 1 - A_t$ for all $t < T^*$ we set $\theta(t) = 1$ for all $t < T^*$ so that producers receive no gains from trade. From (13) the buyer's bargaining power that implements q^* when T^* has been reached is θ^* defined in (34). Incentive feasibility means $\theta^* \in [0, 1]$, which holds if and only if (32) holds. From (14) agents stop mining when A^* is reached if

$$r \leq \frac{(1 - A^*)[u(q^*) - q^*]}{q^*} [\alpha\sigma A^* - \lambda(\bar{A} - A^*)] - \lambda(\bar{A} - A^*).$$

This inequality can be rearranged to give (33). ■

Proof of Proposition 8. Part 1. Consider a steady state such that $q^s = q^*$ and $\phi^s = d/r$. Such a steady state requires that $\omega(q^*) \leq \phi^s A^s$. Such an equilibrium exists if the curve representing (41) is located below the curve representing (42) when $A = r\omega(q^*)/d$. Alternatively, the left side of (41) is greater than the right side when $\phi = d/r$ and $A = r\omega(q^*)/d$, i.e., (43) holds. From (37)

$$\frac{\dot{\phi}}{\phi} = \frac{r(\phi - \phi^s)}{\phi} - \alpha(1 - m)\sigma\theta \left[\frac{u'(q_t) - 1}{(1 - \theta)u'(q_t) + \theta} \right].$$

In the neighborhood of the steady state $q_t = q^*$ and the second term on the right side is equal to 0. The only solution to $\dot{\phi} = r(\phi - \phi^s)$ is such that $\phi_t = \phi^s$. Any other path violates $\lim_{t \rightarrow +\infty} e^{-rt}\phi_t = 0$ or $\phi_t \geq d/r$. Solving $\dot{\phi} = r(\phi - \phi^s)$ in backward time leads to $\phi_t = d/r$ for all t . Using that $\lambda(\bar{A} - A^s)\phi = \alpha\sigma(1 - \theta)[u(q^*) - q^*]$ at the steady state, it follows that $\lambda(\bar{A} - A_t)\phi > \alpha\sigma(1 - \theta)[u(q^*) - q^*]$ for all t such that $A_t < A^s$, and hence $m_t = 1$ for all t such that $A_t < A^s$. See the phase diagram in the right panel of Figure 10.

Part 2. If (43) does not hold then the unique steady state features $q^s < q^*$ and $\phi^s > d/r$. By the same reasoning as in the proof of Proposition 2, there is a unique equilibrium leading to the steady state. By (37) and (40) the slope of the trajectory in the (A, ϕ) space is

$$\frac{\partial \phi}{\partial A} = \frac{\dot{\phi}}{\dot{A}} = \frac{\frac{1}{m} \left[r\phi - d - \left(\frac{\alpha\sigma\theta[u'(q)-1]}{(1-\theta)u'(q)+\theta} \right) \phi \right] + \frac{\alpha\sigma\theta[u'(q)-1]}{(1-\theta)u'(q)+\theta} \phi}{\lambda(\bar{A} - A)}. \quad (91)$$

Since q_t increases over time and $q^s < q^*$, the bargaining solution implies

$$\omega(q) = (1 - \theta)u(q) + \theta q = \phi A. \quad (92)$$

By the implicit function theorem $\partial q / \partial (\phi A) = 1/\omega'(q)$ and thus $S'(\phi A) = [u'(q) - 1]/\omega'(q)$. By (41), (92) and the definition of S , the slope $\partial \phi / \partial A$ can be written as

$$\frac{\partial \phi / \phi}{\partial A / A} = \frac{\alpha\sigma(1 - \theta) \frac{[u'(q)-1]}{\omega'(q)} + \lambda}{\alpha\sigma(1 - \theta) \left[1 - \frac{\omega(q)[u'(q)-1]}{\omega'(q)[u(q)-q]} \right] \frac{[u(q)-q]}{\omega(q)}}}.$$

Let $m_I(q, \phi)$ be the measure of miners implied by the indifference locus. To solve for $m_I(q, \phi)$ replace $\partial \phi / \partial A$ in the above equation by (91). Then by (41) and (92)

$$\begin{aligned} & \left\{ 1 - \frac{\omega(q)[u'(q) - 1]}{\omega'(q)[u(q) - q]} \right\} \left\{ \frac{r - d/\phi - \frac{\alpha\sigma\theta[u'(q)-1]}{\omega'(q)}}{m_I(q, \phi)} + \frac{\alpha\sigma\theta[u'(q) - 1]}{\omega'(q)} \right\} \\ &= \alpha\sigma(1 - \theta) \frac{[u'(q) - 1]}{\omega'(q)} + \lambda. \end{aligned}$$

The right side falls strictly in q by the concavity of u . The left side rises in ϕ because $d \geq 0$. The first term in the large square bracket is strictly negative when $q < q^s$ and $\phi < \phi^s$ by the definition of the steady state. Therefore, the left side rises strictly in m_I . If the elasticity of $u(q)$ falls in q then the expression in the large bracket rises in q , and thus the left side rises in q . Altogether as q and ϕ increase, $m_I(q, \phi)$ must fall strictly to balance the equation.

The rest of the proof is similar to that of Proposition 2. Suppose the dynamical system starts from the steady state and goes backward in time. By the proof logic leading to Proposition 2, the economy lies in the regime with $m \in [0, 1]$ if and only if $m_I(q_t, \phi_t) \leq 1$. Since q_t and ϕ_t increase over time and $m_I(q, \phi)$ falls as q and ϕ increase, the system has at most one regime switch. Since $m_t = 1$ when A is sufficiently small, as t increases from 0 the equilibrium either stays in the regime with $m_t = 1$ or has exactly one regime switch from $m_t = 1$ to $m_t < 1$. The economy has $m_t = 1$ until it reaches the steady state if and only if the slope of

the trajectory when $m = 1$,

$$\frac{\partial \phi / \phi}{\partial A / A} = \alpha \sigma \theta \frac{S(A\phi)}{\lambda(\bar{A} - A)\phi} \nu(A\phi),$$

is larger than the slope of the locus where agents are indifferent between occupations, i.e.,

$$\frac{\partial \phi / \phi}{\partial A / A} = \frac{\nu(A\phi) + A/(\bar{A} - A)}{1 - \nu(A\phi)}.$$

This happens when (44) does not hold. Otherwise, the equilibrium features $m_t < 1$ in the neighborhood of the steady state.

Part 3. From (42) if $d = 0$ then q^s solves

$$r = \alpha \sigma \theta \left\{ \frac{u'(q) - 1}{(1 - \theta)u'(q) + \theta} \right\}.$$

A unique solution exists if $r < \alpha \sigma \theta / (1 - \theta)$. The characterization of the unique equilibrium leading to the steady state is similar to Part 2. ■

Proof of Proposition 11. For given π , q is uniquely determined by (37). Assuming $u'(0) = +\infty$, the solution is positive if $(r + \pi)(1 - \theta) < \alpha \sigma \theta$. Given q , a is determined by the bargaining solution $a = (1 - \theta)u(q) + \theta q$. Given q and a , the measure of miners is given by (58), i.e.,

$$m = \frac{\pi a}{\alpha \sigma [u(q) - q] - ia + \pi a}.$$

Notice from (37) that a maximizes $\alpha \sigma \theta [u(q) - q] - ia$. Hence, $\alpha \sigma [u(q) - q] - ia > 0$ and $m \in (0, 1)$ for all $\pi > 0$ such that $a > 0$. In the neighborhood of $\pi = 0^+$,

$$\frac{\partial m}{\partial \pi} = \frac{a}{\alpha \sigma [u(q) - q] - ra} > 0.$$

It is easy to check from (37) that a^* is a decreasing function of i , and hence $\partial a^* / \partial \pi < 0$. By the bargaining solution $a = (1 - \theta)u(q) + \theta q$ we also have $\partial q / \partial \pi < 0$. ■

Proof of Proposition 12. The total welfare is

$$W \equiv \alpha \sigma \beta (1 - m) [u(q) - q] = \min \{ \alpha \sigma (1 - m), m \lambda \} [u(q) - q].$$

Both m and q are continuous in π . If $\pi = 0$ then agents have no incentive to mine and thus $W = 0$. Moreover $\lim_{\pi \rightarrow \infty} W = 0$ because agents do not carry money when $\pi \rightarrow \infty$. For all $\pi \in (0, +\infty)$, $W > 0$ because

$m \in (0, 1)$ as explained in the proof of Proposition 11. Hence, W reaches its maximum when π is strictly positive but finite.

It is clearly suboptimal to choose π such that $m\lambda > \alpha\sigma(1 - m)$ in equilibrium. So consider the case $m\lambda \leq \alpha\sigma(1 - m)$. The confirmation probability is given by

$$\beta = \frac{\lambda m}{(1 - m)\alpha\sigma} \quad (93)$$

and the welfare is

$$W = \lambda m[u(q) - q].$$

By (93), (37) and (58), we have

$$i = \frac{[u'(q) - 1]q}{u(q) - q} \left(\frac{\pi}{m} + r \right). \quad (94)$$

The right side falls in q because $\epsilon(q)$ is decreasing in q . Therefore, (94) defines a negative relationship between q and m . By (37) and (93)

$$i = \frac{\lambda m}{1 - m} [u'(q) - 1]. \quad (95)$$

Equation (95) defines a positive relationship between q and m . Therefore, the equilibrium is unique for a given i . It is easy to show that as λ rises, q , λm , β and W rise, and m falls. Similarly one can show m increases in i .

Now we argue that there exists λ^* such that if $\lambda > \lambda^*$ then the optimal inflation rate implements $\beta = 1$ and thus $m = m^*$. We focus on positive inflation, namely $i \in [r, \infty)$. Assume $i = i'$ for some arbitrary $i' \in [r, \infty)$. As λ rises we have $m = m^*$ when λ reaches some finite λ' by (94) and (95). For $i > i'$ and $\lambda \geq \lambda'$, we also have $\beta = 1$ because m is non-decreasing in i and β is non-decreasing in λ as argued above. This implies $\beta = 1$ for all $i \geq i'$, provided that $\lambda \geq \lambda'$.

Now consider $i \in [r, i']$. For each i in this interval, there exists a finite λ_i such that if $\lambda \geq \lambda_i$ then $\beta = 1$ by (94) and (95). Since $[r, i']$ is a closed and compact interval, by Weierstrass's extreme value theorem there exists a finite $\lambda'' \equiv \max_{i \in [r, i']} \lambda_i$ such that if $\lambda \geq \lambda''$ then $\beta = 1$ for all $i \in [r, i']$. Finally let $\lambda^* \equiv \max\{\lambda', \lambda''\}$. If $\lambda \geq \lambda^*$, then the optimal π necessarily implements $\beta = 1$ and thus $m = m^*$. ■

Appendix B: Search while mining

In the main text we described the decision to mine money as an occupation choice and the cost to mine as the foregone opportunities in the trading sector. In the following we allow miners to search for trading opportunities. We will show that this possibility can change the dynamics of prices depending on the efficiency of the mining technology.

Suppose now that agents who choose to mine can still receive opportunities to produce. Specifically, miners meet a money holders at Poisson arrival rate $\alpha\eta A$ where $\eta \leq 1$ while non-miners meet money holders at rate αA . It follows that a money holder meets someone who can produce at rate

$$\frac{m\alpha\eta A + (1 - A - m)\alpha A}{A} = \alpha [1 - A - m(1 - \eta)], \quad (96)$$

and he likes the good offered with probability σ . The case studied so far was $\eta = 0$. Another polar case is $\eta = 1$ where agents can engage in mining without foregoing any trading opportunity. In that case, the buyer's matching rate is simply $\alpha(1 - A)$.

Agents' value functions solve:

$$rV_1 = \alpha\sigma [1 - A - m(1 - \rho)] \theta [u(q) - q] + \dot{V}_1 \quad (97)$$

$$rV_0 = \alpha\sigma A\rho(1 - \theta) [u(q) - q] + \max \{ \alpha\sigma A(1 - \rho)(1 - \theta) [u(q) - q], \lambda (\bar{A} - A) \omega(q) \} + \dot{V}_0. \quad (98)$$

The key novelty in (98) is that the opportunity cost of mining has been multiplied by $1 - \rho$. In particular, if $\rho = 1$ there is no opportunity cost of mining and all agents without money mine. Subtracting (98) from (97) the value of money solves:

$$\begin{aligned} r\omega(q) = & \left\{ 1 - \left[1 + \rho \left(\frac{1 - \theta}{\theta} \right) \right] A - m(1 - \rho) \right\} \theta \alpha \sigma [u(q) - q] \\ & - \max \{ \alpha\sigma A(1 - \rho)(1 - \theta) [u(q) - q], \lambda (\bar{A} - A) \omega(q) \} + \omega'(q)\dot{q}. \end{aligned} \quad (99)$$

The law of motion for A is:

$$\dot{A} = m\lambda (\bar{A} - A). \quad (100)$$

The locus of pairs (A, q) such that agents are indifferent between mining or not is given by:

$$A = \mu(q) \equiv \frac{\lambda \bar{A} \omega(q)}{\alpha\sigma(1 - \rho)(1 - \theta) [u(q) - q] + \lambda \omega(q)}$$

The μ -locus shifts to the right as ρ increases and it becomes vertical at $A = \bar{A}$ when $\rho = 1$. Let us start first with steady-state equilibria.

Proposition 13 (*Search while mining*) *There exists a steady-state monetary equilibrium iff*

$$r < \frac{\alpha\sigma}{1-\theta} \left[\frac{\alpha\sigma\theta(1-\eta) + \lambda(\theta - \bar{A})}{\alpha\sigma(1-\eta) + \lambda} \right]. \quad (101)$$

The steady-state money supply, A^s , increases with η while the value of money, q^s , decreases with η .

Suppose $\eta = 1$. There exists a monetary equilibrium if $r < \alpha\sigma(\theta - \bar{A})/(1-\theta)$ and it is such that A^s tends to $\bar{A} < \theta$. For all $A_0 < \bar{A}$ the unique equilibrium leading to the steady state is such that: A increases over time until it reaches $\bar{A}$; q increases over time if $\lambda > r/(\theta - \bar{A})$, decreases if $\lambda < r/(\theta - \bar{A})$, and remains constant if $\lambda = r/(\theta - \bar{A})$.

Proof. By the same reasoning as in Section 3.2, q^s solves (12),

$$r\omega(q) = (\theta - A)\alpha\sigma[u(q) - q],$$

and A^s is the smallest root to

$$\lambda(\bar{A} - A)(\theta - A) - A(1-\eta)(1-\theta)r = 0. \quad (102)$$

It is easy to check that A^s increases with ρ while q^s decreases with η . Moreover, as η approaches to 1, A^s approaches to $\min\{\theta, \bar{A}\}$. By the same reasoning as in the proof of Proposition 1 there exists a steady-state monetary equilibrium iff

$$\lim_{q \rightarrow 0} \{r\omega(q) - [\theta - \mu(q)]\alpha\sigma[u(q) - q]\} < 0.$$

Dividing by $\omega(q) > 0$ this condition can be reexpressed as:

$$\lim_{q \rightarrow 0} \left\{ r - \alpha\sigma \frac{[\theta - \mu(q)][u(q) - q]}{\omega(q)} \right\} < 0.$$

Using that $\lim_{q \rightarrow 0} \{[u(q) - q]/\omega(q)\} = 1/(1-\theta)$ and $\lim_{q \rightarrow 0} \mu(q) = \lambda\bar{A}/[\alpha\sigma(1-\eta) + \lambda]$ the condition above can be reexpressed as (101). In particular, when $\eta = 1$,

$$r < \frac{\alpha\sigma}{1-\theta} (\theta - \bar{A}).$$

In that case a necessary condition for a steady-state monetary equilibrium is $\bar{A} < \theta$. Hence, $A^s = \theta < \bar{A}$.

The condition $\alpha\sigma(\theta - \bar{A}) > r(1-\theta)$ guarantees the existence of a steady-state monetary equilibrium when $\eta = 1$. The system of ODEs, (99) and (100), becomes:

$$\begin{aligned} \omega'(q)\dot{q} &= [r + \lambda(\bar{A} - A)]\omega(q) - (\theta - A)\alpha\sigma[u(q) - q] \\ \dot{A} &= \lambda(1 - A)(\bar{A} - A) \end{aligned}$$

Linearizing the system around the steady state we obtain:

$$\begin{pmatrix} \dot{q} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} \frac{r\omega'(q^s) - (\theta - \bar{A})\alpha\sigma[u'(q^s) - 1]}{\omega'(q^s)} & \frac{-\lambda\omega(q^s) + \alpha\sigma[u(q^s) - q^s]}{\omega'(q^s)} \\ 0 & -\lambda(1 - \bar{A}) \end{pmatrix} \begin{pmatrix} q - q^s \\ A - A^s \end{pmatrix}.$$

If $(\theta - \bar{A})\alpha\sigma > r(1 - \theta)$ then $r\omega'(q^s) > (\theta - \bar{A})\alpha\sigma[u'(q^s) - 1]$. It follows that the determinant of the Jacobian matrix is negative, i.e., the steady state is a saddle point. The negative eigenvalue is $e_1 = -\lambda(1 - \bar{A})$ and the associated eigenvector is

$$\vec{v}_1 = \begin{pmatrix} \frac{[\lambda - r/(\theta - \bar{A})]\omega(q^s)}{[r + \lambda(1 - \bar{A})]\omega'(q^s) - (\theta - \bar{A})\alpha\sigma[u'(q^s) - 1]} \\ 1 \end{pmatrix}$$

where we used that $r\omega(q^s) = (\theta - \bar{A})\alpha\sigma[u(q^s) - q^s]$. The first component of $\vec{v}_1$ is of the same sign as $\lambda - r/(\theta - \bar{A})$. The solution to the linearized system is

$$\begin{pmatrix} q - q^s \\ A - A^s \end{pmatrix} = Ce^{-\lambda(1 - \bar{A})t} \vec{v}_1,$$

where C is some constant. Hence, in the neighborhood of the steady state,

$$\frac{\partial q}{\partial A} = \frac{[\lambda - r/(\theta - \bar{A})]\omega(q^s)}{[r + \lambda(1 - \bar{A})]\omega'(q^s) - (\theta - \bar{A})\alpha\sigma[u'(q^s) - 1]},$$

which is of the same sign as $\lambda - r/(\theta - \bar{A})$. If $\lambda > r/(\theta - \bar{A})$, then the saddle path in the neighborhood of the steady state is upward sloping, i.e., q and A increase over time. We can show that this result holds globally since the equation of the q -isocline is:

$$\frac{\omega(q)}{u(q) - q} = \frac{(\theta - A)\alpha\sigma}{r + \lambda(\bar{A} - A)}.$$

The q -isocline is upward sloping when $\lambda > r/(\theta - \bar{A})$. See left panel of Figure 14. By the same reasoning, if $\lambda < r/(\theta - \bar{A})$, then the saddle path is downward sloping and along the equilibrium path, q decreases while A increases. See middle panel of Figure 14. Finally, if $\lambda = r/(\theta - \bar{A})$, then the q -isocline is horizontal. In that case q is constant over time. See right panel of Figure 14. ■

According to (101) the set of parameter values for which a steady-state monetary equilibrium exists shrinks as η increases. If agents can meet trading partners more frequently while mining, then the opportunity cost of mining is lower and the incentives to mine are greater, which leads to a higher supply of money. But for a monetary equilibrium to exist, the money supply cannot be too large. A higher η also reduces the value of money. In the limiting case where $\eta = 1$, then there is no opportunity cost to engage in mining and all

Figure 14: Phase diagrams when agents can mine while searching for trading partners ($\eta = 1$).

agents without money mine, $m = 1 - A$. At the steady state the money supply is equal to the maximum stock of money that could be mined, $\bar{A}$. We now turn to the transition dynamics for this special case.

Proposition 13 shows that when there is no opportunity cost of mining, the correlation between the value of money and the money stock along the transitional path depends on the efficiency of the mining technology.²¹ If the mining intensity is high, the value of money increases with the money supply. If the mining intensity is low, then the opposite correlation prevails and the value of money decreases as the money supply increases. Finally, there is a mining rate such that the price level is constant, the value of money is independent of the money stock.

Finally, the path for q_t can be non-monotone when η is close but less than 1. As η falls below one, the q -isocline shifts downward. Hence, the steady state, (A^s, q^s) , is now located above the q -isocline, which implies that close to the steady state q increases. If λ is small, as in the middle panel of panel of Figure 14, the q -isocline is downward sloping and the upward trajectory of $[A_t, q_t]$ must cross it. As a result, for low values of A , q is decreases and it increases as A approaches the steady state.

²¹ While Proposition 13 focuses on the unique equilibrium leading to the steady state, there is also a continuum of equilibria where the value of money vanishes asymptotically. In the left panel of Figure 14, when λ is high, the value of money increases first and then decreases. In the middle and right panels, when λ is low, the value of money is monotone decreasing in time.

Appendix C: Endogenous mining intensity

Suppose that agents do not have to give up their occupation to engage in mining, $\rho = 1$, but they have to suffer a disutility cost. Moreover, we let agents choose the intensity with which they mine, λ . The flow cost of the mining technology is $c(\lambda)$ and for simplicity we assume it is quadratic, $c(\lambda) = \bar{c}\lambda^2/2$ with $\bar{c} > 0$.

An agent with one unit of asset will not engage in mining since money holdings have a unit upper bound. Therefore, V_1 solves the Bellman equation (1). The lifetime expected utility of an agent without money solves

$$rV_0 = \alpha\sigma A(-q + V_1 - V_0) + \max_{\lambda \geq 0} \{-c(\lambda) + \lambda(\bar{A} - A)(V_1 - V_0)\} + \dot{V}_0. \quad (103)$$

An agent without money receives an opportunity to produce at Poisson arrival rate $\alpha\sigma A$. While searching for trading partners, the agent engages in mining by choosing λ . The optimal mining intensity solves

$$\bar{c}\lambda = (\bar{A} - A)(V_1 - V_0). \quad (104)$$

The marginal cost of mining on the right side is equalized to the marginal gain on the right side. The mining intensity increases with the amount of asset that is still in the ground and the value of money.

In order to determine terms of trade in pairwise meetings we assume $\theta = 1$, buyers have all the bargaining power. Then $q = V_1 - V_0$ and from (104),

$$\lambda = \frac{(\bar{A} - A)q}{\bar{c}}. \quad (105)$$

From (1) and (103) the value of money solves the following ODE:

$$rq = \alpha\sigma(1 - A)[u(q) - q] - \frac{[(\bar{A} - A)q]^2}{2\bar{c}} + \dot{q}. \quad (106)$$

After substituting λ by its expression given by (105) the law of motion for the money supply is:

$$\dot{A} = \frac{(1 - A)(\bar{A} - A)^2 q}{\bar{c}}. \quad (107)$$

An equilibrium is a pair of time paths, (q_t, A_t) , that satisfies (106), (107), and the initial condition $A(0) = A_0$.

Proposition 14 (*Endogenous mining intensity.*) *Assume $\bar{A} < 1$. There is a unique monetary steady state and it is such that*

$$A^s = \bar{A} \quad (108)$$

$$rq^s = \alpha\sigma(1 - \bar{A})[u(q^s) - q^s]. \quad (109)$$

For any $A_0 < \bar{A}$ there exists a unique equilibrium and it is such that A increases over time while q decreases over time.

Proof. We select the steady state with the lowest value for A . Setting $\dot{A} = \dot{q} = 0$ in (106) and (107) we obtain (108)-(109). Let us turn to non-stationary equilibria. In order to determine the global solution we represent the phase diagram of (106)-(107). The A -isoclines are $A = \bar{A}$ and $A = 1$. The equation of the q -isocline is

$$rq + \frac{(\bar{A} - A)^2 q^2}{2\bar{c}} = \alpha\sigma(1 - A)[u(q) - q].$$

For all $A < \bar{A}$, the left side is strictly convex while the right side is strictly concave in q . Moreover, for any $A < \bar{A}$, there is a unique positive solution and it is such that q decreases with A . The phase diagram is represented in Figure 15. The steady state is a saddle point and there is unique (saddle) path leading to it.

■

At the steady state all the money in the ground has been digged out and the value of money is independent of the mining technology.

In contrast to the model with mining as an occupation choice, here the value of money decreases over time, i.e., there is a negative correlation between the value of money and its stock.

Figure 15: Costly moning: Phase diagram

Appendix D: General matching function

Suppose now that only buyers (money holders) and producers participate in the matching process according to a constant returns to scale matching function. The matching probability of a buyer is $\alpha(\tau)$ where $\tau = (1 - A - m)/A$ is market tightness expressed as the ratio of sellers to buyers. As is standard, we assume that $\alpha' > 0$, $\alpha'' < 0$, $\alpha'(0) = +\infty$, $\alpha'(+\infty) = 0$. A matching function that satisfies these properties is the Cobb Douglas matching function.

The HJB equations of agents with and without money are:

$$rV_1 = \alpha(\tau)\sigma\theta[u(q) - q] + \dot{V}_1 \quad (110)$$

$$rV_0 = \max \left\{ \frac{\alpha(\tau)}{\tau}\sigma(1 - \theta)[u(q) - q], \lambda(\bar{A} - A)\omega(q) \right\} + \dot{V}_0. \quad (111)$$

The novelty is that the matching rate of a buyer is $\alpha(\tau)$ while the matching rate of a seller is $\alpha(\tau)/\tau$. Using that $\lim_{\tau \rightarrow 0} \alpha(\tau)/\tau = +\infty$, it follows that $\tau > 0$ in equilibrium, i.e., $m < 1 - A$. The goods market is always active and

$$\max \left\{ \frac{\alpha(\tau)}{\tau}\sigma(1 - \theta)[u(q) - q], \lambda(\bar{A} - A)\omega(q) \right\} = \frac{\alpha(\tau)}{\tau}\sigma(1 - \theta)[u(q) - q]. \quad (112)$$

Subtracting (111) from (110) the value of money solves:

$$r\omega(q) = \left[\alpha(\tau)\sigma\theta - \frac{\alpha(\tau)}{\tau}\sigma(1 - \theta) \right] [u(q) - q] + \omega'(q)\dot{q}. \quad (113)$$

From (112) market tightness in the goods market solves:

$$\frac{\alpha(\tau)}{\tau}\sigma(1 - \theta)[u(q) - q] \geq \lambda(\bar{A} - A)\omega(q), \quad " = " \text{ if } \tau < \frac{1 - A}{A}.$$

Solving for τ we obtain:

$$\tau(\omega, A) = \min \left\{ g^{-1} \left[\frac{\lambda(\bar{A} - A)\omega}{\sigma(1 - \theta)S(\omega)} \right], \frac{1 - A}{A} \right\}. \quad (114)$$

where $S(\omega) \equiv u[q(\omega)] - q(\omega)$ and $g(\tau) \equiv \alpha(\tau)/\tau$. For all (ω, A) such that $\frac{\lambda(\bar{A} - A)\omega}{\sigma(1 - \theta)S(\omega)} \geq g\left(\frac{1 - A}{A}\right)$, $m > 0$ and $\tau(\omega, A)$ is decreasing in ω and increasing in A . Moreover, $\tau(+\infty, A) = 0$ and $\tau(0, A) > 0$. The money supply evolves according to

$$\dot{A} = [1 - A(1 + \tau)]\lambda(\bar{A} - A), \quad (115)$$

where we used that $1 - A(1 + \tau) = m$.

We summarize the equilibrium by a system of two ODEs in ω and A :

$$\dot{\omega} = r\omega - \{\alpha[\tau(\omega, A)]\sigma\theta - g[\tau(\omega, A)]\sigma(1 - \theta)\}S(\omega) \quad (116)$$

$$\dot{A} = \{1 - A[1 + \tau(\omega, A)]\}\lambda(\bar{A} - A) \quad (117)$$

The locus of the points such that $\dot{A} = 0$ corresponds to all pairs (ω, A) such that $\tau(\omega, A) = (1 - A)/A$. From (114) it is given by:

$$\frac{\lambda(\bar{A} - A)\omega}{\sigma(1 - \theta)S(\omega)} \leq g\left(\frac{1 - A}{A}\right). \quad (118)$$

Condition (118) at equality gives a positive relationship between ω and A . As ω approaches 0, A tends to the solution to $\lambda(\bar{A} - A) = \sigma g\left(\frac{1 - A}{A}\right)$. As ω tends to $+\infty$, A tends to $\bar{A}$. This locus is represented by a red upward-sloping curve in Figure 16.

The locus of the points such that $\dot{\omega} = 0$ and $\dot{A} > 0$ is such that

$$r\frac{\omega}{S(\omega)} = \{\alpha[\tau(\omega, A)]\sigma\theta - g[\tau(\omega, A)]\sigma(1 - \theta)\}. \quad (119)$$

The left side is increasing in ω while the right side is decreasing in ω but increasing in A . For given A there is a unique ω solution to (119) provided that

$$r(1 - \theta) < \{\alpha[\tau(0, A)]\sigma\theta - g[\tau(0, A)]\sigma(1 - \theta)\},$$

where $\tau(0, A)$ is the solution to $g(\tau) = \lambda(\bar{A} - A)/\sigma$. If this condition holds for $A = 0$, then it holds for all A . Hence, we assume

$$r(1 - \theta) < [\alpha(\tau_0)\sigma\theta - g(\tau_0)\sigma(1 - \theta)] \text{ where } \tau_0 = g^{-1}[\lambda(\bar{A} - A)/\sigma]. \quad (120)$$

Assuming this condition is satisfied, the ω -isocline is upward sloping as illustrated in Figure 16. As A goes to zero, ω tends to a positive value.

There is a unique steady state such that agents are indifferent between mining and not mining and it solves

$$g(\tau) = g\left(\frac{1 - A}{A}\right) = \frac{\lambda(\bar{A} - A)\omega}{\sigma(1 - \theta)S(\omega)} \quad (121)$$

$$r\frac{\omega}{S(\omega)} = [\alpha(\tau)\sigma\theta - g(\tau)\sigma(1 - \theta)]. \quad (122)$$

Equation (121) specifies the market tightness such that agents are indifferent between mining or participating in the goods market. Equation (122) gives the value of money given market tightness. Combining (121) and

Figure 16: Phase diagram under matching function satisfying Inada conditions

(122), steady-state market tightness solves:

$$\sigma(1 - \theta) \left[\frac{r}{\lambda \left(\bar{A} - \frac{1}{1+\tau} \right)} + 1 \right] g(\tau) = \alpha(\tau) \sigma \theta. \quad (123)$$

It is easy to check that there is a unique $\tau^s \in \left(0, \frac{1-\bar{A}}{\bar{A}}\right)$ solution to this equation. The supply of money at the steady state is then $A^s = 1/(1 + \tau^s)$. The equilibrium is monetary if (120) holds. The existence of a unique steady state guarantees that the A -isocline and ω -isocline only intersect once, i.e., the ω -isocline is located above the A -isocline as illustrated in Figure 16.

In Figure 16 we represent the phase diagram of the dynamic system (116)-(117) and its arrows of motion. It can easily be checked that the steady state is a saddle point and given the initial condition $A_0 = 0$ there is a unique path leading to it. Along that path the value of money increases over time. There is also a continuum of other equilibria where the value of money vanishes asymptotically. We summarize our results in the following proposition.

In order to characterize the path for market tightness, we can rewrite (119) as

$$r \frac{\omega(\tau, A)}{S[\omega(\tau, A)]} = [\alpha(\tau) \sigma \theta - g(\tau) \sigma(1 - \theta)],$$

where $\omega(\tau, A)$ is defined implicitly by $\tau = \tau(\omega, A)$. Assuming $m > 0$, ω is a decreasing function of τ and an increasing function of A . Hence, the τ -isocline is upward sloping. The A -isocline becomes $A = 1/(1 + \tau)$. By the same reasoning as above, the saddle path is upward sloping, which means that τ increases over time.

Appendix E: Interest-bearing/commodity monies

Suppose the money is either a commodity that provides some direct utility, e.g., gold or silver, or it is a financial asset that pays interest. We denote $d > 0$ the dividend flow enjoyed by each money holder. The Bellman equation of an agent with one unit of money becomes:

$$rV_1 = d + \alpha\sigma(1 - A - m)\theta[u(q) - q] + \dot{V}_1. \quad (124)$$

The only novelty is the first term on the right side representing the dividend flow. The Bellman equation for an agent without money is unchanged. It follows that the dynamic equation for the value of money is:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma(1 - A - m)\theta[u(q) - q] \\ &\quad - \max\{\alpha\sigma A(1 - \theta)[u(q) - q], \lambda(\bar{A} - A)\omega(q)\} + \omega'(q)\dot{q}. \end{aligned} \quad (125)$$

A steady-state equilibrium, (q^s, A^s) , solves:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma(\theta - A)[u(q) - q] \\ A &= \frac{\lambda\bar{A}\omega(q)}{\alpha\sigma(1 - \theta)[u(q) - q] + \lambda\omega(q)} \end{aligned} \quad (126)$$

The first equation gives a negative relationship between q and A while the second equation gives a positive relationship between A and q . So there is a unique steady state and $\partial q^s / \partial d > 0$ and $\partial A^s / \partial d > 0$.

Let us turn to transitional dynamics. Suppose $m < 1 - A$, the trajectory follows $A = \mu(q)$ as in the baseline model. Suppose $m = 1 - A$. Then:

$$\dot{q} = \frac{[r + \lambda(\bar{A} - A)]\omega(q) - d}{\omega'(q)} \quad (127)$$

$$\dot{A} = (1 - A)\lambda(\bar{A} - A). \quad (128)$$

The slope $\partial q / \partial A = \dot{q} / \dot{A}$ falls in d for any given (A, q) , but one can show that $\dot{q} > 0$ in equilibrium. If $\dot{q} = 0$ at certain time t , then $\dot{q} < 0$ for after t by (127). The equilibrium cannot change regime after t as a regime switch requires both trajectories to have the same slope and the locus $A = \mu(q)$ is always upward sloping. In the regime $m = 1 - A$, $\dot{q} = \alpha\sigma(1 - A)\theta[u(q) - q] / \omega'(q) > 0$ when $q \approx q^s$ by (125) and (126). Therefore $\dot{q} = 0$ is impossible at all time.

By the proof of Proposition 2, mining and production co-exist near the steady state if only if

$$\left. \frac{\partial q}{\partial A} \right|_{m=1-A} > \left. \frac{\partial q}{\partial A} \right|_{m \in (0, 1-A)} \iff \frac{\mu'(q^s) / \mu(q^s)}{\omega'(q^s) / \omega(q^s)} > \frac{1 - \theta}{\theta}.$$

Figure 17: Production and purchasing power of gold.

As d increases there are two opposing effects. Since $\partial q / \partial A|_{m=1-A}$ falls in d for any given (A, q) , it is more likely that $m = 1 - A$ near the steady state when d is large. On the other hand q^s and A^s increase in d and therefore agents have less incentive to mine around the steady state. The net effect is ambiguous in general.

Figure 18: Production and purchasing power of silver.

Appendix F: Competing private monies

Under the assumption $\lambda^g \bar{A}^g \geq \lambda^u \bar{A}^u$, when A^g and A^u are close to 0, then only silver is mined, $A = A^g$.

The indifference condition between occupations is

$$\alpha\sigma(1-\theta)[u(q) - q] = \lambda^g(\bar{A}^g - A)\phi, \quad (129)$$

and the law of motion of the total asset supply is

$$\dot{A} = m\lambda^g(\bar{A}^g - A). \quad (130)$$

When the supply of silver is sufficiently large, agents have incentives to mine gold as well. Whenever the two assets are mined, $\lambda^g(\bar{A}^g - A^g) = \lambda^u(\bar{A}^u - A^u)$, which implies

$$\bar{A}^g - A^g = \frac{\lambda^u}{\lambda^g + \lambda^u}(\bar{A} - A).$$

Figure 19: Difficulty level of Bitcoin mining and price of Bitcoin.

Figure 20: Fixing the growth rate of money can induce endogenous fluctuations in prices.

The fraction of silver remaining in the ground is a constant fraction of the total quantity of undug assets. The indifference condition between production and mining can be rewritten as

$$\alpha\sigma(1-\theta)[u(q)-q] = \frac{\lambda^g\lambda^u}{\lambda^g+\lambda^u}(\bar{A}-A)\phi. \quad (131)$$

Note that this condition is identical to the one in the one-asset economy where the effective mining rate is $\lambda = \lambda^g\lambda^u/(\lambda^g+\lambda^u)$. Agents will remain indifferent between mining silver or gold if $\lambda^g\dot{A}^g = \lambda^u\dot{A}^u$, which implies $\lambda^gm^g = \lambda^um^u$, where m^g is the measure of silver miners, m^u is the measure of gold miners, and $m = m^g + m^u$. The growth of the total supply of assets, $\dot{A} = m^g\lambda^g(\bar{A}^g - A^g) + m^u\lambda^u(\bar{A}^u - A^u)$, is equal to

$$\dot{A} = \frac{\lambda^u\lambda^g}{\lambda^g+\lambda^u}m(\bar{A}-A), \quad (132)$$

where we used that $m^g = \lambda^um/(\lambda^g+\lambda^u)$.

The phase diagram corresponding to this dual asset economy is represented in Figure 21. The indifference condition, (129), corresponds to the upward-sloping purple curve labeled MG . The indifference condition, (131), corresponds to the red upward-sloping curve labeled $M2$. For an equilibrium starting from $A^u = A^g = 0$, the relevant indifference condition is the frontier of the yellow area where $m = 1$, i.e.,

$$\alpha\sigma(1-\theta)[u(q)-q] = \max\left\{\lambda^g(\bar{A}^g - A)\phi, \frac{\lambda^u\lambda^g}{\lambda^g+\lambda^u}(\bar{A}-A)\phi\right\}.$$

The two indifference loci, MG and $M2$, intersect for a positive ϕ , as illustrated in Figure 21, only if $\lambda^g\bar{A}^g > \lambda^u\bar{A}^u(\alpha\sigma + \lambda^g)/\alpha\sigma$. Otherwise, $M2$ is located to the right of MG . The level of assets at which agents transition from mining silver only to mining both gold and silver is $\hat{A} \equiv (\lambda^g\bar{A}^g - \lambda^u\bar{A}^u)/\lambda^g$.

Proof of Proposition 9. There is production of both gold and silver along the equilibrium path if $\hat{A} < A^s$. Graphically, this condition is satisfied if the intersection of the loci MG and $M2$, $(\hat{A}, \hat{\phi})$, is located below the locus $\phi^s(A)$. See Figure 21. Let $\hat{\ell} \equiv \hat{\phi}\hat{A}$ denote the aggregate liquidity at the intersection of MG and $M2$. At the intersection $A^u = 0$ and the agents are indifferent between producing or mining gold, therefore $\alpha\sigma(1-\theta)S(\hat{\ell}) = \hat{\ell}\lambda^u\bar{A}^u/\hat{A}$. By the definition of $\hat{A}$, $\hat{\ell}$ is the largest solution of

$$\alpha\sigma(1-\theta)S(\hat{\ell}) = \hat{\ell}\frac{\lambda^g\lambda^u\bar{A}^u}{\lambda^g\bar{A}^g - \lambda^u\bar{A}^u}.$$

This solution is strictly positive if $\lambda^g\bar{A}^g > \lambda^u\bar{A}^u(\alpha\sigma + \lambda^g)/\alpha\sigma$ and $\hat{\ell} = 0$ otherwise. Moreover, $\hat{\ell}$ increases with $(\lambda^g\bar{A}^g - \lambda^u\bar{A}^u)/\lambda^g\lambda^u\bar{A}^u$. By (42) the condition $\hat{A} < A^s$ is equivalent to:

$$r < \frac{d}{\hat{\phi}} + \alpha\sigma\theta S'(\hat{\ell}) \iff r < \frac{d\lambda^u\bar{A}^u}{\alpha\sigma(1-\theta)S(\hat{\ell})} + \alpha\sigma\theta S'(\hat{\ell}).$$

Figure 21: Phase diagram for a dual asset economy

The right side is decreasing in $\hat{\ell}$ and it tends to infinity as $\hat{\ell}$ goes to 0. Hence, this condition is satisfied provided that $(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u) / \lambda^g \lambda^u \bar{A}^u$ is sufficiently low, i.e.,

$$\frac{\lambda^g \bar{A}^g - \lambda^u \bar{A}^u}{\lambda^g \lambda^u \bar{A}^u} < \kappa_0$$

for $\kappa_0 > 0$. The threshold κ_0 is endogenous but by construction it does not depend on λ^g and $\bar{A}^g$. This implies that, fixing all parameters except $\bar{A}^g$, the inequality is satisfied as $\bar{A}^g \rightarrow \lambda^u \bar{A}^u / \lambda^g$ from above. Similarly the inequality is satisfied when λ^g is sufficiently small. The construction of the equilibrium going backward from the steady state is as described in earlier proposition and is therefore omitted. ■